International Journal of Maps in Mathematics

*Volume **9**, Issue **1**, 2026, Pages:**10–25***

E-ISSN: 2636-7467

www.simadp.com/journalmim

INNOVATIVE AGGREGATION METHODS: SEMI-LINDELÖF PERFECT FUNCTIONS AND SEMI-PERFECT FUNCTIONS IN BITOPOLOGICAL SPACES, TOGETHER WITH THEIR UTILIZATION

ALI A. ATOOM AND MOHAMMAD A. BANI ABDELRAHMAN *

Abstract. Due to the significance of topological spaces in analysis and particular fields, numerous scholars use distinct frameworks to broaden topological space, encompassing the concept of topology. The mathematical description of perfect functions in bitopological spaces emerged as one of most profoundly prominent improvements. Precisely a consequence, we investigate different collections of operator strategies for creating paired semi perfect functions in this work. Accordance relates to the connections between specific kinds of pair semi perfect functions and related traditional topologies Function analysis allows us to explore properties and applications of classical topological concepts with regard for misalignment. The present study suggests and examines several new classes of perfect functions, such as pairwise semi-perfect functions and pairwise semi-Lindelöf perfect functions, within the framework of fundamental topological spaces. Through the use of cases and other instances, the characteristics they have and how they connect to various jobs are examined. It covers the computation of the Cartesian combination among finite intersection.

Keywords: Pairwise compact spaces, Pairwise semi-Lindelofness, Perfect functions, Pairwise semi normal, Applications in bitopological spaces.

MSC (2020): 54C10, 54D20.

1. Introduction

The characteristics of typical issues include large amounts of information with different degrees of complexity. Afterwards, it is necessary to establish innovative mathematical methods to cope with them. There have been some theorized extending topological configurations. Due to the topological space's significance in analysis and different uses, such as artificial intelligence, economics, physics, and statistics. Topological spaces provide a framework to examine continuity, convergence, and compactness, which are fundamental across various fields see ([3, 12, 13, 19, 22, 23, 29, 30, 37, 41]). Kelly [21] introduced the notion of bitopological spaces in 1963. Thereafter, many topologists have focused on applying well-known topological concepts to bitopological spaces. In 1969, Fletcher [16] further introduced the notion of pairwise compactness for bitopological spaces. Later, this concept was extended

Received: 2024.10.16 *Revised: 2025.04.02* *Accepted: 2025.09.17*

* Corresponding author

Ali A. Atoom (aliatoom@anu.edu.jo) ⋄ ORCID:0000-0001-5249-1823

Mohammad A. Bani Abdelrahman (maabdelrahman991@gmail.com) ⋄ ORCID:0009-0001-4394-4576

to pairwise Lindelöf spaces by Reilly [36] in 1973. In 1980s, Dorsett and Hanna [14, 15, 20] studied the concept of semi-compactness. The concept of semi-open sets and semi-continuity was introduced by Levine [26], and the notions were extended to bitopological spaces in 1977 by Maheshwari and Prasad [27]. The interrelations between different open sets in topological spaces were studied by Khedr [24] in 1992. The concepts of pairwise semi-Lindelöf and pairwise semi-Lindelöf bitopological spaces were defined by Balasubramanian [9] in 2009. A set $\mathscr{X}$ equipped with two topologies ξ_1 and ξ_2 is called a bitopological space and denoted by $(\mathscr{X}, \xi_1, \xi_2)$. Subsequently, the notions of semi perfect functions in topological spaces were defined by Atoom et al. [5]. A function $\varrho : (\mathscr{X}, \xi_1, \xi_2) \to (\mathscr{Y}, \mu_1, \mu_2)$ is called pairwise perfect if ϱ is pairwise continuous, pairwise closed, and for each $y \in \mathscr{Y}$, $\varrho^{-1}(y)$ is pairwise compact. They also defined the notion of pairwise semi-perfect functions in bitopological spaces, provided some characterizations of pairwise semi-perfect functions and pairwise M-semi perfect functions, and studied the images and inverse images of certain bitopological properties under these functions. Let $(\mathscr{X}, \xi)$ be a topological space, and let $\mathscr{D}$ be a subset of $\mathscr{X}$. We denote the closure of $\mathscr{D}$ (resp., the interior of $\mathscr{D}$) by $\mathrm{Cl}(\mathscr{D})$ (resp., $\mathrm{Int}(\mathscr{D})$). A subset $\mathscr{D}$ of $(\mathscr{X}, \xi)$ is called semi-open [26] if $\mathscr{D} \subseteq Cl(Int(\mathscr{D}))$. The complement of a semi-open set is called semi-closed. The family of all semi-open sets in $(\mathscr{X}, \xi)$ is denoted by $\mathrm{S}(\xi)$. A $\mathscr{D}$ space $(\mathscr{X}, \xi)$ is said to be semi-compact [28] (resp., semi-Lindelöf) if every cover of $\mathscr{X}$ by semi-open sets has a finite (resp., countable) subcover. Let $\mathscr{D}$ be a subset of a bitopological space $(\mathscr{X}, \xi_1, \xi_2)$. The closure of $\mathscr{D}$ and the interior of $\mathscr{D}$ with respect to ξ_i are denoted by $Cl_i(\mathscr{D})$ and $\mathrm{Int}_i(\mathscr{D})$, respectively. Also, the topologies on $\mathscr{D}$ inherited from ξ_1 and ξ_2 will be denoted by $\xi_1\mathscr{D}$ and $\xi_2\mathscr{D}$, respectively. The intersection of all semi-closed sets containing $\mathscr{D}$ is called the semi-closure [15] of $\mathscr{D}$, denoted by $\mathrm{sCl}(\mathscr{D})$. The semi-closure of $\mathscr{D}$ with respect to $\mathrm{S}(\xi_i)$ is denoted by $s\ Cl_i(\mathscr{D})$. A function $\varrho : (\mathscr{X}, \xi_1, \xi_2) \to (\mathscr{Y}, \mu_1, \mu_2)$ is said to be pairwise continuous (resp., pairwise open, pairwise closed, a pairwise homeomorphism) if $\varrho_1 : (\mathscr{X}, \xi_1) \to (\mathscr{Y}, \mu_1)$ and $\varrho_2 : (\mathscr{X}, \xi_2) \to (\mathscr{Y}, \mu_2)$ are continuous (resp., open, closed, homeomorphism) functions. In recent years, numerous authors have examined the relationships between other topological and analytical concepts. Compactness, pairwise semi-compactness, and pairwise semi-Lindelöfness have been refined to develop new notions within the structure of pairwise perfect functions and pairwise Lindelöf perfect functions in topological spaces, including product properties for generalized pairwise Lindelöf spaces [39], Difference Compactness in Bitopological Spaces [4] pairwise weakly regular-Lindelöf spaces [25], Omega Closed Functions in Bitopological Spaces [8], mappings on pairwise para-Lindelöf bitopological spaces [11], some results on nearly pairwise compact spaces [32], KC-bitopological spaces [35], notes on p_1-Lindelöf spaces which are not contra second-countable spaces in bitopology [2], Study the structure of difference Lindelöf topological spaces and their properties [6], two forms of pairwise Lindelöfness and some results related to hereditary class in a bigeneralized topological spaces [1], some results on pairwise locally compact bitopological spaces [40], pairwise locally compact and compactification in double topological spaces [31], Pairwise α-perfect functions [7], and Semi-Compact and Semi-Lindelöf Spaces via Neutrosophic Crisp Set Theory [38]. Additionally, the concept of providing stronger and weaker forms of functions is becoming increasingly prevalent in research. The structure of the paper consists of several sections. In Section 2, we introduces the essential ideas and terms that are crucial to acknowledging semi-perfect functions in bitopological spaces. In Section 3 several kinds of pairwise semi-perfect functions(briefly; $\widetilde{P_w}\widetilde{S_e}\widetilde{P}$) and their relationships are covered. We further explain and offer an alternative viewpoint on the many $\widetilde{P_w}\widetilde{S_e}\widetilde{P}$ function forms in Section 4, also we investigate and establish the idea of $\widetilde{P_w}\widetilde{S_e}\widetilde{P}$ functions. Furthermore we look into the characteristics of pairwise semi-Lindelöf perfect functions(briefly; $\widetilde{P_w}\widetilde{S_e}\widetilde{L}\widetilde{P}$). In Section 5, we examine a wide range of applications and discuss how they could lead to changes in multi-variable systems and enhance efficiency across various study areas.

