International Journal of Maps in Mathematics

Volume ***8****, Issue* ***2****, 2025, Pages:****481-492***

E-ISSN: 2636-7467

www.simadp.com/journalmim

ON HYPERCYCLICITY OF WEIGHTED COMPOSITION OPERATORS ON STEIN MANIFOLDS

FIROOZ PASHAIE *, MOHAMMAD REZA AZIMI, AND SEYYED MOHAMMAD SHAHIDI

Abstract. In this manuscript, we study the hypercyclicity of weighted composition operators defined on the set of holomorphic complex functions on a connected Stein n-manifold $\mathbf{M}$. We show that a weighted composition operator $\mathbf{C}_{\psi,\omega}$ (associated to a holomorphic self-map ψ and a holomorphic function ω on $\mathbf{M}$) is hypercyclic with respect to an increasing sequence $(n_l)_l$ of natural numbers if and only if at every $p \in \mathbf{M}$ we have $\omega(p) \neq 0$ and the self-map ψ is injective without any fixed points in $\mathbf{M}$, $\psi(\mathbf{M})$ is a Runge domain and for every $\mathbf{M}$-convex compact subset $C \subset \mathbf{M}$ there is a positive integer number k such that the sets C and $\psi^{[n_k]}(C)$ are separable in $\mathbf{M}$.

Keywords: Holomorphic, Composition operators, Hypercyclic, Convex.

2020 Mathematics Subject Classification: Primary: 47B33, 47B38, Secondary: 32H50.

1. Introduction

Let U be a domain in the complex plane $\mathbb{C}$, and $\mathbb{H}(\mathrm{U})$ be the space of holomorphic complex functions in U. The space $\mathbb{H}(\mathrm{U})$ is endowed with the topology of locally uniform convergence, under which it becomes a complete separable metric space. We are interested in proving the existence of dense orbits for composition operators on $\mathbb{H}(\mathrm{U})$. If ψ is a holomorphic self-map on U, then the composition operator associated to ψ is defined as $\mathbf{C}_{\psi}(f) = f \circ \psi$ for every $f \in \mathbb{H}(\mathrm{U})$.

Received: 2024.10.26 *Revised: 2025.02.22* *Accepted: 2025.03.14*

* Corresponding author

Firooz Pashaie ⋄ f_pashaie@maragheh.ac.ir ⋄ https://orcid.org/0000-0002-3020-7649

Mohammad Reza Azimi ⋄ mhr.azimi@maragheh.ac.ir⋄ https://orcid.org/0000-0002-7337-1617

Seyyed Mohammad Shahidi ⋄ muhamd.shahidi@gmail.com ⋄ https://orcid.org/0009-0000-2150-9074.

The first step has been taken in 1929 by Birkhoff ([7]) when he proved that there exists an entire function $\lambda : \mathbb{C} \mapsto \mathbb{C}$ such that $\{\lambda \circ t_n\}_{n\in\mathbf{N}}$ forms a dense set in $\mathbb{H}(\mathbb{C})$, where $\{t_n\}_{n=1}^{\infty}$ is the sequence of $\mathbb{C}$-automorphisms defined by $t_n : z \mapsto z + n$. The function λ is called universal.

Gethner and Shapiro have studied universal vectors for operators on spaces of holomorphic functions in 1987 ([13]). In 90s, the subject of cyclic composition operators has been discussed by many researchers ([8, 9, 10, 14]). In the same decade, some generalizations to hypercyclic operators have also been studied ([15, 20, 21]).

In 2001, Shapiro studied the dynamics of linear operators ([22]) which followed by Grose-Erdman in 2003 ([16]). As a concrete example, Bernal-Gonzales has studied the universal entire functions for affine endomorphisms on $\mathbb{C}^n$ in 2005.

A class of linear fractional maps of the ball and its composition operators has been considered by Bayart in 2007 ([5]). One can find the continuation of research progress on the hypercyclicity of operators in the references [6, 11, 18, 24]. Between them, the manuscript [24] has a special importance because it discuss on the hypercyclicity of composition operators associated to some holomorphic self-maps defined on an important class of complex manifolds namely Stein manifolds. The important properties of Stein manifolds can be found in [24].

The weighted composition operators associated to some holomorphic self-maps have been interested in some recent researches (see for instance [1, 2, 3, 4, 23]). Also, in [19], the authors have studied the dynamics of weighted composition operators on Stein manifolds, where the maps and functions are defined on a Stein manifold.

In this paper, we consider a holomorphic self-map $\psi \in \mathcal{O}(\mathbf{M})$ defined on a connected Stein n-manifold $\mathbf{M}$ and a holomorphic function $\omega \in \mathbb{H}(\mathbf{M})$. We study the hypercyclicity of weighted composition operator $\mathbf{C}_{\psi,\omega} : \mathbb{H}(\mathbf{M}) \to \mathbb{H}(\mathbf{M})$ defined by rule $\mathbf{C}_{\psi,\omega}(f) := \omega \cdot (f \circ \psi)$ with respect to an increasing sequence of natural numbers.

