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# OPERATORS ASSOCIATED WITH OF GOLDEN RIEMANNIAN STRUCTURES ON TANGENT AND COTANGENT BUNDLES

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Abstract. In this paper, operators were applied to vertical and horizontal lifts with respect to the Golden Riemannian structures on tangent and cotangent bundles, respectively.

## 1. Introduction

For a manifold $M$, let $\varphi$ be a $(1,1)-$tensor field on $M$. If the polynomial $X^2 - X - 1$ is the minimal polynomial for a structure $\varphi$ satisfying $\varphi^2 - \varphi - 1 = 0$, then $\varphi$ is called a Golden structure on $M$ and $(M, \varphi)$ is a Golden manifold [1, 5, 6]. This structure was inspired by the Golden Ratio, which was described by Johannes Kepler $(1571 - 1630)$. The number $\eta = \frac{1+\sqrt{5}}{2} \approx 1.618...$, which is a solution of the equation $x^2 - x - 1 = 0$, is the Golden ratio. We note that for Golden structures, $\varphi \neq aI$, where $a \in R$. If $\varphi = aI$, $a = \frac{1+\sqrt{5}}{2}$, then its minimal polynomial is $X - a$. However, the minimal polynomial of the Golden structure $\varphi$ is $X^2 - X - 1$.

Let $(M, g)$ be a Riemannian manifold endowed with the Golden structure $\varphi$ such that [1, 5, 6]

$$g(\varphi X, Y) = g(X, \varphi Y) \tag{1.1}$$

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for all $X, Y \in \Im_0^1(M)$. If we substitute $\varphi X$ into $X$ in (1.1), the equation (1.1) may also be written as

$$g(\varphi X, \varphi Y) = g\left(\varphi^2 X, Y\right) = g((\varphi + 1) X, Y) = g(\varphi X, Y) + g(X, Y). \tag{1.2}$$

The Riemannian metric (1.1) is called $\varphi$-compatible and $(M, \varphi, g)$ is named Golden Riemannian manifold. Such Riemannian metrics are also referred to as pure metrics [7, 11].

Let $\varphi$ be a (1.1)-tensor field on $M$, i.e. $\varphi \in \Im_1^1(M)$. A tensor field $t$ of type $(r, s)$ is called a pure tensor field with respect to $\varphi$ if

$$\begin{aligned} t\left(\varphi X_1, X_2, ..., X_s; \overset{1}{\xi}, \overset{2}{\xi}, ..., \overset{r}{\xi}\right) &= t\left(X_1, \varphi X_2, ..., X_s; \overset{1}{\xi}, \overset{2}{\xi}, ..., \overset{r}{\xi}\right) \\ &\;\;... \\ &= t\left(X_1, X_2, ..., \varphi X_s; \overset{1}{\xi}, \overset{2}{\xi}, ..., \overset{r}{\xi}\right) \\ &= t\left(X_1, X_2, ..., X_s; {}'\varphi\overset{1}{\xi}, \overset{2}{\xi}, ..., \overset{r}{\xi}\right) \\ &= t\left(X_1, X_2, ..., X_s; \overset{1}{\xi}, \overset{2}{\xi}, ..., {}'\varphi\overset{r}{\xi}\right) \end{aligned}$$

for any $X_1, X_2, ..., X_s \in \Im_0^1(M)$ and $\overset{1}{\xi}, \overset{2}{\xi}, ..., \overset{r}{\xi} \in \Im_1^0(M)$, where ${}'\varphi$ is the adjoint operator of $\varphi$ defined by

$$\left({}'\varphi\xi\right)(X) = \xi(\varphi X) = (\xi \circ \varphi)(X), X \in \Im_0^1(M), \xi \in \Im_1^0(M).$$

We define an operator

$$\phi_\varphi : \Im_s^0(M) \to \Im_{s+1}^0(M)$$

applied to the pure tensor field $t$ of type $(0, s)$ with respect to $\varphi$ by [12]

$$\begin{aligned} (\phi_\varphi t)(X, Y_1, ..., Y_s) &= (\varphi X) t(Y_1, ..., Y_s) - X t(\varphi Y_1, ..., Y_s) \\ &\quad + \sum_{\lambda=1}^{s} t(Y_1, ..., (L_{Y_\lambda}\varphi) X, ..., Y_s) \end{aligned}$$

for any $X, Y_1, ..., Y_s \in \Im_0^1(M)$, where $L_Y$ denotes the Lie differentiation with respect to $Y$.