2. Preliminaries

This section introduces the essential ideas and terms that are crucial to acknowledging semi-perfect functions in bitopological spaces.

Definition 2.1. *[33] In a bitopological space $(\mathscr{X}, \xi_1, \xi_2)$, a collection of subsets $\mathscr{D}$ is considered $\xi_1\xi_2$-semi-open($\xi_1\xi_2$-$\widetilde{S_e}\widetilde{O}$) if $\mathscr{D} \subset S(\xi_1) \cup S(\xi_2)$. Additionally, if $\mathscr{D} \cap S(\xi_1) \neq \emptyset$ and $\mathscr{D} \cap S(\xi_2) \neq \emptyset$, then $\mathscr{D}$ is further classified as pairwise semi-open ($\widetilde{P_w}\widetilde{S_e}\widetilde{O}$).*

Definition 2.2. *[33] A bitopological space $(\mathscr{X}, \xi_1, \xi_2)$ is considered $\widetilde{P_w}\widetilde{S_e}\widetilde{C_{pt}}$ if every $\widetilde{P_w}\widetilde{S_e}$ cover of $\mathscr{X}$ has a finite subcover. Similarly, it is considered pairwise M-semi compact(briefly; $\widetilde{P_w}Z^* - \widetilde{S_e}\widetilde{C_{pt}}$) if every $\xi_1\xi_2$-$\widetilde{S_e}\widetilde{O}$ cover of $\mathscr{X}$ has a finite subcover. It is obvious that every $\widetilde{P_w}Z^* - \widetilde{S_e}\widetilde{C_{pt}}$) space is also $\widetilde{P_w}\widetilde{S_e}\widetilde{C_{pt}}$, but the opposite is not true.*

Definition 2.3. *[10] Consider $(\mathscr{X}, \xi_1, \xi_2)$ be a bitopological space. A subset $\mathscr{D}$ of $\mathscr{X}$ is defined as a semi open set ($\widetilde{S_e}\widetilde{O}$) if $\mathscr{D} \subset cl_\kappa(int_\kappa \mathscr{D})$. Additionally, $\mathscr{D}$ is considered semi closed if $int_\kappa(cl_\kappa(\mathscr{D})) \subset \mathscr{D}$, for $\kappa = 1, 2$.*

Definition 2.4. *[10] A function $\varrho : (\mathscr{X}, \xi_1, \xi_2) \to (\mathscr{Y}, \mu_1, \mu_2)$ is termed pairwise semi continuous(brf. $\widetilde{P_w}\widetilde{S_e}\widetilde{C_{nt}}$) if $\varrho^{-1}(V)$ is a $\widetilde{S_e}\widetilde{O}$ (or $\widetilde{S_e}\widetilde{C}$) set for every $\widetilde{P_w}\widetilde{S_e}\widetilde{O}$ (or $\widetilde{P_w}\widetilde{S_e}\widetilde{C}$) set V in $\mathscr{Y}$.*

Definition 2.5. *[10] A function $\varrho : (\mathscr{X}, \xi_1, \xi_2) \to (\mathscr{Y}, \mu_1, \mu_2)$ is referred to as $\widetilde{P_w}\widetilde{S_e}\widetilde{C}$ if both $\varrho_1 : (\mathscr{X}, \xi_1) \to (\mathscr{Y}, \mu_1)$ and $\varrho_2 : (\mathscr{X}, \xi_2) \to (\mathscr{Y}, \mu_2)$ are $\widetilde{S_e}\widetilde{C}$ functions. That is if ϱ_1 is $\widetilde{S_e}\widetilde{C}$ in $S(\xi_1)$, then $\varrho(F_1)$ is $\widetilde{S_e}\widetilde{C}$ in $S(\mu_1)$. Similarly, if F_2 is $\widetilde{S_e}\widetilde{C}$ in $S(\xi_2)$, then $\varrho(F_2)$ is $\widetilde{S_e}\widetilde{C}$ in $S(\mu_2)$.*

Theorem 2.1. *[33] A bitopological space $(\mathscr{X}, \xi_1, \xi_2)$ is pairwise semi-compact ($\widetilde{P_w}\widetilde{S_e}\widetilde{C_{pt}}$) iff every ξ_κ-semi-closed subset of $\mathscr{X}$ is semi-compact ($\widetilde{S_e}\widetilde{C_{pt}}$) with respect to the topology ξ_ι, where $\kappa, \iota \in \{1, 2\}$ and $\kappa \neq \iota$.*

Definition 2.6. *[34] If $\check{U}$ and $\check{N}$ are $\xi_1\xi_2$-$\widetilde{S_e}\widetilde{O}$ covers of the bitopological space $(\mathscr{X}, \xi_1, \xi_2)$, then $\check{U}$ is said to be a refinement of $\check{N}$ if each $\mathscr{U}_\kappa \in \check{U} \cap S(\xi_\kappa)$ is contained in some $\mathscr{V}_\kappa \in \check{N} \cap S(\xi_\kappa)$, $\kappa = 1, 2$.*

Definition 2.7. *[34] A collection $\mathscr{D}$ of subsets of a space $(\mathscr{X}, \xi)$ is considered locally finite in $(\mathscr{X}, S(\xi))$ if for any $x \in \mathscr{X}$, there's a $\widetilde{S_e}\widetilde{O}$ set $\mathscr{U}$ such that $x \in \mathscr{U}$ and $\mathscr{U}$ intersects at most finitely many elements of $\mathscr{D}$.*

Definition 2.8. *[34] In a bitopological space $(\mathscr{X}, \xi_1, \xi_2)$. Let $\mathscr{U}$ and $\mathscr{N}$ be $\xi_1\xi_2$-$\widetilde{S_e}\widetilde{O}$ covers. The cover $\mathscr{U}$ is considered a refinement of $\mathscr{N}$ if any $\mathscr{U} \in \mathscr{U} \cap S(\xi_\kappa)$, there exists a $\mathscr{V} \in \mathscr{N} \cap S(\xi_\kappa)$, for $\kappa = 1, 2$.*

