International Journal of Maps in Mathematics

*Volume **8**, Issue **1**, 2025, Pages:**326-333***

E-ISSN: 2636-7467

www.simadp.com/journalmim

UNIT TANGENT SPHERE BUNDLES WITH THE KALUZA-KLEIN METRIC SATISFYING SOME COMMUTATIVE CONDITIONS

MURAT ALTUNBAŞ *

Abstract. Let (M, g) be an $n-$dimensional Riemannian manifold and T_1M its tangent sphere bundle with the contact metric structure $(\tilde{G}, \eta, \phi, \xi)$, where $\tilde{G}$ is the Kaluza-Klein metric. Let $\tilde{S}$ be the Ricci operator and h be the structural operator on T_1M. In this paper, we find some conditions for the relations $\tilde{S}h = h\tilde{S}$ and $\tilde{S}\phi h = \phi h\tilde{S}$ to be satisfied.

Keywords: Tangent sphere bundle, Kaluza-Klein metric, Ricci operator, contact metric structure.

2010 Mathematics Subject Classification: 53C25, 53D10.

1. Introduction

Let (M, g) be a Riemannian manifold and T_1M its tangent sphere bundle. There can be defined a lot of metrics on T_1M such as the Sasaki metric [9], the Kaluza-Klein metric [10], the Kaluza-Klein type metric [1] and a general natural metric ([5],[12]). All of these metrics are sub-classes of the $g-$natural metric introduced in [2]. These metrics are restrictions of the metrics on the tangent bundle TM to T_1M. Remark that the restrictions of the two well-known metrics on TM, say the Cheeger-Gromoll metric [15] and a metric with two parameter ([6], [13]), yield nothing new thank to isometries to the Sasaki metric (see [8] and [14]).

In [11], Cho and Chun studied the commutativity of the Ricci operator with the structural operator in (T_1M, g^s) by considering the base manifold M is conformally flat. Here, g^s denotes the Sasaki metric.

Received:2024.10.17 *Accepted:2024.12.31*

* Corresponding author

Murat Altunbaş ⋄ maltunbas@erzincan.edu.tr ⋄ https://orcid.org/0000-0002-0371-9913.

In this paper, we endow the unit tangent sphere bundle T_1M with the Kaluza-Klein metric $\tilde{G}$ and give a contact metric structure (η, ϕ, ξ) associated to the metric $\tilde{G}$. We investigate the commutative properties of the Ricci operator $\tilde{S}$ with h and ϕh, where h is the structural operator on T_1M.

2. Preliminaries

The aim of this section is to report some fundamental facts about contact metric manifolds. Every manifolds are supposed to be smooth and connected. We can refer to [7] for a survey about contact metric geometry.

Given a $(2n-1)$ dimensional differentiable manifold $\bar{M}$. If $\bar{M}$ acknowledges a global 1-form η such that $\eta \wedge (d\eta)^n \neq 0$, then it is called a contact manifold. When η is given, there is a vector field ξ (so called characteristic vector field) such that $\eta(\xi) = 1$ and $d\eta(\xi, \bar{A}) = 0$ for all vector fields $\bar{A}$ on $\bar{M}$. Moreover, a Riemannian metric $\bar{g}$ is called an associated metric if there exists a $(1,1)-$tensor such that

$$\eta(\bar{A}) = g(\bar{A}, \xi), \; d\eta(\bar{A}, \bar{B}) = \bar{g}(\bar{X}, \phi\overline{B}), \; \phi^2\bar{A} = -\bar{A} + \eta(\bar{A})\xi, \tag{2.1}$$

for all vector fields $\bar{A}, \bar{B}$ on $\bar{M}$. It follows that

$$\phi\xi = 0, \; \eta \circ \phi = 0, \; \bar{g}(\phi\bar{A}, \phi\overline{B}) = \bar{g}(\bar{A}, \bar{B}) - \eta(\bar{A})\eta(\bar{B}).$$

The quartet $(\eta, \bar{g}, \phi, \xi)$ satisfying (2.1) is called a contact metric structure and the quintet $(\bar{M}, \eta, \bar{g}, \phi, \xi)$ a contact metric manifold.

Let $\bar{M}$ be a contact metric manifold. The structural operator h is defined by $h = \frac{1}{2}L_\xi\phi$, where L is the Lie derivative operator. The operator h is self-adjoint and it satisfies the following relations:

$$h\xi = 0, \; h\phi = -\phi h, \; \bar{\nabla}_{\bar{A}}\xi = -\phi\overline{A} - \phi h\overline{A}, \; (\bar{\nabla}_\xi h)\phi = -\phi(\bar{\nabla}_\xi h),$$

where $\bar{\nabla}$ is the Levi-Civita connection of $\bar{g}$.