1.1. **The Golden structure on tangent bundle $T(M)$.** Golden structure on a Riemannian manifold is important because this structure has relation with pure Riemannian metrics with respect to the structure. Pure metrics with respect to certain structures were studied by various authors (for example see [7, 8], etc.). Since Riemannian Golden and almost product structures are related to each other (see Theorem 2.3 in [4]), the method of $\phi-$operator used in the theory of almost product structures can be transferred to Golden structures.

**Definition 1.1.** *The Sasaki metric on the tangent bundle $T(M)$ is defined by*

$$^{S}g\left(X^{H},Y^{H}\right)=\left(g\left(X,Y\right)\right)^{V}, \tag{1.3}$$

$$^{S}g\left(X^{V},Y^{H}\right)={}^{S}g\left(X^{H},Y^{V}\right)=0 \tag{1.4}$$

$$^{S}g\left(X^{V},Y^{V}\right)=\left(g\left(X,Y\right)\right)^{V}, \tag{1.5}$$

for all $X,Y\in\Im_{0}^{1}(M)$ (see [13], p. 155-175). It is obvious that the Sasaki metric $^{S}g$ is contained in the class of the so-called $g-$natural metrics on the tangent bundle.

**Definition 1.2.** *The Golden structure $\widetilde{J}$ on tangent bundle $T(M)$, which implies $\widetilde{J}^{2}-\widetilde{J}-I=0$, defined by* [4]

$$\begin{aligned}\widetilde{J}\left(X^{H}\right)&=\frac{1}{2}\left(X^{H}+\sqrt{5}X^{V}\right)\\ \widetilde{J}\left(X^{V}\right)&=\frac{1}{2}\left(X^{V}+\sqrt{5}X^{H}\right)\end{aligned} \tag{1.6}$$

*for all $X,Y\in\Im_{0}^{1}(M)$.*

**Theorem 1.1.** *Let $(M,g)$ be a Riemannian manifold and let $T(M)$ be its tangent bundle equipped with the Sasaki metric $^{S}g$ and the Golden structure $\widetilde{J}$ defined by (1.6). The triple $\left(T(M),\widetilde{J},{}^{S}g\right)$ is a Golden Riemannian manifold.*

### 1.2. The Golden structure on cotangent bundle $T^{*}(M)$.

**Definition 1.3.** *A Sasakian metric $^{S}g$ is defined on $T^{*}(M)$ by the three equations*

$$^{S}g\left(\omega^{V},\theta^{V}\right)=\left(g^{-1}\left(\omega,\theta\right)\right)^{V}=g^{-1}\left(\omega,\theta\right)\circ\pi, \tag{1.7}$$

$$^{S}g\left(\omega^{V},Y^{H}\right)=0, \tag{1.8}$$

$$^{S}g\left(X^{H},Y^{H}\right)=\left(g\left(X,Y\right)\right)^{V}=g\left(X,Y\right)\circ\pi \tag{1.9}$$

for any $X,Y\in\Im_{0}^{1}(M)$ and $\omega,\theta\in\Im_{1}^{0}(M)$. Since any tensor field of type $(0,2)$ on $T^{*}(M)$ is completely determined by its action on vector fields of type $X^{H}$ and $\omega^{V}$ (see [13], p. 280), it follows that $^{S}g$ is completely determined by the equations (1.7),(1.8) and (1.9).

**Definition 1.4.** *The Golden structure $\tilde{\varphi}$ on $T^*(M)$ defined by* [4]

$$\begin{aligned} \tilde{\varphi}X^H &= \frac{1}{2}\left(X^H + \sqrt{5}\tilde{X}^V\right) \\ \tilde{\varphi}\omega^V &= \frac{1}{2}\left(\omega^V + \sqrt{5}\tilde{\omega}^H\right) \end{aligned} \tag{1.10}$$

*for any $X \in \Im_0^1(M)$ and $\omega \in \Im_1^0(M)$, where $\tilde{X} = g \circ X \in \Im_1^0(M)$, $\tilde{\omega} = g^{-1} \circ \omega \in \Im_0^1(M)$. Also note that ${}^S g$ is pure with respect to $\tilde{\varphi}$.*

## 2. Main Results

### 2.1. **The Tachibana operators applied to vertical and horizontal lifts with respect to the Golden structure on tangent bundle and cotangent bundle.**

**Definition 2.1.** *Let $\varphi \in \Im_1^1(M)$, and $\Im(M) = \sum_{r,s=0}^{\infty} \Im_s^r(M)$ be a tensor algebra over R. A map $\phi_\varphi \mid_{r+s\rangle 0} : \overset{*}{\Im}(M) \to \Im(M)$ is called a Tachibana operator or $\phi_\varphi$ operator on $M$ if*