Definition 2.9. *[34] A collection $\mathscr{D}$ of subsets of a space $(\mathscr{X}, \xi)$ is considered locally finite in $(\mathscr{X}, S(\xi))$ if, for any $x \in \mathscr{X}$, there exists a $\widetilde{S_e}\widetilde{O}$ set $\mathscr{U}$ such that $\mathscr{X} \in \mathscr{U}$ and $\mathscr{U}$ intersects with at most finitely many elements of $\mathscr{D}$.*

Definition 2.10. *[34] In a bitopological space $(\mathscr{X}, \xi_1, \xi_2)$, the collection $S(\xi_1)$ is considered semi-regular(brf. $\widetilde{S_e}\widetilde{R_{eg}}$) with respect to $S(\xi_2)$ if, for every $x \in \mathscr{X}$ and every ξ_1-$\widetilde{S_e}\widetilde{C}$ set f such that $\mathscr{X} \notin f$, there exists a ξ_1-$\widetilde{S_e}\widetilde{O}$ set $\mathscr{U}$ and a ξ_2-$\widetilde{S_e}\widetilde{O}$ set $\mathscr{V}$ such that $\mathscr{X} \in \mathscr{U}$, $f \subseteq \mathscr{V}$, and $\mathscr{U} \cap \mathscr{V} = \emptyset$. The space $(\mathscr{X}, \xi_1, \xi_2)$ is referred to as $\widetilde{P_w}\widetilde{S_e}\widetilde{R_{eg}}$ if both sets of $\widetilde{S_e}\widetilde{O}$ sets are $\widetilde{S_e}\widetilde{R_{eg}}$ with respect to each other.*

Definition 2.11. *[7] A function $\varrho : (\mathscr{X}, \xi_1, \xi_2) \to (\mathscr{Y}, \mu_1, \mu_2)$ is said to be $\widetilde{P_w}\widetilde{L}\widetilde{P}$, if ϱ is $\widetilde{P_w}\widetilde{C}$, $\widetilde{P_w}\widetilde{C_{nt}}$, and for every $y \in \mathscr{Y}$, $\varrho^{-1}(y)$ is pairwise Lindelöf $\widetilde{P_w}\widetilde{L}$.*

3. A Taxonomy Of Semi Perfect Functions

This section explores the various types of perfect functions and the relationships between them.

Definition 3.1. *A function $\varrho : (\mathscr{X}, \xi_1, \xi_2) \to (\mathscr{Y}, \mu_1, \mu_2)$ is considered $\widetilde{P_w}\widetilde{S_e}\widetilde{P}$, if ϱ is $\widetilde{P_w}\widetilde{S_e}\widetilde{C_{nt}}$, $\widetilde{P_w}\widetilde{S_e}\widetilde{C}$, and any $y \in \mathscr{Y}$, $\varrho^{-1}(y)$ is $\widetilde{P_w}\widetilde{S_e}\widetilde{C_{pt}}$.*

Theorem 3.1. *Given that $\varrho : (\mathscr{X}, \xi_1, \xi_2) \to (\mathscr{Y}, \mu_1, \mu_2)$ is a $\widetilde{P_w}\widetilde{S_e}\widetilde{L}\widetilde{P}$ function, for every a pairwise semi Lindelöf ($\widetilde{P_w}\widetilde{S_e}\widetilde{L}$) $\mathscr{X} \subseteq \mathscr{Y}$, $\varrho^{-1}(\mathscr{X})$ represents $\widetilde{P_w}\widetilde{S_e}\widetilde{L}$.*

Theorem 3.2. *Every $\widetilde{P_w}\widetilde{P}$ function is also $\widetilde{P_w}\widetilde{L}\widetilde{P}$; however, the opposite is not always true.*

Proof. while every $\widetilde{P_w}\widetilde{C_{pt}}$ space is $\widetilde{P_w}\widetilde{L}$, it follows that every $\widetilde{P_w}\widetilde{P}$ function results in a $\widetilde{L}\widetilde{P}$. ☐

As we will show in an example, the reverse isn't always the case:

Example 3.1. *Consider $(\mathbb{R}, \xi_s, \xi_{dis})$ be a bitopological space, where ξ_s is the standard topology on $\mathbb{R}$ and ξ_{dis} is the discrete topology. Consider $U_\iota = (\iota, \iota + 2), \iota \in \mathbb{Z}$ as an open cover of $\mathbb{R}$ under ξ_s that covers $\mathbb{R}$, that is,*

$$\mathbb{R} = \bigcup_{\iota \in \mathbb{Z}} (\iota, \iota + 2).$$

Since the cover is countable and there exists a countable subcover, it follows that $(\mathbb{R}, \xi_s)$ is Lindelöf.

Now, consider $(\mathbb{R}, \xi_{dis})$. In ξ_{dis}, every singleton set $\{x\}$ is open. Consider the open cover $\check{U} = \{\{x\} \mid x \in \mathbb{R}\}$ of $\mathbb{R}$ under ξ_{dis}. Since $\check{U}$ contains all points of $\mathbb{R}$, it is an open cover. However, no finite subcover of $\check{U}$ can cover $\mathbb{R}$, because each singleton only covers one point. Thus, $(\mathbb{R}, \xi_{dis})$ is not compact.

Theorem 3.3. *Every $\widetilde{P_w}\widetilde{S_e}\widetilde{L}\widetilde{P}$ function is also $\widetilde{P_w}\widetilde{L}$. however, but reverse does not necessarily true.*

Proof. Since every $\widetilde{P_w}\widetilde{S_e}\widetilde{L}$ space is $\widetilde{P_w}\widetilde{L}$ space, a $\widetilde{P_w}\widetilde{S_e}\widetilde{L}\widetilde{P}$ function is $\widetilde{P_w}\widetilde{L}\widetilde{P}$ function. ☐

However, as the following example demonstrates, that the opposite not necessarily true:

Example 3.2. *Consider $(\mathbb{R}, \xi_s, \xi)$ be a bitopological space, where ξ_s is the standard topology on $\mathbb{R}$ and ξ is the Sorgenfrey topology with basic open sets $[a, b)$. The space $(\mathbb{R}, \xi_s)$ is Lindelöf. Meaning that for any open cover $\mathscr{U} = \{U_\alpha\}$, there exists a countable subcover. To illustrate this, consider an arbitrary open cover:*

$$\mathscr{U} = \{(n, n+2) \mid n \in \mathbb{Z}\}$$

has the countable subcover $\{(n, n+2) \mid n \in \mathbb{Z}\}$ itself, since $\mathbb{Z}$ is countable.

Now, consider the ξ-semi open cover:

$$\mathscr{C} = \{[x, x+1) \mid x \in \mathbb{R}\}.$$

This covers $\mathbb{R}$ because $\forall y \in \mathbb{R},\ y \in [y, y+1)$. So, $\mathscr{C}$ has no countable subcover.

To examine it; Suppose for contradiction there exists a countable subcover:

$$\mathscr{T} = \{[x_m, x_m + 1) \mid m \in \mathbb{N}\}.$$

Let $Y = \sup\{x_m \mid m \in \mathbb{N}\}$. Two cases: If $Y = +\infty$, this contradicts $\mathbb{N}$ being countable. If $Y < +\infty$, then $Y + 1 \notin \bigcup_{m\in\mathbb{N}}[x_m, x_m + 1)$.