Let (M, g) be an $n-$Riemannian manifold with the Levi-Civita connection ∇. The tangent bundle TM of M is a $2n-$dimensional manifold with the projection map $\pi : TM \to M, \; \pi(p, u) = u$. The $g-$natural metric G on TM is defined by, [4]

$$\begin{cases} G(A^h, B^h) = (\alpha_1 + \alpha_3)(r^2)g(A, B) + (\beta_1 + \beta_3)(r^2)g(A, u)g(B, u), \\ G(A^h, B^v) = \alpha_2(r^2)g(A, B) + \beta_2(r^2)g(A, u)g(B, u), \\ G(A^v, B^v) = \alpha_1(r^2)g(A, B) + \beta_1(r^2)g(A, u)g(B, u), \end{cases}$$

for all vector fields A, B on M, where $\alpha_i, \beta_i : \mathbb{R}^+ \to \mathbb{R}$, $i = 1, 2, 3$ are smooth functions, $r^2 = g(u, u)$ and A^h, A^v denote the horizontal lift and the vertical lift of A, respectively.

The unit tangent sphere bundle T_1M is a hypersurface of TM defined by $T_1M = \{(x, u) \in TM : g(u, u) = 1\}$. By definition, the $g-$natural metric on T_1M is the restriction of the $g-$natural metric of TM to its hypersurface T_1M. The $g-$natural metric on T_1M is defined by

$$\begin{cases} G(A^h, B^h) = (a + c)g(A, B) + \beta(r^2)g(A, u)g(B, u), \\ G(A^h, B^v) = bg(A, B), \\ G(A^v, B^v) = ag(A, B), \end{cases}$$

where $a, b, c \in \mathbb{R}$ and $\beta : [0, \infty) \to \mathbb{R}$. The vector field $N = \frac{1}{\sqrt{(a+c+d)\varphi}}[-bu^h + (a + c + d)u^v]$ is the unit normal vector field, where $d = \beta(1)$ and $\varphi = a(a + c + d) - b^2$. The tangential lift A^t is given by $A^t = A^v - \sqrt{\frac{\varphi}{a+c+d}}g(A, u)N$. Inasmuch as the tangent space $(T_1M)_{(x,u)}$ of T_1M at (x, u) is produced by vectors of the form A^h and B^t, the Riemannian metric g on T_1M, induced from G, is established by

$$\begin{cases} G(A^h, B^h) = (a + c)g(A, B) + dg(A, u)g(B, u), \\ G(A^h, B^t) = bg(A, B), \\ G(A^t, B^t) = ag(A, B) - \frac{\varphi}{a+c+d}g(A, u)g(B, u), \end{cases} \tag{2.2}$$

for all vector fields A, B on M. The particular cases of the metric G are listed below:

i) The Sasaki metric if $a = 1,\ b = c = d = 0$,

ii) The Cheeger-Gromoll type metric if $b = d = 0,\ a = 1/2^m,\ c = 1 - a$,

iii) The Kaluza-Klein type metric if $b = 0$ (and $a,\ a + c > 0,\ a + c + d > 0$),

iv) The Kaluza-Klein metric if $b = d = 0$ (and $a,\ a + c > 0$).

In this paper, we deal with the Kaluza-Klein metric. From (2.2), the Kaluza-Klein metric $\tilde{G}$ is defined by

$$\begin{cases} \tilde{G}(A^h, B^h) = (a + c)g(A, B), \\ \tilde{G}(A^h, B^t) = 0, \\ \tilde{G}(A^t, B^t) = a(g(A, B) - g(A, u)g(B, u)). \end{cases} \tag{2.3}$$

The Levi-Civita connection $\tilde{\nabla}$ of $\tilde{G}$ is given by, [10]

$$\begin{aligned} \tilde{\nabla}_{A^h}B^h &= (\nabla_A B)^h - \frac{1}{2}(R(A, B)u)^t, \\ \tilde{\nabla}_{A^h}B^t &= (\nabla_A B)^t + \frac{a}{2(a + c)}(R(u, A)B)^h, \\ \tilde{\nabla}_{A^t}B^h &= \frac{a}{2(a + c)}(R(u, A)B)^h, \\ \tilde{\nabla}_{A^t}B^t &= -g(B, u)A^t. \end{aligned} \tag{2.4}$$