We prove that $\mathbf{C}_{\psi,\omega}$ is hypercyclic if and only if for every $p \in \mathbf{M}$, $\omega(p) \neq 0$ and ψ is univalent without fixed points in $\mathbf{M}$, $\psi(\mathbf{M})$ is a Runge domain and for every compact holomorphically convex set $C \subset \mathbf{M}$ there is an integer n such that $C \cap \psi^{[n]}(C) = \emptyset$ and their sum is $\mathbf{M}$-convex.

In the study of hypercyclicity of $\mathbf{C}_{\psi,\omega}$, which is connected with some approximation theorems, one can use two well-known theorems namely the Runge Theorem and Oka-Weil Theorem.

2. Preliminaries

In this section, we present the preliminary concepts and notations from [1, 2, 5, 6, 17, 24]. We denote the family of all open subsets of a given topological space X by $\mathbf{Op}(X)$ and the family of all compact subsets of X by $\mathbf{Cp}(X)$. As usual, $\mathcal{C}(X,Y)$ denotes the set of all continuous maps between two topological spaces X and Y.

Definition 2.1. For every $C \in \mathbf{Cp}(X)$ and $U \in \mathbf{Op}(Y)$, the set of functions $f \in \mathcal{C}(X,Y)$ satisfying condition $f(C) \subset U$ is denoted by $\mathcal{V}(C,U)$. The topology generated by subbase

$$\triangle := \{\mathcal{V}(C,U) | C \in \mathbf{Cp}(X), U \in \mathbf{Op}(Y)\}$$

is called the *compact-open topology* on $\mathcal{C}(X,Y)$.

We note that $\triangle$ does not always form a base for a topology on $\mathcal{C}(X,Y)$. The compact-open topology (which is applied in homotopy theory and functional analysis) was introduced by Ralph Fox in 1945 [12].

A continuous map $f \in \mathcal{C}(X,Y)$ is said to be *proper* if each connected component of $f^{-1}(K)$ is compact for every $K \in \mathbf{Cp}(Y)$.

Definition 2.2. Let X be a topological vector space and $\{\alpha_r : X \to X\}_{r=1}^{\infty}$ be a sequence of continuous self-maps on X.

(1) $\{\alpha_r\}_{r=1}^{\infty}$ is called *topologically transitive* if for every non-empty $U, V \in \mathbf{Op}(X)$ there exists r_0 such that $\alpha_{r_0}(U) \cap V \neq \emptyset$.

(2) A point $p \in X$ is said to be an *universal element* for $\{\alpha_r\}_{r=1}^{\infty}$ if the sequence $\{\alpha_r(p)\}_{r=1}^{\infty}$ of points is dense in X.

(3) A point $p \in X$ is said to be an *weakly universal element* for $\{\alpha_r\}_{r=1}^{\infty}$ if the sequence $\{\alpha_r(p)\}_{r=1}^{\infty}$ of points is dense in X with respect to the weak topology of X.

(4) The sequence $\{\alpha_r\}_{r=1}^{\infty}$ is said to be *universal* if it admits a universal element.

(5) The sequence $\{\alpha_r\}_{r=1}^{\infty}$ is said to be *weakly universal* if it admits a weakly universal element.

Definition 2.3. Let X be a topological vector space and $\alpha : X \to X$ be a continuous self-map on X.

(1) The iterations of α is defined by $\alpha^{[1]} = \alpha$, $\alpha^{[2]} = \alpha \circ \alpha$ and $\alpha^{[r+1]} = \alpha \circ \alpha^{[r]}$ for integer number $r \geq 2$.

(2) We say that α is *hypercyclic with respect to an increasing sequence* $\{r_k\}_{k=1}^{\infty} \subset \mathbb{N}$ if the sequence $\{\alpha^{[r_k]}\}_{k=1}^{\infty}$ is universal.

(3) We say that α is *weakly hypercyclic with respect to an increasing sequence* $\{r_k\}_{k=1}^{\infty} \subset \mathbb{N}$ if the sequence $\{\alpha^{[r_k]}\}_{k=1}^{\infty}$ is weakly universal.

(4) α is called *hypercyclic* if it is hypercyclic with respect to the full sequence $\{r\}_{r=1}^{\infty}$.

(5) α is called *weakly hypercyclic* if it is hypercyclic with respect to the full sequence $\{r\}_{r=1}^{\infty}$.

Here, we recall an essential theorem from [15] which gives a necessary and sufficient condition for topological transitivity of a sequence of continuous linear maps on a separable Fréchet space using the set of its universal elements. Remember that, a Fréchet space is a complete locally convex metrizable topological vector space.

Theorem 2.1. Let $\mathbf{F}$ be separable Fréchet space and $\{\alpha_r\}_{r=1}^{\infty}$ be a sequence of continuous self-maps on $\mathbf{F}$. This sequence is topologically transitive if and only if the set of its universal elements is dense in $\mathbf{F}$. Moreover, in this case the set of universal elements for $\{\alpha_r\}_{r=1}^{\infty}$ is a dense G_δ-subset of $\mathbf{F}$.

Also, we recall another useful theorem from [15] in this context.