Both cases yield contradictions. Thus, $(\mathbb{R}, \xi)$ is not semi-Lindelöf.

Theorem 3.4. *Every $\widetilde{P_w}\widetilde{S_e}\widetilde{P}$ function is $\widetilde{P_w}\widetilde{S_e}\widetilde{L}\widetilde{P}$ function but the converse need not to be true.*

Proof. Since every $\widetilde{P_w}\widetilde{S_e}\widetilde{C_{pt}}$ is also $\widetilde{P_w}\widetilde{S_e}\widetilde{L}$, so $\widetilde{P_w}\widetilde{S_e}\widetilde{P}$ function is $\widetilde{P_w}\widetilde{S_e}\widetilde{L}\widetilde{P}$ function. □

Example 3.3. *Consider the bitopological space $(\mathbb{S}, \mu_s, \mu_2)$, where $\mathbb{S} = \mathbb{R}$, μ_s is the standard topology on $\mathbb{S}$ and μ_2 is the discrete topology on $\mathbb{S}$. A set $D \subseteq \mathbb{S}$ is semi-open in μ_s if $D \subseteq \mathrm{cl}(\mathrm{int}(D))$. For example, intervals $[a, b)$ are semi-open in μ_s. The space $(\mathbb{S}, \mu)$ is semi-Lindelöf because any semi open cover $\mathscr{V} = \{V_\beta \mid \beta \in B\}$ has a countable subcover $\{V_{\beta_i} \mid i \in \mathbb{N}\}$. Now, consider the μ_2-semi open cover*

$$\mathscr{C} = \{(\ell, \ell+1) \cup \{\ell+1\} \mid \ell \in \mathbb{Z}\}.$$

Each set is semi open because $(\ell, \ell+1)$ is open in μ_s, $\{\ell+1\}$ is open in μ_2 and $(\ell, \ell+1) \subseteq \mathrm{cl}(\mathrm{int}((\ell, \ell+1) \cup \{\ell+1\}))$.

Hence, $\mathscr{C}$ has no finite subcover.

To examine it, Suppose for contradiction there exists a finite subcover:

$$\{(\ell_k, \ell_k+1) \cup \{\ell_k+1\} \mid k = 1, \ldots, n\}.$$

Let $M = \max\{\ell_k\}$. Then,

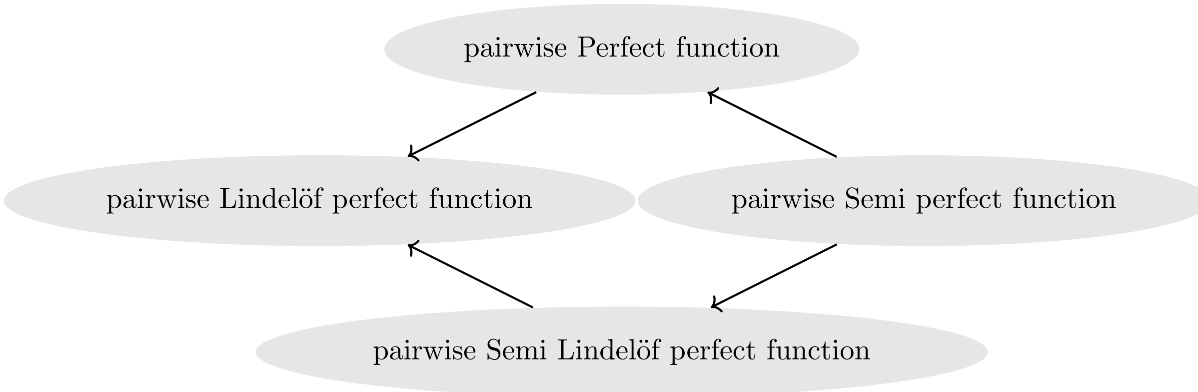
$$M + 1 \notin \bigcup_{k=1}^{n} (\ell_k, \ell_k+1) \cup \{\ell_k+1\}$$

contradicting $\mathscr{C}$ being a cover.

Theorem 3.5. *Every $\widetilde{P_w}\widetilde{S_e}\widetilde{P}$ function is a $\widetilde{P_w}\widetilde{P}$ function. However, the reverse is not necessarily true.*

Proof. Since every $\widetilde{P_w}\widetilde{S_e}\widetilde{C_{pt}}$ space is $\widetilde{P_w}\widetilde{C_{pt}}$ space, it follows that every $\widetilde{P_w}\widetilde{S_e}\widetilde{P}$ function is also pairwise perfect. □

We have the following diagram in which the converses of each implication need not be true as shown by the examples stated above. Moreover, the examples show that the pairwise semi perfect function are independent.



4. Essential Findings on Pairwise Semi-Lindelöf Perfect Functions

Definition 4.1. *A function $\varrho : (\mathscr{X}, \xi_1, \xi_2) \to (\mathscr{Y}, \mu_1, \mu_2)$ is considered $\widetilde{P_w}\widetilde{S_e}\widetilde{P}$, if it is $\widetilde{P_w}\widetilde{S_e}\widetilde{C_{nt}}$, $\widetilde{P_w}\widetilde{S_e}\widetilde{C}$, and for every $y \in \mathscr{Y}$, $\varrho^{-1}(y)$ is $\widetilde{P_w}\widetilde{S_e}\widetilde{C_{pt}}$.*

Definition 4.2. *A function $\varrho : (\mathscr{X}, \xi_1, \xi_2) \to (\mathscr{Y}, \mu_1, \mu_2)$ is considered $\widetilde{P_w}\widetilde{S_e}\widetilde{L}\widetilde{P}$, if ϱ is $\widetilde{P_w}\widetilde{S_e}\widetilde{C_{nt}}$, $\widetilde{P_w}\widetilde{S_e}\widetilde{C}$, and for every $y \in \mathscr{Y}$, $\varrho^{-1}(y)$ is $\widetilde{P_w}\widetilde{S_e}\widetilde{L}$.*

Definition 4.3. *A function $\varrho : (\mathscr{X}, \xi_1, \xi_2) \to (\mathscr{Y}, \mu_1, \mu_2)$ is considered $\widetilde{P_w}\widetilde{S_e}Z^* - \widetilde{P}$, if ϱ is $\widetilde{P_w}\widetilde{S_e}\widetilde{C_{nt}}$, $\widetilde{P_w}\widetilde{S_e}\widetilde{C}$, and for every $y \in \mathscr{Y}$, $\varrho^{-1}(y)$ is $\widetilde{P_w}Z^* - \widetilde{S_e}\widetilde{C_{pt}}$.*

Theorem 4.1. *If $\varrho : (\mathscr{X}, \xi_1, \xi_2) \to (\mathscr{Y}, \mu_1, \mu_2)$ is a$\widetilde{P_w}\widetilde{S_e}\widetilde{P}$ function, then for every $\widetilde{P_w}\widetilde{S_e}\widetilde{C_{pt}}$ subset $\mathscr{Z} \subseteq \mathscr{Y}$, the inverse image $\varrho^{-1}\mathscr{Z}$ is also $\widetilde{P_w}\widetilde{S_e}\widetilde{C_{pt}}$.*