$a)$ $\phi_\varphi$ is linear with respect to constant coefficient,

$b)$ $\phi_\varphi : \overset{*}{\Im}(M) \to \Im_{s+1}^r(M)$ for all r and s,

$c)$ $\phi_\varphi(K \overset{C}{\otimes} L) = (\phi_\varphi K) \otimes L + K \otimes \phi_\varphi L$ for all $K, L \in \overset{*}{\Im}(M)$,

$d)$ $\phi_{\varphi X} Y = -(L_Y \varphi)X$ for all $X, Y \in \Im_0^1(M)$ where $L_Y$ is the Lie derivation with respect to $Y$,

$e)$

$$\begin{aligned} (\phi_{\varphi X}\eta)Y &= (d(\imath_Y \eta))(\varphi X) - (d(\imath_Y(\eta o \varphi)))X + \eta((L_Y\varphi)X) \\ &= \phi X(\imath_Y \eta) - X(\imath_{\varphi Y}\eta) + \eta((L_Y\varphi)X) \end{aligned} \tag{2.11}$$

for all $\eta \in \Im_1^0(M)$ and $X, Y \in \Im_0^1(M)$, where $\imath_Y \eta = \eta(Y) = \eta \overset{C}{\otimes} Y, \overset{*}{\Im}{}_s^r(M)$ the module of all pure tensor fields of type $(r, s)$ on $M$ according to the affinor field $\varphi$ [2, 3, 9](see [10] for applied to pure tensor field).

**Theorem 2.1.** *Let $L_X$ the operator Lie derivation with respect to $X$ , $\widetilde{J}$ be the Golden structure on tangent bundle $T(M)$, which implies $\widetilde{J}^2 - \widetilde{J} - I = 0$, defined by (1.6) and $\phi_{\widetilde{J}}$*

*the Tachibana operator on $M$. We get the following formulas*

$$\begin{aligned} i)\ \phi_{\tilde{J}X^V}Y^H &= \frac{\sqrt{5}}{2}((R(Y,X)U)^V+(\nabla_XY)^H), \\ ii)\ \phi_{\tilde{J}X^H}Y^H &= -\frac{\sqrt{5}}{2}((\nabla_XY)^V+(R(Y,X)U)^H), \\ iii)\ \phi_{\tilde{J}X^V}Y^V &= \frac{\sqrt{5}}{2}(\hat{\nabla}_XY)^V, \\ iv)\ \phi_{\tilde{J}X^H}Y^V &= -\frac{\sqrt{5}}{2}(\hat{\nabla}_XY)^H, \end{aligned}$$

where $R$ is the curvature tensor of $\nabla$, $X,Y\in\Im_0^1(M)$ and $\widetilde{J}\in\Im_1^1(M)$.

**Proof.** $i)$

$$\begin{aligned} \phi_{\tilde{J}X^V}Y^H &= -\left(L_{Y^H}\tilde{J}\right)X^V=-L_{Y^H}\tilde{J}X^V+\tilde{J}L_{Y^H}X^V \\ &= -L_{Y^H}\frac{1}{2}\left(X^V+\sqrt{5}X^H\right)+\tilde{J}\left(\tilde{\nabla}_YX\right)^V \\ &= -\frac{1}{2}\left[Y^H,X^V\right]-\frac{\sqrt{5}}{2}\left[Y^H,X^H\right]+\frac{1}{2}\left(\tilde{\nabla}_YX\right)^V+\frac{\sqrt{5}}{2}\left(\tilde{\nabla}_YX\right)^H \\ &= -\frac{1}{2}\left(\tilde{\nabla}_YX\right)^V-\frac{\sqrt{5}}{2}\left([Y,X]^H-(R(Y,X)U)^V\right)+\frac{1}{2}\left(\tilde{\nabla}_YX\right)^V \\ &\quad +\frac{\sqrt{5}}{2}\left(\tilde{\nabla}_YX\right)^H \\ &= -\frac{\sqrt{5}}{2}[Y,X]^H+\frac{\sqrt{5}}{2}(R(Y,X)U)^V+\frac{\sqrt{5}}{2}((\nabla_XY)^H+[Y,X]^H) \\ &= \frac{\sqrt{5}}{2}((R(Y,X)U)^V+(\nabla_XY)^H) \end{aligned}$$