Theorem 2.2. Let $\mathbf{F}$ be separable Fréchet space and $\{\alpha_r\}_{r=1}^{\infty}$ be a sequence of continuous self-maps on $\mathbf{F}$. If α_r has dense range in $\mathbf{F}$ for each $r \in \mathbb{N}$ and the sequence $\{\alpha_r\}_{r=1}^{\infty}$ is commuting (i.e. for every $r, s \in \mathbb{N}$, we have $\alpha_r \circ \alpha_s = \alpha_s \circ \alpha_r$), then the set of universal elements of $\{\alpha_r\}_{r=1}^{\infty}$ is empty or dense in $\mathbf{F}$.

The hypercyclicity of a bounded linear map α on a Fréchet space $\mathbf{F}$ means that for a vector $\mathbf{v} \in \mathbf{F}$, its orbit (i.e. $Orb(\alpha, \mathbf{v}) = \{\alpha^{[r]}(\mathbf{v})\}_{r=1}^{\infty}$) is dense in $\mathbf{F}$. By these theorems we get a corollary that allows us to investigate topological transitivity instead of hypercyclicity. Also, Theorem 3 in [15] has a similar argument.

Corollary 2.1. Let X be a separable Fréchet space, let $\alpha : X \to X$ be a continuous map, and let $\{r_k\}_{k=1}^{\infty} \subset \mathbb{N}$ be an increasing sequence. Then, α is hypercyclic w.r.t. $\{r_k\}_{k=1}^{\infty}$ if and only if the sequence $\{\alpha^{[r_k]}\}_{k=1}^{\infty}$ is topologically transitive.

Now, we introduce the Stein manifold which plays main role in this paper.

Definition 2.4. A complex manifold $\mathbf{M}$ of (finite) dimension n is called a *Stein manifold*, if it satisfies the following four conditions:

(1) $\mathbf{M}$ admits a *compact exhaustion*, which means that, there is a sequence $(C_r)_{r=1}^{\infty}$ of compact subsets of $\mathbf{M}$ such that $\mathbf{M} = \bigcup_{r=l}^{\infty} C_r$ and for each r, $C_r \subset (C_{r+1})^0$.

(2) $\widehat{C}_{\mathbf{M}} \in \mathbf{Cp}(\mathbf{M})$ for every $C \in \mathbf{Cp}(\mathbf{M})$, where

$$\widehat{C}_{\mathbf{M}} := \{p \in \mathbf{M} : |f(p)| \leq \sup_{C} |f|, \forall f \in \mathcal{O}(\mathbf{M})\}$$

is the holomorphic hull of C.

(3) $\mathbb{H}(\mathbf{M})$ separates points in $\mathbf{M}$, i.e. for each two distinct points $p, q \in \mathbf{M}$, there exists $f \in \mathbb{H}(\mathbf{M})$ with $f(p) \neq f(q)$,

(4) For each $p \in \mathbf{M}$ there exists a map $F \in \mathcal{O}(\mathbf{M}, \mathbb{C}^n)$ such that the derivative of F at p is an isomorphism.

Definition 2.5. Let $\mathbf{M}$ be a Stein n-manifold.

(1) A $C \in \mathbf{Cp}(\mathbf{M})$ is said to be $\mathbf{M}$-convex (equivalently, holomorphically convex) if $\widehat{C}_{\mathbf{M}} = C$.

(2) In special case $\mathbf{M} = \mathbb{C}^n$, $\widehat{C}_{\mathbf{M}}$ is denoted with shorter symbol $\widehat{C}$ and is called the polynomial hull of C.

(3) A $C \in \mathbf{Cp}(\mathbb{C}^n)$ is called polynomially convex if $C = \widehat{C}$.

For two finite-dimensional complex manifolds $\mathbf{M}$, $\mathbf{N}$, the notation $\mathcal{O}(\mathbf{M}, \mathbf{N})$ denotes the set of all holomorphic maps $\phi : \mathbf{M} \to \mathbf{N}$. In special cases, we use simple notations $\mathcal{O}(\mathbf{M}) := \mathcal{O}(\mathbf{M}, \mathbf{M})$ and $\mathbb{H}(\mathbf{M}) := \mathcal{O}(\mathbf{M}, \mathbb{C})$. A holomorphic function on an open subset of the complex plane is called univalent if it is injective.

Definition 2.6.

(1) We say that a sequence of holomorphic maps $\{\phi_k \in \mathcal{O}(\mathbf{M}, \mathbf{N})\}_{k=1}^{\infty}$ is *compactly divergent* (in $\mathcal{O}(\mathbf{M}, \mathbf{N})$) if for each $C \in \mathbf{Cp}(\mathbf{M})$ and $K \in \mathbf{Cp}(\mathbf{N})$ there is k_0 such that $\phi_k(C) \cap K = \emptyset$ for all $k \geqslant k_0$.

(2) The sequence $\{\phi_k \in \mathcal{O}(\mathbf{M}, \mathbf{N})\}_{k=1}^{\infty}$ is said to be *run-away* (in $\mathcal{O}(\mathbf{M}, \mathbf{N})$) if for each $C \in \mathbf{Cp}(\mathbf{M})$ and $K \in \mathbf{Cp}(\mathbf{N})$, there is k_0 such that $\phi_{k_0}(C) \cap K = \emptyset$. In the case $\mathbf{M} = \mathbf{N}$, it is always enough to consider the situation when $C = K$.