Proof. Let's consider $\check{U} = \{\mathscr{U}_\alpha : \alpha \in \Lambda\}$ be a pairwise open cover of $(\mathscr{X}, \xi_1, \xi_2)$, since ϱ is a $\widetilde{P_w}\widetilde{S_e}\widetilde{P}$ function, then $\forall y \in \mathscr{Y}$, $\varrho^{-1}(y)$ is $\widetilde{P_w}\widetilde{S_e}\widetilde{C_{pt}}$. Thus,this implies there exist finite subsets Λ_y, Λ_y^* of Λ such that

$$\varrho^{-1}(y) \subseteq \bigcup_{\alpha \in \Lambda_y} \{\mathscr{V}_\alpha : \alpha \in \Lambda_y\} \cup \bigcup_{\alpha \in \Lambda_y^*} \{W_\alpha : \alpha \in \Lambda_y^*\},$$

where $\{\mathscr{V}_\alpha : \alpha \in \Lambda_y\}$ is semi open in $S(\xi_1)$, and $\{W_\alpha : \alpha \in \Lambda_y\}$ is semi open in $S(\xi_2)$.

Let $O_y = \mathscr{Y} - \varrho(\mathscr{X} - \bigcup_{\alpha \in \Lambda_y} \mathscr{V}_\alpha)$ be a $\widetilde{S_e}\widetilde{O}$ set in $S(\mu_1)$ containing y, and $O_y^* = \mathscr{Y} - \varrho(\mathscr{X} - \bigcup_{\alpha \in \Lambda_y^*} W_\alpha)$ be a $\widetilde{S_e}\widetilde{O}$ set in $S(\mu_2)$ containing y, where $\varrho^{-1}(O_y) \subseteq \bigcup_{\alpha \in \Lambda_y} \mathscr{V}_\alpha$ and $\varrho^{-1}(O_y^*) \subseteq \bigcup_{\alpha \in \Lambda_y^*} W_\alpha$. Define

$$\mathscr{O} = \{O_y : y \in \mathscr{Y}\} \cup \{O_y^* : y \in \mathscr{Y}\}$$

as a $\widetilde{P_w}\widetilde{S_e}\widetilde{O_p}$ cover of $\mathscr{Y}$. $\mathscr{O}$ is a $\widetilde{P_w}\widetilde{S_e}\widetilde{O_p}$ cover of $\mathscr{Z}$.

Since $\mathscr{Z}$ is $\widetilde{P_w}\widetilde{S_e}\widetilde{C_{pt}}$, we have $\mathscr{Z} \subseteq \bigcup_{i=1}^n O_{y_i} \cup \bigcup_{j=1}^m O_{y_j}^*$.

Thus,

$$\varrho^{-1}(\mathscr{Z}) \subseteq \bigcup_{i=1}^{n} \varrho^{-1}(O_{y_i}) \cup \bigcup_{j=1}^{m} \varrho^{-1}(O_{y_j}^*).$$

This is a subset of the union of a finite number of $\check{U}$, that is, $\varrho^{-1}(\mathscr{Z})$ is $\widetilde{P_w}\widetilde{S_e}\widetilde{C_{pt}}$.

□

Corollary 4.1. *A $\widetilde{P_w}\widetilde{S_e}\widetilde{C_{pt}}$ space remains invariant under the inverse of a $\widetilde{P_w}\widetilde{S_e}\widetilde{P}$ function.*

Using a parallel reasoning, we can conclude these accompanying claims:

Theorem 4.2. *If $\varrho : (\mathscr{X}, \xi_1, \xi_2) \to (\mathscr{Y}, \mu_1, \mu_2)$ is a $\widetilde{P_w}Z^* - \widetilde{S_e}\widetilde{P}$ function, then for every $\widetilde{P_w}\widetilde{S_e}\widetilde{C_{pt}}$ subset $\mathscr{Z} \subseteq \mathscr{Y}$, $\varrho^{-1}(\mathscr{Z})$ is $\widetilde{P_w}Z^* - \widetilde{S_e}\widetilde{C_{pt}}$.*

Proof. Theorem 4.1 provides the justification for this claim. □

Corollary 4.2. *$\widetilde{P_w}Z^* - \widetilde{S_e}\widetilde{C_{pt}}$ space remains invariant under the inverse of a $\widetilde{P_w}Z^* - \widetilde{S_e}\widetilde{P}$ function.*

Theorem 4.3. *given that $\varrho : (\mathscr{X}, \xi_1, \xi_2) \to (\mathscr{Y}, \mu_1, \mu_2)$ is a $\widetilde{P_w}Z^* - \widetilde{S_e}\widetilde{L}\widetilde{P}$ function, then for every $\widetilde{P_w}\widetilde{S_e}\widetilde{L}$ subset $\mathscr{Z} \subseteq \mathscr{Y}$, $\varrho^{-1}(\mathscr{Z})$ is $\widetilde{P_w}Z^* - \widetilde{S_e}\widetilde{L}$.*

Corollary 4.3. *A $\widetilde{P_w}\widetilde{S_e}\widetilde{L}$ space is remains invariant under the inverse of a $\widetilde{P_w}Z^* - \widetilde{S_e}\widetilde{L}\widetilde{P}$.*

Theorem 4.4. *Given that $\varrho : (\mathscr{X}, \xi_1, \xi_2) \to (\mathscr{Y}, \mu_1, \mu_2)$ is a $\widetilde{P_w}\widetilde{S_e}\widetilde{L}\widetilde{P}$ function, for every $\widetilde{P_w}\widetilde{S_e}\widetilde{L}$ $\mathscr{Z} \subseteq \mathscr{Y}$, $\varrho^{-1}(\mathscr{Z})$ is a $\widetilde{P_w}\widetilde{S_e}\widetilde{L}$.*

Remark 4.1. *The composition of two $\widetilde{P_w}\widetilde{S_e}\widetilde{P}$ functions does not always results in a $\widetilde{P_w}\widetilde{S_e}\widetilde{P}$ functions.*

Example 4.1. *Consider the bitopological spaces: $(\mathscr{X}, \xi_1, \xi_2)$ where $\mathscr{X} = \{a, b, c\}$, $\xi_1 = \{\mathscr{X}, \emptyset, \{a\}, \{a, b\}\}$, and $\xi_2 = \{\mathscr{X}, \emptyset, \{c\}\}$. $(\mathscr{Y}, \mu_1, \mu_2)$ where $\mathscr{Y} = \{1, 2, 3\}$, $\mu_1 = \{\mathscr{Y}, \emptyset, \{1\}, \{2\}, \{1, 2\}\}$, and $\mu_2 = \{\mathscr{Y}, \emptyset, \{3\}\}$. $(\mathscr{Z}, \delta_1, \delta_2)$ where $\mathscr{Z} = \{x, y, z\}$, $\delta_1 = \{\mathscr{Z}, \emptyset, \{x\}\}$, and $\delta_2 = \{\mathscr{Z}, \emptyset, \{y, z\}\}$.*