$ii)$

$$\begin{aligned} \phi_{\tilde{J}X^H}Y^H &= -\left(L_{Y^H}\tilde{J}\right)X^H=-L_{Y^H}\tilde{J}X^H+\tilde{J}L_{Y^H}X^H \\ &= -L_{Y^H}(\frac{1}{2}X^H+\frac{\sqrt{5}}{2}X^V)+\tilde{J}([Y,X]^H-(R(Y,X)U)^V) \\ &= -\frac{1}{2}\left[Y^H,X^H\right]-\frac{\sqrt{5}}{2}\left[Y^H,X^V\right]+\tilde{J}[Y,X]^H-\tilde{J}(R(Y,X)U)^V \end{aligned}$$

$$\begin{aligned} &= -\frac{1}{2}(\left([Y,X]^H-(R(Y,X)U)^V\right)-\frac{\sqrt{5}}{2}[Y,X]^V+(\nabla_XY)^V) \\ &\quad +\frac{1}{2}[Y,X]^H+\frac{\sqrt{5}}{2}[Y,X]^V-\frac{1}{2}(R(Y,X)U)^V-\frac{\sqrt{5}}{2}(R(Y,X)U)^H \\ &= -\frac{1}{2}[Y,X]^H+\frac{1}{2}(R(Y,X)U)^V-\frac{\sqrt{5}}{2}[Y,X]^V-\frac{\sqrt{5}}{2}(\nabla_XY)^V \\ &\quad +\frac{1}{2}[Y,X]^H+\frac{\sqrt{5}}{2}[Y,X]^V-\frac{1}{2}(R(Y,X)U)^V-\frac{\sqrt{5}}{2}(R(Y,X)U)^H \\ &= -\frac{\sqrt{5}}{2}\left((\nabla_XY)^V+(R(Y,X)U)^H\right) \end{aligned}$$

*iii)*

$$\begin{aligned}
\phi_{\tilde{J}X^V} Y^V &= -\left(L_{Y^V}\tilde{J}\right) X^V = -L_{Y^V}\tilde{J}X^V + \tilde{J}L_{Y^V}X^V \\
&= -L_{Y^V}(\frac{1}{2}X^V + \frac{\sqrt{5}}{2}X^H) \\
&= -\frac{1}{2}L_{Y^V}X^V - \frac{\sqrt{5}}{2}L_{Y^V}X^H \\
&= \frac{\sqrt{5}}{2}\left[X^H, Y^V\right] \\
&= \frac{\sqrt{5}}{2}(\hat{\nabla}_X Y)^V
\end{aligned}$$

*iv)*

$$\begin{aligned}
\phi_{\tilde{J}X^H} Y^V &= -\left(L_{Y^V}\tilde{J}\right) X^H = -L_{Y^V}\tilde{J}X^H + \tilde{J}L_{Y^V}X^H \\
&= -L_{Y^V}(\frac{1}{2}X^H + \frac{\sqrt{5}}{2}X^V) + \tilde{J}([Y,X]^V - (\nabla_Y X)^V) \\
&= -\frac{1}{2}L_{Y^V}X^H - \frac{\sqrt{5}}{2}L_{Y^V}X^V + \frac{1}{2}[Y,X]^V + \frac{\sqrt{5}}{2}[Y,X]^H \\
&\quad -\frac{1}{2}(\nabla_Y X)^V - \frac{\sqrt{5}}{2}(\nabla_Y X)^H \\
&= \frac{\sqrt{5}}{2}([Y,X] - (\nabla_Y X))^H \\
&= -\frac{\sqrt{5}}{2}(\hat{\nabla}_X Y)^H
\end{aligned}$$

**Theorem 2.2.** *Let $\nabla_X$ be the operator covariant derivation with respect to $X$, $\tilde{\varphi}$ be the Golden structure on $T^*(M)$ defined by (1.10) and $\phi_{\tilde{\varphi}}$ the Tachibana operator on $M$. We get the following formulas*

$$\begin{aligned}
i)\ \phi_{\tilde{\varphi}X^H}Y^H &= -\frac{\sqrt{5}}{2}\left(g\left(\nabla_Y X\right)\right)^V + \frac{\sqrt{5}}{2}\left(gL_Y X\right)^V + \frac{\sqrt{5}}{2}\left(g^{-1}\left(PR\left(Y,X\right)\right)\right)^V, \\
ii)\ \phi_{\tilde{\varphi}\omega^V}Y^H &= -\frac{\sqrt{5}}{2}\left(\left(L_Y g^{-1}\right)\circ\omega\right)^H - \frac{\sqrt{5}}{2}\left(g^{-1}\left(L_Y\omega\right)\right)^H \\
&\quad -\frac{\sqrt{5}}{2}\left(P\left(R\left(Y, g^{-1}\circ\omega\right)\right)\right)^V + \frac{\sqrt{5}}{2}\left(g^{-1}\left(\nabla_Y\omega\right)\right)^H, \\
iii)\ \phi_{\tilde{\varphi}\omega^V}\theta^V &= \frac{\sqrt{5}}{2}(\nabla_{(g^{-1}\circ\omega)}\theta)^V, \\
iv)\ \phi_{\tilde{\varphi}X^H}\omega^V &= -\frac{\sqrt{5}}{2}\left(g^{-1}\left(\nabla_X\omega\right)\right)^H,
\end{aligned}$$