For an orthonormal basis $e_1, e_2, ..., e_n = u$, the Ricci tensor $\widetilde{Ric}$ of $\tilde{G}$ is computed as, [3]

$$\begin{aligned} \widetilde{Ric}(A^h, B^h) &= Ric(A,B) - \frac{a}{2(a+c)}\sum_{i=1}^{n} g(R(u,e_i)A, R(u,e_i)B), \\ \widetilde{Ric}(A^t, B^h) &= \frac{a}{2(a+c)}((\nabla_u Ric)(A,B) - (\nabla_A Ric)(u,B)), \\ \widetilde{Ric}(A^t, B^t) &= (n-2)(g(A,B) - g(A,u)g(B,u)) \\ &\quad + \frac{a^2}{4(a+c)^2}\sum_{i=1}^{n} g(R(u,A)e_i, R(u,B)e_i), \end{aligned} \tag{2.5}$$

where Ric denotes the Ricci tensor of g.

From [2], we have a contact metric structure $(\tilde{G}, \eta, \phi, \xi)$ on T_1M satisfying

$$\xi = 2u^h,\ \phi A^t = \frac{1}{4(a+c)}(-A^h + \frac{1}{2}g(A,u)\xi),\ \phi A^h = \frac{1}{4a}A^t, \quad \eta(A^h) = \frac{1}{2}g(A,u),\ \eta(A^t) = 0. \tag{2.6}$$

We also have

$$\begin{aligned} hA^t &= \frac{1}{4a}A^t - \frac{1}{4(a+c)}(R_uA)^t, \\ hA^h &= -\frac{1}{4a}A^h + \frac{1}{8a}g(A,u)\xi + \frac{1}{4(a+c)}(R_uA)^h, \end{aligned} \tag{2.7}$$

where $R_u = R(\cdot, u)u$ is the Jacobi operator related with the unit vector u.

3. T_1M satisfying some commutative conditions

Let (M, g) be a Riemannian manifold and T_1M be its unit tangent sphere bundle with the Kaluza-Klein metric $\tilde{G}$ (2.2). Denote the Ricci operator of $\tilde{G}$ by $\tilde{S}$. First, we suppose that $\tilde{S}h = h\tilde{S}$. Then, using (2.5), (2.7) and self-adjoint property of the Jacobi operator R_u, we occur

$$\begin{aligned} 0 &= \tilde{G}(\tilde{S}hA^t - h\tilde{S}A^t, B^t) \\ &= \widetilde{Ric}(hA^t, B^t) - \widetilde{Ric}(A^t, hB^t) \\ &= \frac{a^2}{16(a+c)^3}\sum_{i=1}^{n}[g(R(u,A)e_i, R(u,R_uB)e_i) - g(R(u,B)e_i, R(u,R_uA)e_i)], \end{aligned} \tag{3.8}$$

$$\begin{aligned}
0 &= \tilde{G}(\tilde{S}hA^t - h\tilde{S}A^t, B^h) \\
&= \widetilde{Ric}(hA^t, B^h) - \widetilde{Ric}(A^t, hB^h) \\
&= \frac{1}{4(a+c)}[(\nabla_u Ric)(A,B) - (\nabla_A Ric)(u,B)] \\
&\quad -\frac{a}{8(a+c)^2}[(\nabla_u Ric)(R_u A, B) - (\nabla_{R_u A} Ric)(u,B)] \\
&\quad -\frac{a}{8(a+c)^2}[(\nabla_u Ric)(A, R_u B) - (\nabla_A Ric)(u, R_u B)] \\
&\quad -\frac{1}{8(a+c)} g(B,u)[(\nabla_u Ric)(A,u) - (\nabla_u Ric)(u,u)],
\end{aligned} \tag{3.9}$$

$$\begin{aligned}
0 &= \tilde{G}(\tilde{S}hA^h - h\tilde{S}A^h, B^h) \\
&= \widetilde{Ric}(hA^h, B^h) - \widetilde{Ric}(A^h, hB^h) \qquad (3.10)\\
&= \frac{1}{4a} g(A,u)[Ric(B,u) - \frac{a}{2(a+c)} \sum_{i=1}^{n} g(R(u,e_i)u, R(u,e_i)B)] \\
&\quad -\frac{1}{4a} g(B,u)[Ric(A,u) - \frac{a}{2(a+c)} \sum_{i=1}^{n} g(R(u,e_i)A, R(u,e_i)u)] \\
&\quad +\frac{1}{4(a+c)}[Ric(R_u A, B) - \frac{a}{2(a+c)} \sum_{i=1}^{n} g(R(u,e_i)R_u A, R(u,e_i)B)] \\
&\quad -\frac{1}{4(a+c)}[Ric(A, R_u B) - \sum_{i=1}^{n} g(R(u,e_i)A, R(u,e_i)R_u B)].
\end{aligned}$$

Thus, T_1M fulfills $\tilde{S}h = h\tilde{S}$ if and only if M fulfills (3.8)-(3.10).