When $\mathbf{M}$ and $\mathbf{N}$ admit compact exhaustions, the sequence $\{\phi_k\}_{k=1}^{\infty}$ is run-away if and only if it has a compactly divergent subsequence.

A holomorphic map $f \in \mathcal{O}(\mathbf{M}, \mathbf{N})$ between to complex manifold is called *regular* if its derivative is a monomorphism at each point of $\mathbf{M}$.

A *Runge domain* in a Stein Manifold $\mathbf{M}$ is a domain $U \subset \mathbf{M}$ such that every function $f \in \mathbb{H}(U)$ can be approximated uniformly on U by a sequence of members of $\mathbb{H}(\mathbf{M})$. By the well-known Oka-Weil theorem, on every compact $\mathbf{M}$-convex subset $C \subset \mathbf{M}$, every holomorphic function (i.e. holomorphic on a neighborhood of C) can be approximated uniformly by functions from $\mathbb{H}(\mathbf{M})$.

Remark 2.1. By condition (1) of Definition 2.4, a Stein manifold $\mathbf{M}$ has a compact exhaustion $\{C_k\}_{k=1}^{\infty}$ such that $\bigcup_{k=l}^{\infty} C_k = \mathbf{M}$ and for each k, $C_k \subset (C_{k+1})^0$. So, we can take a sequence of semi-norms $\{p_k : \mathbb{H}(\mathbf{M}) \to \mathbb{R}\}_{k=1}^{\infty}$ defined by $p_k(f) := \sup\{|f(p)| p \in C_k\}$, which gives the topology of $\mathbb{H}(\mathbf{M})$. So, $\mathbb{H}(\mathbf{M})$ with this topology is a separable Fréchet space (see [23, 24]). This observation allows us to use Corollary 2.1 for the space $X = \mathbb{H}(\mathbf{M})$, with $\mathbf{M}$ being a connected Stein manifold.

Remark 2.2. By theorem from [24], a domain U in a connected Stein manifold $\mathbf{M}$ is a Runge domain if and only if every compact subset $C \subset U$ satisfies $\widehat{C}_{\mathbf{M}} = \widehat{C}_U$. Also, that condition is equivalent to equality $\widehat{C}_{\mathbf{M}} \cap U = \widehat{C}_U$ for every compact subset $C \subset U$.

For every locally compact topological space X, the usual compactification with one point $\infty_X \notin X$ is denoted by $X_c = X \cup \{\infty_X\}$.

It is clear that, if a continuous self-map α defined on a topological vector space X is hypercyclic, then any universal element of $\{\alpha^{[r]}\}_{r=1}^{\infty}$ is a hypercyclic vector. Finally, we have a useful lemma which guarantees that the adjoint operator of a weakly hypercyclic operator on a topological vector space dose not have any eigenvector.

Lemma 2.1. The adjoint operator of a weakly hypercyclic operator on a topological vector space does not have any eigenvector.

Proof. Let α be a weakly hypercyclic linear self-map on a topological vector space X. Clearly, α is 1-weakly. Hence, α^* does not have any eigenvectors by Proposition 3.2 in [11]. □

The following well-known theorems ([24]) characterizes the Runge domains in a Stein manifold $\mathbf{M}$ in the language of holomorphic hulls.

Theorem 2.3. Let U be a Stein manifold which is a domain of a connected Stein manifold $\mathbf{M}$. Then, the following conditions are equivalent:

(1) The domain U is a Runge domain in $\mathbf{M}$.
(2) $\widehat{C}_{\mathbf{M}} = \widehat{C}_U$ for every compact subset $C \subset U$.

(3) $\widehat{C}_{\mathbf{M}} \cap U = \widehat{C}_U$ for every compact subset $C \subset U$.

Theorem 2.4. Let C and D be two compact subsets of a connected Stein manifold $\mathbf{M}$. Then the following conditions are equivalent:

(1) C and D are separable in $\mathbf{M}$.

(2) There exist open and disjoint subsets $U, V \subset \mathbf{M}$ such that $\widehat{C}_{\mathbf{M}} \subset U$, $\widehat{D}_{\mathbf{M}} \subset V$ and $(\widehat{C \cup D})_{\mathbf{M}} \subset U \cup V$.

(3) $\widehat{C}_{\mathbf{M}} \cap \widehat{D}_{\mathbf{M}} = \emptyset$ and $(\widehat{C \cup D})_{\mathbf{M}} = \widehat{C}_{\mathbf{M}} \cup \widehat{D}_{\mathbf{M}}$.

In particular, if C and D are disjoint and $\mathbf{M}$-convex, then $C \cup D$ is $\mathbf{M}$-convex if and only if C and D are separable in $\mathbf{M}$.

Corollary 2.2. Let C and D be two disjoint compact subsets of a connected Stein manifold $\mathbf{M}$ such that $C \cup D$ is $\mathbf{M}$-convex. Then C and D are both $\mathbf{M}$-convex.

3. Main results

In this section, we choose a $\psi \in \mathcal{O}(\mathbf{M})$ and a weight function $\omega \in \mathbb{H}(\mathbf{M})$ on a connected Stein n-manifold $\mathbf{M}$. Some necessary conditions for hypercyclicity of the weighted composition operator $\mathbf{C}_{\psi,\omega}$ with respect to an increasing sequence of natural numbers $\{n_k\}_{k=1}^{\infty}$ are presented.