where $R$ is the curvature tensor of $\nabla$, $\tilde{\varphi}\in\Im_1^1(M)$, $X,Y\in\Im_0^1(M)$ and $\omega\in\Im_1^0(M)$, $\tilde{X}=g\circ X\in\Im_1^0(M)$, $\tilde{\omega}=g^{-1}\circ\omega\in\Im_0^1(M)$.

**Proof.** $i)$

$$\begin{aligned}
\phi_{\tilde{\varphi}X^H}Y^H &= \left(L_{Y^H}\tilde{\varphi}\right)X^H = -L_{Y^H}\tilde{\varphi}X^H + \tilde{\varphi}L_{Y^H}X^H \\
&= -L_{Y^H}\frac{1}{2}X^H + \frac{\sqrt{5}}{2}\tilde{X}^V + \tilde{\varphi}\left([Y,X]^H + \left(P\left(R\left(Y,X\right)\right)\right)^V\right) \\
&= -\frac{1}{2}\left[Y^H, X^H\right] - \frac{\sqrt{5}}{2}\left[Y^H, \tilde{X}^H\right] + \tilde{\varphi}([Y,X]^H + \left(P\left(R\left(Y,X\right)\right)\right)^V) \\
&= -\frac{1}{2}[Y,X]^H - \frac{1}{2}\left(PR\left(Y,X\right)\right)^V - \frac{\sqrt{5}}{2}\left(\nabla_Y\left(g\circ X\right)\right)^V + \frac{1}{2}[Y,X]^H \\
&\quad + \frac{\sqrt{5}}{2}[Y,X]^V + \frac{1}{2}\left(PR\left(Y,X\right)\right)^V + \frac{\sqrt{5}}{2}\left(PR\left(Y,X\right)\right)^V \\
&= -\frac{\sqrt{5}}{2}\left(g\left(\nabla_Y X\right)\right)^V + \frac{\sqrt{5}}{2}\left(gL_Y X\right)^V + \frac{\sqrt{5}}{2}\left(g^{-1}\left(PR\left(Y,X\right)\right)\right)^V
\end{aligned}$$

$ii)$

$$\begin{aligned}
\phi_{\tilde{\varphi}\omega^V}Y^H &= -\left(L_{Y^H}\tilde{\varphi}\right)\omega^V = -L_{Y^H}\tilde{\varphi}\omega^V + \tilde{\varphi}L_{Y^H}\omega^V \\
&= -L_{Y^H}\frac{1}{2}(\omega^V + \sqrt{5}\tilde{\omega}^H) + \tilde{\varphi}\left(\nabla_Y\omega\right)^V \\
&= -\frac{1}{2}L_{Y^H}\omega^V - \frac{\sqrt{5}}{2}L_{Y^H}\tilde{\omega}^H + \frac{1}{2}\left(\nabla_Y\omega\right)^V + \frac{\sqrt{5}}{2}\left(g^{-1}\left(\nabla_Y\omega\right)\right)^H \\
&= -\frac{1}{2}\left(\nabla_Y\omega\right)^V - \frac{\sqrt{5}}{2}(\left[Y, g^{-1}\circ\omega\right]^H + \left(P\left(R\left(Y, g^{-1}\circ\omega\right)\right)\right)^V) \\
&\quad + \frac{1}{2}\left(\nabla_Y\omega\right)^V + \frac{\sqrt{5}}{2}\left(g^{-1}\left(\nabla_Y\omega\right)\right)^H \\
&= -\frac{\sqrt{5}}{2}\left(\left(L_Y g^{-1}\right)\circ\omega\right)^H - \frac{\sqrt{5}}{2}\left(g^{-1}\left(L_Y\omega\right)\right)^H \\
&\quad - \frac{\sqrt{5}}{2}\left(P\left(R\left(Y, g^{-1}\circ\omega\right)\right)\right)^V + \frac{\sqrt{5}}{2}\left(g^{-1}\left(\nabla_Y\omega\right)\right)^H
\end{aligned}$$