Let

$$\varrho : \mathscr{X} \to \mathscr{Y}, \quad \varrho(a) = 1,\ \varrho(b) = 2,\ \varrho(c) = 3$$

$$\psi : \mathscr{Y} \to \mathscr{Z}, \quad \psi(1) = x,\ \psi(2) = y,\ \psi(3) = z$$

ϱ and ψ are $\widetilde{P_w}\widetilde{S_e}\widetilde{C_{nt}}$.
A function is pairwise semi continuous if the inverse of $\widetilde{P_w}\widetilde{S_e}\widetilde{O}$ set is semi-open in ξ_1 or ξ_2. For ϱ: $\varrho^{-1}(\{1, 2\}) = \{a, b\}$ is ξ_1-open (hence semi-open), but not ξ_2-semi open. $\varrho^{-1}(\{3\}) = \{c\}$ is ξ_2-open (hence semi open), but not ξ_1-semi open. Thus, ϱ is $\widetilde{P_w}\widetilde{S_e}\widetilde{C_{nt}}$.
For ψ: $\psi^{-1}(\{x, y\}) = \{1, 2\}$ is μ_1-open (hence semi-open), but not μ_2-semi open. Thus, ψ is $\widetilde{P_w}\widetilde{S_e}\widetilde{C_{nt}}$.

Now, Consider the $\widetilde{P_w}\widetilde{S_e}\widetilde{O}$ set $V = \{x, y\}$ in $\mathscr{Z}$:

$$(\psi \circ \varrho)^{-1}(V) = \varrho^{-1}(\psi^{-1}(V)) = \varrho^{-1}(\{1, 2\}) = \{a, b\}.$$

Check semi-openness in ξ_1 and ξ_2; In ξ_1: $\{a, b\}$ is open (hence semi open). In ξ_2: $\{a, b\}$ is not semi open because $\mathrm{cl}_{\xi_2}(\mathrm{int}_{\xi_2}(\{a, b\})) = \mathrm{cl}_{\xi_2}(\emptyset) = \emptyset \not\supseteq \{a, b\}$. Therefore, $(\psi \circ \varrho)^{-1}(V)$ is not semi-open in ξ_2, violating Definition 2.4. Thus, $\psi \circ \varrho$ is not $\widetilde{P_w}\widetilde{S_e}\widetilde{C_{nt}}$.

Remark 4.2. *the composition of two $\widetilde{P_w}Z^* - \widetilde{S_e}\widetilde{P}$ functions does not necessarily yield another$\widetilde{P_w}Z^* - \widetilde{S_e}\widetilde{P}$ function.*

Theorem 4.5. *the composition of two* $\widetilde{P_w}Z^* - \widetilde{S_e}\widetilde{P}$ *functions does not necessarily yield another* $\widetilde{P_w}Z^* - \widetilde{L}\widetilde{C_{pt}}$ *function.*

Proof. Let $(\mathscr{X}, \xi_1, \xi_2)$ where $\mathscr{X} = \{a, b, c\}$.$(\mathscr{Y}, \mu_1, \mu_2)$ where $\mathscr{Y} = \{1, 2, 3\}$.$(\mathscr{Z}, \delta_1, \delta_2)$ where $\mathscr{Z} = \{x, y, z\}$.
Consider $\xi_1 = \{\mathscr{X}, \emptyset, \{a\}, \{a, b\}\}$. $\xi_2 = \{\mathscr{X}, \emptyset, \{c\}\}$. $\mu_2 = \{\mathscr{Y}, \emptyset, \{3\}\}$.
Define $\varrho(a) = 1,\ \varrho(b) = 2,\ \varrho(c) = 3$. $\psi(1) = x,\ \psi(2) = y,\ \psi(3) = z$
For ϱ: $\widetilde{P_w}\widetilde{S_e}\widetilde{C_{nt}}$: $\varrho^{-1}(\{1, 2\}) = \{a, b\}$ is ξ_1-semi-open. $\varrho^{-1}(\{3\}) = \{c\}$ is ξ_2-semi-open.
$\widetilde{P_w}\widetilde{S_e}\widetilde{C}$: $\varrho(\{a, b\}) = \{1, 2\}$ is μ_1-semi-closed. For ψ: $\widetilde{P_w}\widetilde{S_e}\widetilde{C_{nt}}$: $\psi^{-1}(\{x, y\}) = \{1, 2\}$ is μ_1-semi-open.
Now, consider $V = \{x, y\} \in \delta_1$-$\widetilde{S_e}\widetilde{O}$, then,

$$(\psi \circ \varrho)^{-1}(V) = \varrho^{-1}(\psi^{-1}(V)) = \{a, b\}.$$

Hence, composition fails $\widetilde{P_w}\widetilde{S_e}\widetilde{C_{nt}}$ in ξ_2 since the topology mismatch between μ_2 and δ_2. □

Theorem 4.6. *given that* ϱ*:(*$\mathscr{X}$*,* ξ_1*,* ξ_2*)* $\to$ *(*$\mathscr{Y}$*,* μ_1*,* μ_2*) is a* $\widetilde{P_w}\widetilde{S_e}\widetilde{P}$ *function, and* ω*:(*$\mathscr{Y}$*,* μ_1*,* μ_2*)* $\to$ *(*$\mathscr{Z}$*,* δ_1*,* δ_2*) is a pairwise perfect function,* $\omega \circ \varrho$ *is a* $\widetilde{P_w}\widetilde{S_e}\widetilde{P}$ *function.*

Proof. Consider $\mathscr{H}$ be any δ_1-open set in $\mathscr{Z}$. Since ω is a pairwise perfect function, then $\omega^{-1}(\mathscr{H})$ is a μ_1-open set in $\mathscr{Y}$. Since ϱ is a pairwise semi perfect function, then $\varrho^{-1}(\omega^{-1}(\mathscr{H}))$ is a ξ_1-$\widetilde{S_e}\widetilde{O}$ set in $\mathscr{X}$.

Similarly, let $\mathscr{B}$ be any δ_2-open set in $\mathscr{Z}$. Then, $\omega \circ \varrho$ is a $\widetilde{P_w}\widetilde{S_e}\widetilde{P}$ function. □

Using a parallel reasoning, we can conclude these accompanying claims:

Theorem 4.7. *given that* $\varrho : (\mathscr{X}, \xi_1, \xi_2) \to (\mathscr{Y}, \mu_1, \mu_2)$ *is* $\widetilde{P_w}\widetilde{S_e}\widetilde{L}\widetilde{P}$ *function and* ω*:(*$\mathscr{Y}$*,* μ_1*,* μ_2*)* $\to$ *(*$\mathscr{Z}$*,* δ_1*) is pairwise perfect function,* $\omega \circ \varrho$ *is* $\widetilde{P_w}\widetilde{S_e}\widetilde{L}\widetilde{P}$ *function.*

Corollary 4.4. *If* $\varrho : (\mathscr{X}, \xi_1, \xi_2) \to (\mathscr{Y}, \mu_1, \mu_2)$ *is a* $\widetilde{P_w}Z^* - \widetilde{S_e}\widetilde{P}$ *function and* $\omega : (\mathscr{Y}, \mu_1, \mu_2) \to (\mathscr{Z}, \delta_1, \delta_2)$ *is a pairwise perfect function, then* $\omega \circ \varrho$ *is a* $\widetilde{P_w}Z^* - \widetilde{S_e}\widetilde{P}$ *function.*