Proposition 3.1. Let $\{n_k\}_{k=1}^{\infty}$ be an increasing sequence of natural numbers, $\mathbf{M}$ be a connected Stein n-manifold, $\omega \in \mathbb{H}(\mathbf{M})$ and $\psi \in \mathcal{O}(\mathbf{M})$. If the weighted composition operator $\mathbf{C}_{\psi,\omega}$ is hypercyclic with respect to $\{n_k\}_{k=1}^{\infty}$, then the following conditions hold:

(1) $\omega \neq 0$ on $\mathbf{M}$ and ψ has no fixed point in $\mathbf{M}$.

(2) ψ is injective.

(3) $\psi(\mathbf{M})$ is a Runge domain w.r.t. $\mathbf{M}$.

(4) The sequence $\{\psi^{[n_k]}\}_{k=1}^{\infty}$ is run-away.

Proof.

(1) Remember that $\mathbb{H}(\mathbf{M})$ is a separable Fréchet space and the point evaluation linear functional $\mathcal{E}_p : \mathbb{H}(\mathbf{M}) \to \mathbb{C}$ (at each point $p \in \mathbf{M}$) defined by $\mathcal{E}_p(h) := h(p)$ is continuous. The adjoint of $\mathbf{C}_{\psi,\omega}$ satisfies the following equality

$$\mathbf{C}^*_{\psi,\omega}(\mathcal{E}_p)(h) = \mathcal{E}_p \circ \mathbf{C}_{\psi,\omega}(h) = \mathcal{E}_p(\omega \cdot (h \circ \psi)) = \omega(p) \cdot (h \circ \psi)(p).$$

So, $\mathbf{C}^*_{\psi,\omega}$ has an eigenvalue if $\omega(p) = 0$ or $\psi(p) = p$ and then, in these two cases $\mathbf{C}_{\psi,\omega}$ can not be hypercyclic.

(2) Since $\mathbf{C}_{\psi,\omega}$ is hypercyclic with respect to $\{n_k\}_{k=1}^{\infty}$, it admits a hypercyclic vector $g \in \mathbb{H}(\mathbf{M})$. So, for each $h \in Orb(\mathbf{C}_{\psi,\omega}, g)$ there exists a positive integer k such that

$$h = (\mathbf{C}^{[n_k]}_{\psi,\omega}(g)) = \prod_{j=0}^{n_k-1} \mathbf{C}^{[j]}_{\psi}(\omega) \cdot \mathbf{C}^{[n_k]}_{\psi} g = \omega \cdot (\prod_{j=1}^{n_k-1} \omega \circ \psi^j) \cdot (g \circ \psi^{n_k}).$$

Assuming $\psi(p) = \psi(q)$ for two distinct points $p, q \in \mathbf{M}$, we get $\frac{1}{\omega(p)} h(p) = \frac{1}{\omega(q)} h(q)$ and then

$$\frac{1}{\omega(p)} \mathcal{E}_p(h) = \frac{1}{\omega(q)} \mathcal{E}_q(h) \tag{3.1}$$

for every $h \in Orb(\mathbf{C}_{\psi,\omega}, g)$. So, by continuity of $\frac{1}{\omega(p)}\mathcal{E}_p$ and $\frac{1}{\omega(q)}\mathcal{E}_q$, it follows that the equality (3.1) holds for every $h \in \overline{Orb(\mathbf{C}_{\psi,\omega}, g)} = \mathbb{H}(\mathbf{M})$. Therefore, $\frac{1}{\omega(p)}\mathcal{E}_p = \frac{1}{\omega(q)}\mathcal{E}_q$ on $\mathbb{H}(\mathbf{M})$.

Now, putting $g = 1$, we get $\frac{1}{\omega(p)}\mathcal{E}_p(1) = \frac{1}{\omega(q)}\mathcal{E}_q(1)$ which gives $\omega(p) = \omega(q)$. Therefore, the equality $h(p) = h(q)$ holds for every $g \in \mathbb{H}(\mathbf{M})$, which by condition (3) in Definition 2.4, implies that $p = q$. So, ψ is injective.

(3) It is enough to prove that the subset of restrictions $\{h|_{\psi(\mathbf{M})} : h \in \mathcal{O}(\mathbf{M})\}$ is dense in $\mathcal{O}(\psi(\mathbf{M}))$.

If $h \in \mathcal{O}(\psi(\mathbf{M}))$, then $h \circ \psi$ is holomorphic on $\mathbf{M}$, so there is a subsequence $\{n_{l_k}\}_{k=1}^{\infty}$ of $\{n_k\}_{k=1}^{\infty}$ such that $g \circ \psi^{[n_{l_k}]} \to g \circ \psi$ on $\mathbf{M}$ (where, $g \in \mathbb{H}(\mathbf{M})$ is a hypercyclic vector for $C_{\psi,\omega}$ with respect to $\{n_k\}_{k=1}^{\infty}$). Hence $f \circ \psi^{[n_{l_k}-1]} \to h$ on $\psi(\mathbf{M})$, as the mapping ψ is a biholomorphism on its image.