$iii)$

$$\begin{aligned}
\phi_{\tilde{\varphi}\omega^V}\theta^V &= -\left(L_{\theta^V}\tilde{\varphi}\right)\omega^V = -L_{\theta^V}\tilde{\varphi}\omega^V + \tilde{\varphi}L_{\theta^V}\omega^V \\
&= -L_{\theta^V}\frac{1}{2}(\omega^V + \sqrt{5}\tilde{\omega}^H) \\
&= -\frac{1}{2}L_{\theta^V}\omega^V - \frac{\sqrt{5}}{2}L_{\theta^V}\tilde{\omega}^H \\
&= +\frac{\sqrt{5}}{2}\left(\nabla_{\tilde{\omega}}\theta\right)^V \\
&= \frac{\sqrt{5}}{2}(\nabla_{(g^{-1}\circ\omega)}\theta)^V
\end{aligned}$$

*iv)*

$$\begin{aligned}
\phi_{\tilde{\varphi}X^H}\omega^V &= -\left(L_{\omega^V}\tilde{\varphi}\right)X^H = -L_{\omega^V}\tilde{\varphi}X^H + \tilde{\varphi}L_{\omega^V}X^H \\
&= -L_{\omega^V}\frac{1}{2}X^H + \frac{\sqrt{5}}{2}\tilde{X}^V - \tilde{\varphi}\left(\nabla_X\omega\right)^V \\
&= -\frac{1}{2}L_{\omega^V}X^H - \frac{\sqrt{5}}{2}L_{\omega^V}\tilde{X}^V - \frac{1}{2}\left(\nabla_X\omega\right)^V \\
&\quad -\frac{\sqrt{5}}{2}\left(g^{-1}\left(\nabla_X\omega\right)\right)^H \\
&= \frac{1}{2}\left(\nabla_X\omega\right)^V - \frac{1}{2}\left(\nabla_X\omega\right)^V - \frac{\sqrt{5}}{2}\left(g^{-1}\left(\nabla_X\omega\right)\right)^H \\
&= -\frac{\sqrt{5}}{2}\left(g^{-1}\left(\nabla_X\omega\right)\right)^H
\end{aligned}$$

2.2. **The Vishnevskii operators applied to vertical and horizontal lifts with respect to the Golden structure on tangent bundle and cotangent bundle.**

**Definition 2.2.** *Suppose now that $\nabla$ is a linear connection on $M$, and let $\varphi \in \Im_1^1(M)$. We can replace the condition d) of definition 2.1 by*

$$d^{'})\quad \psi_{\varphi X}Y = \nabla_{\varphi X}Y - \varphi\nabla_X Y \tag{2.12}$$

*for any $X, Y \in \Im_0^1(M)$. Then we can consider a new operator by a Vishnevskii operator or $\psi_\varphi-$operator on $M$, we shall mean a map $\psi_\varphi : \overset{*}{\Im}(M) \to \Im(M)$, which satisfies conditions a), b), c), e) of definition 2.1 and the condition $(d^{'})$* [2, 3, 9].

Let $\omega \in \Im_1^0(M)$. Using Definition 2.2, we have

$$\begin{aligned}
\left(\psi_\varphi\omega\right)(X,Y) &= (\psi_{\varphi X}\omega)Y \\
&= (\varphi X)(\iota_Y\omega) - X(\iota_{\varphi Y}\omega) - \omega\left(\nabla_{\varphi X}Y - \varphi\left(\nabla_X Y\right)\right) \\
&= \left(\nabla_{\varphi X}\omega - \nabla_X\left(\omega\circ\varphi\right)\right)Y
\end{aligned} \tag{2.13}$$

for any $X, Y \in \Im_0^1(M)$,where $(\omega\circ\varphi)Y = \omega(\varphi Y)$. From (2.13) we see that $\psi_{\varphi X}\omega = \nabla_{\varphi X}\omega - \nabla_X(\omega\circ\varphi)$ is a 1−form [9].