Let M be a space with constant curvature k. In this case, we observe that it fulfills (3.8)-(3.10). Therefore, we obtain the following proposition.

Proposition 3.1. *Let M be a space with constant curvature k. Then T_1M fulfills the relation $\tilde{S}h = h\tilde{S}$.*

If we consider M as a surface (i.e., 2-dimensional manifold), we obtain the following theorem.

Theorem 3.1. *Let M be a 2-dimensional Riemannian manifold. Then T_1M fulfills the relation $\tilde{S}h = h\tilde{S}$ if and only if M is a space with constant Gaussian curvature.*

Proof. The curvature tensor of a 2-dimensional Riemannian manifold is expressed by the relation $R(A,B)C = \kappa(g(B,C)A - g(A,C)B)$, where κ is a smooth function on M. This relation gives us $R_uA = R(A,u)u = \kappa(A - g(A,u)u)$ and $Ric(A,B) = (n-1)\kappa g(A,B)$. Using these formulas, we see that (3.8) and (3.10) are valid. From (3.9), we deduce

$$\frac{a(1-\kappa)+c}{8(a+c)^2}\{(u\kappa)[2g(A,B)-g(A,u)g(B,u)]-(A\kappa)g(B,u)\}=0. \tag{3.11}$$

Taking $A=B\perp u$ and $\|B\|=1$ in (3.11), we get $\frac{a(1-\kappa)+c}{4(a+c)^2}(u\kappa)=0$. This shows that the Gaussian curvature κ is constant. The converse is true from Proposition 3.1. □

Now, we suppose that the relation $\tilde{S}\phi h=\phi h\tilde{S}$ holds for $(T_1M,\tilde{g})$. From (2.5)-(2.7), we obtain

$$\begin{aligned} 0 &= \tilde{G}(\tilde{S}\phi hA^t-\phi h\tilde{S}A^t,B^t) \\ &= \widetilde{Ric}(\phi hA^t,B^t)+\widetilde{Ric}(A^t,h\phi B^t) \\ &= \frac{a}{2(a+c)}[(\nabla_B Ric)(u,A)-(\nabla_X Ric)(u,B)-(\nabla_u Ric)(A,R_uB) \\ &\quad +(\nabla_u Ric)(R_uA,B)+(\nabla_A Ric)(u,R_uB)-(\nabla_B Ric)(u,R_uA) \\ &\quad +g(A,u)((\nabla_u Ric)(u,B)-(\nabla_B Ric)(u,u) \\ &\quad -g(B,u)((\nabla_u Ric)(u,A)-(\nabla_A Ric)(u,u))\}, \end{aligned} \tag{3.12}$$

$$\begin{aligned} 0 &= \tilde{G}(\tilde{S}\phi hA^t-\phi h\tilde{S}A^t,B^h) \\ &= \widetilde{Ric}(\phi hA^t,B^h)+\widetilde{Ric}(A^t,h\phi B^h) \\ &= (n-2)(g(A,B)-g(A,u)g(B,u))+\frac{a^2}{4(a+c)^2}\sum_{i=1}^n g(R(u,A)e_i,R(u,B)e_i) \\ &\quad -(n-2)g(A,R_uB)-\frac{a^2}{4(a+c)^2}\sum_{i=1}^n g(R(u,A)e_i,R(u,R_uB)e_i) \\ &\quad -Ric(A,B)+\frac{a}{2(a+c)}\sum_{i=1}^n g(R(u,e_i)A,R(u,e_i)B) \\ &\quad +g(A,u)[Ric(B,u)-\frac{a}{2(a+c)}\sum_{i=1}^n g(R(u,e_i)u,R(u,e_i)B)] \\ &\quad +Ric(R_uA,B)-\frac{a}{2(a+c)}\sum_{i=1}^n g(R(u,e_i)R_uA,R(u,e_i)B), \end{aligned} \tag{3.13}$$

$$\begin{aligned} 0 &= \tilde{G}(\tilde{S}\phi hA^h-\phi h\tilde{S}A^h,B^h) \\ &= \widetilde{Ric}(\phi hA^h,B^h)+\widetilde{Ric}(A^h,h\phi B^h) \\ &= \frac{a}{2(a+c)}[\nabla_A Ric(u,B)-(\nabla_B Ric)(u,A)+(\nabla_u Ric)(R_uA,B) \\ &\quad -(\nabla_u Ric)(A,R_uB)-(\nabla_{R_uA} Ric)(u,B)+(\nabla_{R_uB} Ric)(u,A)\}. \end{aligned} \tag{3.14}$$