(4) Let $K \subset \mathbf{M}$ be compact. For each positive integer k, there exists a positive integer n_{l_k} such that $|f \circ \psi^{[n_{l_k}]} - k| \leq \frac{1}{k}$ on K. So, for a big enough k, we have

$$\inf\{|f(z)| : z \in \psi^{[n_{l_k}]}(K)\} = \inf\{|(f \circ \psi^{[n_{l_k}]})(z)| : z \in K\} \geq k - \frac{1}{k} > \sup\{|f(z)| : z \in K\}.$$

Hence, $\psi^{[n_{l_k}]}(K) \cap K = \emptyset$.

□

Remark 3.1. It follows from the equivalence of conditions in Remark 2.2 and theorems 2.3 and 2.4 that ψ maps every $\mathbf{M}$-convex compact $C \subset \mathbf{M}$ onto an $\mathbf{M}$-convex compact set. Also, it implies that for any natural number n the set $\psi^{[n]}(C)$ is $\mathbf{M}$-convex.

It is natural to ask whether the necessary conditions given by Proposition 3.1 are sufficient. In [18], it is shown that if $\mathbf{M}$ is a simply connected or an infinitely connected planar domain or a special type of higher-dimensional Stein manifolds, then the mentioned property holds. But in general the above necessary conditions are not sufficient, as we can see using a simple example $\mathbf{M} = \mathbb{D}_*$ and $\psi(z) = \frac{1}{2}z$ then by Theorems 4.6 the operator C_ψ is not hypercyclic, although it satisfies the conditions (1), (2), (3).

Here, we prefer to re-describe the topology of $\mathcal{O}(\mathbf{M})$ and the concept of topologically transitivity of weighted composition operators.

For every $K \in \mathbf{Cp}(\mathbf{M})$ and $f_0 \in \mathbb{H}(\mathbf{M})$ and positive real number ϵ, the ϵ-neighborhood of f_0 is defined by

$$N_\epsilon^K(f_0) := \{f \in \mathbb{H}(\mathbf{M}) : \forall y \in K, |f(y) - f_0(y)| < \epsilon\}.$$

The family of all such a neighborhoods forms a basis of the topology of $\mathbb{H}(\mathbf{M})$.

With the aim of using Corollary 2.1, so let us first clear the topological transitivity of the sequence $(\mathbf{C}_{\psi,\omega}^{[n_l]})_l$.

Let $\psi \in \mathcal{O}(\mathbf{M})$ be an injective holomorphic self-map and $0 \neq \omega \in \mathbb{H}(\mathbf{M})$. The sequence $(\mathbf{C}_{\psi,\omega}^{[n_l]})_{l=1}^\infty$ is topologically transitive if and only if for every $\epsilon > 0$, $g, h \in \mathbb{H}(\mathbf{M})$ and $K \in \mathbf{Cp}(\mathbf{M})$ there are natural number k and function $f \in \mathbb{H}(\mathbf{M})$ such that $|f - g| < \epsilon$ and $|\mathbf{C}_{\psi,\omega}^{[n_k]}(f) - h| < \epsilon$ on K.

As the mapping ψ is injective and ω in non-zero, the above condition has another form:

$$|f - g| < \epsilon \text{ on } K \text{ and } |f - [\prod_{j=0}^{k-1} \mathbf{C}_\psi^j(\omega)]^{-1} \cdot h \circ \psi^{[-n_k]}| < \epsilon \text{ on } \psi^{[n_k]}(K). \tag{3.2}$$

Since $\mathbf{M}$ is a Stein manifold, we can restrict to considering only $\mathbf{M}$-convex sets.

Theorem 3.1. Let $\mathbf{M}$ be a connected Stein manifold, $\psi \in \mathcal{O}(\mathbf{M})$, $\omega \in \mathbb{H}(\mathbf{M})$ and the weighted composition operator $\mathbf{C}_{\psi,\omega}$ is hypercyclic on $\mathcal{O}(\mathbf{M})$. Then for every $\mathbf{M}$-convex compact subset $C \subset \mathbf{M}$, there exists positive integer n such that $C \cap \psi^{[n]}(C) = \emptyset$ and the set $C \cup \psi^{[n]}(C)$ is $\mathbf{M}$-convex.

Proof. Suppose that $\mathbf{C}_{\psi,\omega}$ is hypercyclic. In view of Corollary 2.1, the condition 3.2 holds. Fix an $\mathbf{M}$-convex compact set $C \subset \mathbf{M}$. By Remark 3.1 we get that the set $\psi^{[n]}(C)$ is $\mathbf{M}$-convex. Using the condition 3.2 for $g = 0$, $h = 1$ and $\epsilon = \frac{1}{2}$, we get that there are $f \in \mathcal{O}(\mathbf{M})$ and $k \in \mathbf{N}$ such that $f(C) \subset \frac{1}{2}\mathbb{D}$ and $\frac{\lambda}{2}(\psi^{[k]}(C)) \subset (1 + \frac{1}{2}\mathbb{D})$ where $\lambda = sup_C[\prod_{j=0}^{k-1} \mathbf{C}_\psi^{[j]}(\omega)]$. This implies that C and $\frac{\lambda}{2}\psi^{[k]}(C)$ are separable in $\mathbf{M}$, so by Lemma 2.9 in [24], the sum $C \cup \frac{\lambda}{2}\psi^{[k]}(C)$ is $\mathbf{M}$-convex. □

Theorem 3.2. Let $\mathbf{M}$ be a connected Stein manifold, $\psi \in \mathcal{O}(\mathbf{M})$, $\omega \in \mathbb{H}(\mathbf{M})$ and the following conditions hold:

(1) for every $p \in \mathbf{M}$, $\omega(p) \neq 0$ and ψ is an injective self-map without fixed point in $\mathbf{M}$.