**Theorem 2.3.** *Let $\nabla^H$ be the horizontal lift of the Levi-Civita connection $\nabla$ in $M$ to $T(M)$ and $\widetilde{J}$ be the Golden structure on tangent bundle $T(M)$, which implies $\widetilde{J}^2 - \widetilde{J} - I = 0$, defined* *by (1.6). $\psi_{\widetilde{J}}$ the Vishnevskii operator on $M$. We get the following formulas*

$$\begin{aligned} i)\ \psi_{\tilde{J}X^V}Y^H &= \frac{\sqrt{5}}{2}\left(\nabla_X Y\right)^H, \\ ii)\ \psi_{\tilde{J}X^V}Y^V &= \frac{\sqrt{5}}{2}\left(\nabla_X Y\right)^V, \\ iii)\ \psi_{\tilde{J}X^H}Y^H &= -\frac{\sqrt{5}}{2}\left(\nabla_X Y\right)^V, \\ iv)\ \psi_{\tilde{J}X^H}Y^V &= -\frac{\sqrt{5}}{2}\left(\nabla_X Y\right)^H, \end{aligned}$$

where $R$ is the curvature tensor of $\nabla$, $X, Y \in \Im_0^1(M)$ and $\widetilde{J} \in \Im_1^1(M)$.

**Proof.** $i)$

$$\begin{aligned} \psi_{\tilde{J}X^V}Y^H &= \nabla^H_{\tilde{J}X^V}Y^H - \tilde{J}\nabla^H_{X^V}Y^H = \nabla^H_{\frac{1}{2}(X^V+\sqrt{5}X^H)}Y^H - \tilde{J}\nabla^H_{X^V}Y^H \\ &= \frac{1}{2}\nabla^H_{X^V}Y^H + \frac{\sqrt{5}}{2}\nabla^H_{X^V}Y^H \\ &= \frac{\sqrt{5}}{2}\left(\nabla_X Y\right)^H \end{aligned}$$

$ii)$

$$\begin{aligned} \psi_{\tilde{J}X^V}Y^V &= \nabla^H_{\tilde{J}X^V}Y^V - \tilde{J}\nabla^H_{X^V}Y^V = \nabla^H_{\frac{1}{2}(X^V+\sqrt{5}X^H)}Y^V - \tilde{J}\nabla^H_{X^V}Y^V \\ &= \frac{1}{2}\nabla^H_{X^V}Y^V + \frac{\sqrt{5}}{2}\nabla_{X^V}Y^V \\ &= \frac{\sqrt{5}}{2}\left(\nabla_X Y\right)^V \end{aligned}$$

$iii)$

$$\begin{aligned} \psi_{\tilde{J}X^H}Y^H &= \nabla^H_{\tilde{J}X^H}Y^H - \tilde{J}\nabla^H_{X^H}Y^H = \nabla^H_{\frac{1}{2}X^H+\frac{\sqrt{5}}{2}X^V}Y^H - \tilde{J}\left(\nabla_X Y\right)^H \\ &= \frac{1}{2}\nabla^H_{X^H}Y^H + \frac{\sqrt{5}}{2}\nabla^H_{X^V}Y^H - \frac{1}{2}\left(\nabla_X Y\right)^H - \frac{\sqrt{5}}{2}\left(\nabla_X Y\right)^V \\ &= \frac{1}{2}\left(\nabla_X Y\right)^H - \frac{1}{2}\left(\nabla_X Y\right)^H - \frac{\sqrt{5}}{2}\left(\nabla_X Y\right)^V \\ &= -\frac{\sqrt{5}}{2}\left(\nabla_X Y\right)^V \end{aligned}$$

$iv)$

$$\begin{aligned} \psi_{\tilde{J}X^H}Y^V &= \nabla^H_{\tilde{J}X^H}Y^V - \tilde{J}\nabla^H_{X^H}Y^V = \nabla^H_{\frac{1}{2}X^H+\frac{\sqrt{5}}{2}X^V}Y^V - \tilde{J}\left(\nabla_X Y\right)^V \\ &= \frac{1}{2}\nabla^H_{X^H}Y^V + \frac{\sqrt{5}}{2}\nabla^H_{X^V}Y^V - \frac{1}{2}\left(\nabla_X Y\right)^V - \frac{\sqrt{5}}{2}\left(\nabla_X Y\right)^H \\ &= -\frac{\sqrt{5}}{2}\left(\nabla_X Y\right)^H \end{aligned}$$