Proposition 4.1. *If the composition* $\omega \circ \varrho$ *of the* $\widetilde{P_w}\widetilde{S_e}\widetilde{C_{nt}}$ *function* $\varrho : (\mathscr{X}, \xi_1, \xi_2) \xrightarrow{onto} (\mathscr{Y}, \mu_1, \mu_2)$ *and the pairwise continuous function* $\omega : (\mathscr{Y}, \mu_1, \mu_2) \xrightarrow{onto} (\mathscr{Z}, \delta_1, \delta_2)$ *is* $\widetilde{P_w}\widetilde{S_e}\widetilde{C}$*, then the function* $\omega : (\mathscr{Y}, \mu_1, \mu_2) \xrightarrow{onto} (\mathscr{Z}, \delta_1, \delta_2)$ *is* $\widetilde{P_w}\widetilde{S_e}\widetilde{C}$*.*

Proof. consider $\mathscr{D}$ be a μ_1-closed set in $\mathscr{Y}$. Then $\varrho^{-1}(\mathscr{D})$ is ξ_1-$\widetilde{S_e}\widetilde{C}$ in $\mathscr{X}$. Since ϱ is a $\widetilde{P_w}\widetilde{S_e}\widetilde{C_{nt}}$ function and $\omega \circ \varrho$ is $\widetilde{P_w}\widetilde{S_e}\widetilde{C}$, then $\omega(\varrho(\varrho^{-1}(\mathscr{D})))$ is δ_1-$\widetilde{S_e}\widetilde{C}$ in $\mathscr{Z}$, i.e., $\omega(\mathscr{D})$ is δ_1-$\widetilde{S_e}\widetilde{C}$ in $\mathscr{Z}$.

Similarly, we can show that if $\mathscr{B}$ is a μ_2-closed set in $\mathscr{Y}$, then $\omega(\mathscr{B})$ is δ_2-$\widetilde{S_e}\widetilde{C}$ in $\mathscr{Z}$. Thus, ω is a $\widetilde{P_w}\widetilde{S_e}\widetilde{C}$ function. □

Theorem 4.8. *Given that* $\varrho : (\mathscr{X}, \xi_1, \xi_2) \xrightarrow{onto} (\mathscr{Y}, \mu_1, \mu_2)$ *is* $\widetilde{P_w}\widetilde{S_e}\widetilde{C_{nt}}$ *function, and pairwise continuous* $\omega : (\mathscr{Y}, \mu_1, \mu_2) \xrightarrow{onto} (\mathscr{Z}, \delta_1, \delta_2)$*, if the composition* $\omega \circ \varrho$ *is* $\widetilde{P_w}\widetilde{S_e}\widetilde{P}$*, then* ω *is* $\widetilde{P_w}\widetilde{S_e}\widetilde{P}$*.*

Proof. For every $z \in \mathscr{Z}$, $\omega^{-1}(z) = \varrho((\omega \circ \varrho)^{-1}(z))$ is $\widetilde{P_w}\widetilde{S_e}\widetilde{C_{pt}}$, because $\omega \circ \varrho$ is $\widetilde{P_w}\widetilde{S_e}\widetilde{P}$. Since ω is $\widetilde{P_w}\widetilde{S_e}\widetilde{C}$ by the previous proposition, we get that ω is $\widetilde{P_w}\widetilde{S_e}\widetilde{P}$. □

By applying similar reasoning, we can derive the following related claims:

Theorem 4.9. *Given that the* $\widetilde{P_w}\widetilde{S_e}\widetilde{C_{nt}}$ *function,* $\varrho : (\mathscr{X}, \xi_1, \xi_2) \xrightarrow{onto} (\mathscr{Y}, \mu_1, \mu_2)$*, and pairwise continuous* $\omega : (\mathscr{Y}, \mu_1, \mu_2) \xrightarrow{onto} (\mathscr{Z}, \delta_1, \delta_2)$*. If* $\omega \circ \varrho$ *is a* $\widetilde{P_w}\widetilde{S_e}\widetilde{L}\widetilde{P}$*, then* $\omega : (\mathscr{Y}, \mu_1, \mu_2) \xrightarrow{onto} (\mathscr{Z}, \delta_1, \delta_2)$ *is* $\widetilde{P_w}\widetilde{S_e}\widetilde{L}\widetilde{P}$*.*

Theorem 4.10. *Given that* $\varrho : (\mathscr{X}, \xi_1, \xi_2) \xrightarrow{onto} (\mathscr{Y}, \mu_1, \mu_2)$ *is a* $\widetilde{P_w}\widetilde{S_e}\widetilde{C}$ *function, for any* $\mathscr{B} \subset \mathscr{Y}$*, the restriction* $\varrho_{\mathscr{B}} : \varrho^{-1}(\mathscr{B}) \to \mathscr{B}$ *is also* $\widetilde{P_w}\widetilde{S_e}\widetilde{C}$*.*

Proof. Given that $\mathscr{B} \subset \mathscr{Y}$. let the function $\varrho : (\mathscr{X}, \xi_1, \xi_2) \xrightarrow{\text{onto}} (\mathscr{Y}, \mu_1, \mu_2)$, and consider $\mathscr{D}$ be a ξ_1-$\widetilde{S_e}\widetilde{C}$ set. Then,

$$\varrho_{\mathscr{B}}(\mathscr{D} \cap \varrho^{-1}(\mathscr{B})) = \varrho(\mathscr{D}) \cap \mathscr{B}$$

which is μ_1-$\widetilde{S_e}\widetilde{C}$ in $\mathscr{B}$. As well, we can show that if $\mathscr{D}$ is a ξ_2-$\widetilde{S_e}\widetilde{C}$ set, then

$$\varrho_{\mathscr{B}}(\mathscr{D} \cap \varrho^{-1}(\mathscr{B})) = \varrho(\mathscr{D}) \cap \mathscr{B}$$

is μ_2-$\widetilde{S_e}\widetilde{C}$ in $\mathscr{B}$. Hence, $\varrho_{\mathscr{B}} : \varrho^{-1}(\mathscr{B}) \to \mathscr{B}$ is $\widetilde{P_w}\widetilde{S_e}\widetilde{C}$. □

Theorem 4.11. *Given that* $\varrho : (\mathscr{X}, \xi_1, \xi_2) \xrightarrow{onto} (\mathscr{Y}, \mu_1, \mu_2)$ *is a* $\widetilde{P_w}\widetilde{S_e}\widetilde{P}$ *function, for every* $\mathscr{B} \subset \mathscr{Y}$*, the restriction* $\varrho_{\mathscr{B}} : \varrho^{-1}(\mathscr{B}) \to \mathscr{B}$ *is* $\widetilde{P_w}\widetilde{S_e}\widetilde{P}$*.*

Proof. This result is an immediate consequence of Theorem 4.10. □

Definition 4.4. *A space* $(\{X\}, \xi_1, \xi_2)$ *is said to* $\widetilde{P_w}\widetilde{S_e}$*−Hausdorff if, for each two distinct points* x *and* y*, there exist a* $\widetilde{S_e} - \xi_1$*-neighbourhood* $\mathscr{U}$ *of* x *and a* $\widetilde{S_e} - \xi_2$*-neighbourhood* $\mathscr{V}$ *of* y *such that* $\mathscr{U} \cap \mathscr{V} = \emptyset$*.*