(2) for every $\mathbf{M}$-convex compact subset $C \subset \mathbf{M}$, there exists positive integer n such that $C \cap \psi^{[n]}(C) = \emptyset$ and the set $C \cup \psi^{[n]}(C)$ is $\mathbf{M}$-convex.

Then, the weighted composition operator $\mathbf{C}_{\psi,\omega}$ is hypercyclic on $\mathbb{H}(\mathbf{M})$.

Proof. Assume that $\{C_n\}_{n=1}^{\infty}$ be an exhaustion of $\mathbf{M}$. Without lose of generality, we can assume that every C_n is $\mathbf{M}$-convex. Since the compact-open topology on $\mathbb{H}(\mathbf{M})$ is independent of the chosen exhaustion, we can endow $\mathbb{H}(\mathbf{M})$ with the topology induced by the semi-norms on $\mathbb{H}(\mathbf{M})$ defined by $p_n(f) := \sup\{|f(p)| : p \in C_n\}$. Let $U, V \subset \mathbb{H}(\mathbf{M})$ be non-empty open sets and fix $f \in U$ and $g \in V$. By definition of compact-open topology of $\mathbb{H}(\mathbf{M})$, there is a closed ball $B \subset \mathbf{M}$ (with respect to the Carathéodory pseudo-distance as can be seen in [24]) and a positive real number ϵ such that, every $h_1 \in U$ satisfies $\sup_{p\in B} |f(p) - h_1(p)| < \epsilon$ and similarly every $h_2 \in V$ satisfies $\sup_{p\in B} |g(p) - h_2(p)| < \epsilon$.

Now, assume that D be another closed ball such that $B \subset D^{\circ}$. Since ψ is an injective self-map without fixed point on $\mathbf{M}$, then the function f is holomorphic on some neighborhood of D, and the function $\frac{g\circ(\psi^{[n_0]})^{-1}}{\prod_{k=1}^{n_0-1}(\omega\circ(\psi^{[k]})^{-1})}$ is holomorphic on some neighborhood of $\psi^{[n_0]}(D)$.

By assumption (2), there exists n_0 such that $D \cap \psi^{[n_0]}(D) = \emptyset$ and the compact set $K := D \cup \psi^{[n_0]}(D)$ is $\mathbf{M}$-convex (by Oka-Weil theorem), there exists a holomorphic function $h \in \mathbb{H}(\mathbf{M})$ such that $\sup_{z\in D} |f(z) - h(z)| < \epsilon$ and

$$\sup_{y\in\psi^{[n_0]}(D)} \left|\frac{g \circ (\psi^{[n_0]})^{-1}}{\prod_{k=1}^{n_0}(\omega \circ (\psi^{[k]})^{-1})}(y) - h(y)\right| < \frac{\epsilon}{M}.$$

where $M := \max_{y\in\psi^{[n_0]}(D)} |\prod_{k=1}^{n_0}(\omega \circ (\psi^{[k]})^{-1})(y)|$.

Hence $\sup_{z\in K} |f(z) - h(z)| < \epsilon$ and

$$\begin{aligned}&\sup_{z\in K} |g(z) - (|K_{\psi,\omega})^{[n_0]}h(z)| \\ &= \sup_{z\in K} \left|\prod_{k=1}^{n_0}(\omega \circ (\psi^{[k]})^{-1})(y)\left(\frac{g \circ (\psi^{[n_0]})^{-1}}{\prod_{k=1}^{n_0}(\omega \circ (\psi^{[k]})^{-1})}(y) - h(y)\right)\right| < \epsilon,\end{aligned}$$

where $y := \psi^{[n_0]}(z)$. This shows that $h \in U$ and $(\mathbf{C}_{\psi,\omega})^{[n_0]}h \in V$, so that $\mathbf{C}_{\psi,\omega}$ is topologically transitive. Since $\mathbb{H}(\mathbf{M})$ is a separable Fréchet space, $\mathbf{C}_{\psi,\omega}$ is hypercyclic. □

Theorem 3.3. Let $\mathbf{M}$ be a connected Stein manifold, $\psi \in \mathcal{O}(\mathbf{M})$ and $\omega \in \mathbb{H}(\mathbf{M})$ and $\{n_l\}_{l=1}^{\infty}$ be an increasing sequence of positive integer numbers. Then the operator $\mathbf{C}_{\psi,\omega}$ is hypercyclic w.r.t. $(n_l)_l$ if and only if for every $p \in \mathbf{M}$, $\omega(p) \neq 0$ and ψ is injective without fixed points in $\mathbf{M}$, $\psi(\mathbf{M})$ is a Runge domain w.r.t. $\mathbf{M}$ and for every $\mathbf{M}$-convex compact subset $C \subset \mathbf{M}$ there is a positive integer number k such that the sets C and $\psi^{[n_k]}(C)$ are separable in $\mathbf{M}$.