**Theorem 2.4.** *Let $\nabla^H$ be the horizontal lift of the Levi-Civita connection $\nabla$ in $M$ to $T^*(M)$ and $\tilde{\varphi}$ be the Golden structure on $T^*(M)$ defined by (1.10) and $\psi_{\tilde{\varphi}}$ the Vishnevskii operator on $M$. We get the following formulas*

$$\begin{aligned}
i)\ \psi_{\tilde{\varphi}\omega^V} Y^H &= \frac{\sqrt{5}}{2}(\nabla_{(g^{-1}\circ\omega)^H} Y)^H, \\
ii)\ \psi_{\tilde{\varphi}\omega^V} \theta^V &= \frac{\sqrt{5}}{2}(\nabla_{(g^{-1}\circ\omega)^H} \theta)^V, \\
iii)\ \psi_{\tilde{\varphi}X^H} Y^H &= -\frac{\sqrt{5}}{2}\left(g\left(\nabla_X Y\right)\right)^V, \\
iv)\ \psi_{\tilde{\varphi}X^H} \omega^V &= -\frac{\sqrt{5}}{2}\left(g^{-1}\left(\nabla_X \omega\right)\right)^H,
\end{aligned}$$

where $R$ is the curvature tensor of $\nabla$, $\tilde{\varphi} \in \Im_1^1(M)$, $X, Y \in \Im_0^1(M)$ and $\omega \in \Im_1^0(M)$, $\tilde{X} = g \circ X \in \Im_1^0(M)$, $\tilde{\omega} = g^{-1} \circ \omega \in \Im_0^1(M)$.

**Proof.** *i)*

$$\begin{aligned}
\psi_{\tilde{\varphi}\omega^V} Y^H &= \nabla^H_{\tilde{\varphi}\omega^V} Y^H - \tilde{\varphi}\nabla^H_{\omega^V} Y^H = \nabla^H_{\frac{1}{2}\omega^V + \frac{\sqrt{5}}{2}(g^{-1}\circ\omega)^H} Y^H \\
&= \frac{1}{2}\nabla^H_{\omega^V} Y^H + \frac{\sqrt{5}}{2}\nabla^H_{(g^{-1}\circ\omega)^H} Y^H \\
&= \frac{\sqrt{5}}{2}(\nabla_{(g^{-1}\circ\omega)^H} Y)^H
\end{aligned}$$

*ii)*

$$\begin{aligned}
\psi_{\tilde{\varphi}\omega^V} \theta^V &= \nabla^H_{\tilde{\varphi}\omega^V} \theta^V - \tilde{\varphi}\nabla^H_{\omega^V} \theta^V = \nabla^H_{\frac{1}{2}\omega^V + \frac{\sqrt{5}}{2}(g^{-1}\circ\omega)^H} \theta^V \\
&= \frac{1}{2}\nabla^H_{\omega^V} \theta^V + \frac{\sqrt{5}}{2}\nabla^H_{(g^{-1}\circ\omega)^H} \theta^V \\
&= \frac{\sqrt{5}}{2}(\nabla_{(g^{-1}\circ\omega)^H} \theta)^V
\end{aligned}$$

*iii)*

$$\begin{aligned}
\psi_{\tilde{\varphi}X^H} Y^H &= \nabla^H_{\tilde{\varphi}X^H} Y^H - \tilde{\varphi}\nabla^H_{X^H} Y^H = \nabla^H_{\frac{1}{2}X^H + \frac{\sqrt{5}}{2}\tilde{X}^V} Y^H - \tilde{\varphi}\left(\nabla_X Y\right)^H \\
&= \frac{1}{2}\left(\nabla_X Y\right)^H + \frac{\sqrt{5}}{2}\nabla^H_{(g\circ X)^V} Y^H - \frac{1}{2}\left(\nabla_X Y\right)^H \\
&\quad -\frac{\sqrt{5}}{2}\left(g \circ \left(\nabla_X Y\right)\right)^V \\
&= -\frac{\sqrt{5}}{2}\left(g\left(\nabla_X Y\right)\right)^V
\end{aligned}$$

*iv)*

$$\begin{aligned}
\psi_{\tilde{\varphi}X^H}\omega^V &= \nabla^H_{\tilde{\varphi}X^H}\omega^V - \tilde{\varphi}\nabla^H_{X^H}\omega^V = \nabla^H_{\frac{1}{2}X^H+\frac{\sqrt{5}}{2}\tilde{X}^V}\omega^V - \tilde{\varphi}\left(\nabla_X\omega\right)^V \\
&= \frac{1}{2}\left(\nabla_X\omega\right)^V + \frac{\sqrt{5}}{2}\nabla^H_{\tilde{X}^V}\omega^V - \frac{1}{2}\left(\nabla_X\omega\right)^V \\
&\quad -\frac{\sqrt{5}}{2}\left(g^{-1}\left(\nabla_X\omega\right)\right)^H \\
&= -\frac{\sqrt{5}}{2}\left(g^{-1}\left(\nabla_X\omega\right)\right)^H
\end{aligned}$$

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