Let's assume that M is a space with constant curvature k. Then, using the relations $R(A,B)C = k(g(B,C)A - g(A,C)B)$, $R_uA = R(A,u)u = k(A - g(A,u)u)$ and $Ric(A,B) = (n-1)kg(A,B)$ in (3.13), we get

$$\frac{2a^2+ac}{2(a+c)^2}k^3 + \frac{2c^2+3ac-2n(a+c)^2}{2(a+c)^2}k^2 + (2n-3)k - n + 2 = 0.$$

So, we have the following theorem.

Theorem 3.2. *Let (M,g) be an $n-$dimensional space with constant curvature k. Then T_1M fulfills the relation $\tilde{S}\phi h = \phi h\tilde{S}$ if and only if the following relation holds:*

$$\frac{2a^2+ac}{2(a+c)^2}k^3 + \frac{2c^2+3ac-2n(a+c)^2}{2(a+c)^2}k^2 + (2n-3)k - n + 2 = 0.$$

Acknowledgments. The authors would like to thank the referee for some useful comments and their helpful suggestions that have improved the quality of this paper.

References

[1] Abbassi, M.T.K., Amri, N. & Calvaruso, G. (2018). Kaluza–Klein type Ricci solitons on unit tangent sphere bundles, Differ. Geom. Appl., 59, 184-203.

[2] Abbassi, M.T.K. & Calvaruso, G. (2007). g-Natural contact metrics on unit tangent sphere bundles, Monatshefte für Mathematik, 151, 89-109.

[3] Abbassi, M.T.K. & Kowalski, O. (2005). On g-natural metrics with constant scalar curvature on unit tangent sphere bundles, Topics in Almost Hermitian Geometry and related fields, 1-29.

[4] Abbassi, M.T.K. & Sarih, M. (2005). On natural metrics on tangent bundles of Riemannian manifolds, Arch Math (Brno) 41, 71-92.

[5] Altunbaş, M. (2022). Some special curves in the unit tangent bundles of surfaces, Journal of Science and Arts, 59(2), 257-264.

[6] Anastasiei, M. (1999). Locally conformal Kaehler structures on tangent bundle of a space form, Libertas Math., 19, 71-76.

[7] Blair, D.E. (2010). Riemannian geometry of contact and symplectic manifolds, 203, Birkhauser, Boston.

[8] Boeckx, E. (2003). When are the tangent sphere bundles of a Riemannian manifold reducible?, Transactions of the American Mathematical Society, 355(7), 2885-2903.

[9] Boeckx, E. & Vanhecke, L. (1997). Characteristic reflections on unit tangent sphere bundles, Houston Journal of Mathematics, (23)(3), 427-448.

[10] Calvaruso, G. & Perrone, D. (2013). Geometry of Kaluza–Klein metrics on the sphere S^3, Annali di Matematica Pura ed Applicata, 192, 879-900.

[11] Cho, J.T. & Chun, S.H. (2021). Unit tangent sphere bundles of conformally flat manifolds, Kodai Mathematical Journal, (44)(2), 307-316.

[12] Druta-Romaniuc, S.L. & Opriou, V. (2010). Tangent sphere bundles of natural diagonal type, Balkan J. Geo. Its Appl., 15(1), 53-67.

[13] Gezer, A. & Altunbaş, M. (2012). Some notes concerning Riemannian metrics of Cheeger Gromoll type, J. Math. Anal. Appl., 396, 119-132.

[14] Munteanu, M.I. (2008). Some aspects on the geometry of the tangent bundles and tangent sphere bundles of a Riemannian manifold, Mediterr. J. Math. 5, 43-59.

[15] Sekizawa, M. (1991). Curvatures of tangent bundles with Kaluza-Klein metric, Tokyo J. Math., 14, 407-417.

Department of Mathematics, Faculty of Arts and Sciences, Erzincan Binali Yildirim University, Erzincan, Turkiye