Proof. Sufficiency in both parts follows from Theorem 3.1 and Theorem 3.2. If the sets C and $\psi^{[n_l]}(C)$ are separable in $\mathbf{M}$, since $\psi(\mathbf{M})$ is a Runge domain in $\mathbf{M}$ and C is $\mathbf{M}$-convex, then $\psi^{[n_l]}(C)$ is $\mathbf{M}$-convex and by a Lemma from [24] their sum is $\mathbf{M}$-convex . Necessity in both parts follows directly from Theorem 3.1. □

Acknowledgments. The authors would like to thank the referee for some useful comments and their helpful suggestions that have improved the quality of this paper.

References

[1] Akbarbaglu, I., & Azimi, M. R. (2022). Universal family of translations on weighted Orlicz spaces. Positivity, 26(1), 1–21.

[2] Akbarbaglu, I., Azimi, M. R., & Kumar, V. (2021). Topologically transitive sequence of cosine operators on Orlicz spaces. Ann. Funct. Anal., 12(1), 3–14.

[3] Azimi, M. R., & Farmani, M. (2022). Subspace-hypercyclicity of conditional weighted translations on locally compact groups. Positivity, 26(3), 1–20.

[4] Azimi, M. R., & Jabbarzadeh, M. R. (2022). Hypercyclicity of weighted composition operators on L^p-spaces. Mediterr. J. Math., 19, Paper No.164, 1–16.

[5] Bayart, F. (2007). A class of linear fractional maps of the ball and its composition operators. Adv. Math., 209, 649–665.

[6] Bayart, F., & Matheron, É. (2009). Dynamics of linear operators. Cambridge Tracts in Mathematics. Cambridge University Press, Cambridge.

[7] Birkhoff, G. D. (1929). Démonstration d'un théorème élémentaire sur les fonctions entières. C.R. Acad. Sci. Paris, 189, 473–475.

[8] Bourdon, P. S., & Shapiro, J. H. (1990). Cyclic composition operators on H_2. Proc. Symp. Pure Math., 51(2), 43–53.

[9] Bourdon, P. S., & Shapiro, J. H. (1997). Cyclic phenomena for composition operators. Mem. Amer. Math. Soc., 596.

[10] Chan, K. C., & Shapiro, J. H. (1991). The cyclic behavior of translation operators on Hilbert spaces of entire functions. Indiana Univ. Math. J., 40, 1421–1449.

[11] Feldman, N. S. (2012). n-Weakly hypercyclic and n-weakly supercyclic operators. J. Funct. Anal., 263, 2255–2299.

[12] Fox, R. H. (1945). On topologies for function spaces. Bull. Amer. Math. Soc., 51(6), 429–433. https://doi.org/10.1090/S0002-9904-1945-08370-0

[13] Gethner, R. M., & Shapiro, J. H. (1987). Universal vectors for operators on spaces of holomorphic functions. Proc. Amer. Math. Soc., 100, 281–288.

[14] Godefroy, G., & Shapiro, J. H. (1991). Operators with dense invariant cyclic vector manifolds. J. Funct. Anal., 98, 229–269.

[15] Grosse-Erdmann, K. G. (1999). Universal families and hypercyclic operators. Bull. Amer. Math. Soc. (N. S.), 36, 345–381.

[16] Grosse-Erdmann, K. G. (2003). Recent developments in hypercyclicity. Rev. R. Acad. Cien. Serie A. Mat., 97(2), 273–286.

[17] Grosse-Erdmann, K. G., & Manguillot, A. P. (2011). Linear chaos. Universitext. Springer, London.

[18] Grosse-Erdmann, K. G., & Mortini, R. (2009). Universal functions for composition operators with nonautomorphic symbol. J. Anal. Math., 107, 355–376.

[19] Pashaie, F., Azimi, M. R., & Shahidi, S. M. Dynamics of weighted composition operators on Stein manifolds. (Submitted).

[20] Salas, H. N. (1995). Hypercyclic weighted shifts. Trans. Amer. Math. Soc., 347, 993–1004.

[21] Shapiro, J. H. (1993). Composition operators and classical function theory. Springer-Verlag, New York.

[22] Shapiro, J. H. (2001). Notes on dynamics of linear operators. http://www.math.msu.edu/shapiro

[23] Yousefi, B., & Rezaei, H. (2007). Hypercyclic property of weighted composition operators. Proc. Amer. Math. Soc., 135, 3263–3271.

[24] Zajac, S. (2016). Hypercyclicity of composition operators in Stein manifolds. Proc. Amer. Math. Soc., 144(9), 3991–4000.

Department of Mathematics, Faculty of Basic Sciences, University of Maragheh, P.O.Box 55181-83111, Maragheh, Iran.

Department of Mathematics, Faculty of Basic Sciences, University of Maragheh, P.O.Box 55181-83111, Maragheh, Iran.

Department of Mathematics, Faculty of Basic Sciences, University of Maragheh, P.O.Box 55181-83111, Maragheh, Iran.