Theorem 4.12. *If* $\varrho : (\mathscr{X}, \xi_1, \xi_2) \xrightarrow{onto} (\mathscr{Y}, \mu_1, \mu_2)$ *is* $\widetilde{P_w}\widetilde{S_e}\widetilde{P}$*, where* $(\mathscr{Y}, \mu_1, \mu_2)$ *is* $\widetilde{P_w}\widetilde{S_e}$*−Hausdorff, and* $(\mathscr{X}, \xi_1, \xi_2)$ *is* $\widetilde{P_w}\widetilde{S_e}\widetilde{C_{pt}}$*, then* ϱ *is* $\widetilde{P_w}\widetilde{S_e}\widetilde{C}$*.*

Proof. If $\mathscr{D}$ is a ξ_1-$\widetilde{S_e}\widetilde{C}$ subset of $(\mathscr{X}, \xi_1, \xi_2)$, it is also ξ_2-$\widetilde{S_e}\widetilde{C_{pt}}$ due to the $\widetilde{P_w}\widetilde{S_e}\widetilde{C_{pt}}$ of $(\mathscr{X}, \xi_1, \xi_2)$. Since ϱ is $\widetilde{P_w}\widetilde{S_e}\widetilde{C_{nt}}$, $\varrho(\mathscr{D})$ is a μ_2-$\widetilde{S_e}\widetilde{C_{pt}}$ subset of $(\mathscr{Y}, \mu_1, \mu_2)$.

Moreover, since $(\mathscr{Y}, \mu_1, \mu_2)$ is $\widetilde{P_w}\widetilde{S_e}$− Hausdorff, it follows that $\varrho(\mathscr{D})$ is a μ_1-$\widetilde{S_e}\widetilde{C}$ set. As well, if $\mathscr{B}$ is a ξ_2-$\widetilde{S_e}\widetilde{C}$ subset of $\mathscr{X}$, then $\varrho(\mathscr{B})$ is a μ_2-$\widetilde{S_e}\widetilde{C}$ subset of $(\mathscr{Y}, \mu_1, \mu_2)$.

□

Using similar reasoning, we can deduce the following related assertions:

Theorem 4.13. *Given that* $\varrho : (\mathscr{X}, \xi_1, \xi_2) \xrightarrow{onto} (\mathscr{Y}, \mu_1, \mu_2)$ *be a* $\widetilde{P_w}\widetilde{S_e}\widetilde{L}\widetilde{P}$*, where* $(\mathscr{X}, \xi_1, \xi_2)$ *is* $\widetilde{P_w}\widetilde{S_e}\widetilde{L}$*, and* $(\mathscr{Y}, \mu_1, \mu_2)$ *is* $\widetilde{P_w}\widetilde{S_e}$*− Hausdorff, then* ϱ *is* $\widetilde{P_w}\widetilde{S_e}\widetilde{C}$*.*

Theorem 4.14. *If* $\varrho : (\mathscr{X}, \xi_1, \xi_2) \xrightarrow{onto} (\mathscr{Y}, \mu_1, \mu_2)$ *is a* $\widetilde{P_w}Z^* - \widetilde{S_e}\widetilde{P}$ *function, where* $(\mathscr{Y}, \mu_1, \mu_2)$ *is* $\widetilde{P_w}\widetilde{S_e}$*- Hausdorff, and* $(\mathscr{X}, \xi_1, \xi_2)$ *is* $\widetilde{P_w}Z^* - \widetilde{S_e}\widetilde{C_{pt}}$*, then* ϱ *is* $\widetilde{P_w}\widetilde{S_e}\widetilde{C}$*.*

Definition 4.5. *A function* $\varrho : (\mathscr{X}, \xi_1, \xi_2) \to (\mathscr{Y}, \mu_1, \mu_2)$ *is termed a* $\widetilde{P_w}\widetilde{S_e}$*-homeomorphism if* ϱ *is* $\widetilde{P_w}\widetilde{S_e}\widetilde{C_{nt}}$ *(or* $\widetilde{P_w}\widetilde{S_e}\widetilde{O}$*), and* ϱ *is a bijection.*

Theorem 4.15. *Consider* $\varrho : (\mathscr{X}, \xi_1, \xi_2) \to (\mathscr{Y}, \mu_1, \mu_2)$ *be a bijective function that is* $\widetilde{P_w}\widetilde{S_e}\widetilde{C_{nt}}$*. If* $(\mathscr{Y}, \mu_1, \mu_2)$ *is a* $\widetilde{P_w}\widetilde{S_e}$*-Hausdorff space and* $(\mathscr{X}, \xi_1, \xi_2)$ *is* $\widetilde{P_w}\widetilde{S_e}\widetilde{C_{pt}}$*, then* ϱ *is a* $\widetilde{P_w}\widetilde{S_e}$*-homeomorphism.*

Proof. It's enough to show that ϱ is $\widetilde{P_w}\widetilde{S_e}\widetilde{C}$. Consider f be a ξ_κ-closed proper subset of $\mathscr{X}$, and hence f is a proper ξ_ι-$\widetilde{S_e}\widetilde{C_{pt}}$ set, where $\kappa \neq \iota$, and $\kappa, \iota = 1, 2$. By using Theorem 2.1, we know that $\varrho(f)$ is μ_ι-$\widetilde{S_e}\widetilde{C_{pt}}$. But since $(\mathscr{Y}, \mu_1, \mu_2)$ is a $\widetilde{P_w}\widetilde{S_e}$-Hausdorff space, $\varrho(f)$ is μ_κ-$\widetilde{S_e}$ closed. Therefore, ϱ is a $\widetilde{P_w}\widetilde{S_e}$-homeomorphism function.

□

Definition 4.6. *A function $\varrho : (\mathscr{X}, \xi_1, \xi_2) \to (\mathscr{Y}, \mu_1, \mu_2)$ is termed $\widetilde{P_w}\widetilde{S_e}$-strongly function (or $\widetilde{P_w}\widetilde{S_e}$-weakly function) if for any $\widetilde{P_w}\widetilde{O}$ cover $\check{U} = \{\mathscr{U}_\alpha : \alpha \in \Lambda\}$, there exists a $\widetilde{P_w}\widetilde{S_e}\widetilde{O_p}$ cover $\check{V} = \{\mathscr{V}_\gamma : \gamma \in \Gamma\}$ of $\mathscr{Y}$ such that each $\varrho^{-1}(\mathscr{V}_\gamma)$ has a finite subcover of $\check{U}$.*

Theorem 4.16. *Given that $\varrho : (\mathscr{X}, \xi_1, \xi_2) \to (\mathscr{Y}, \mu_1, \mu_2)$ be a $\widetilde{P_w}\widetilde{S_e}$-strongly onto function. $(\mathscr{X}, \xi_1, \xi_2)$ is $\widetilde{P_w}\widetilde{S_e}\widetilde{C_{pt}}$ if $(\mathscr{Y}, \mu_1, \mu_2)$ is $\widetilde{P_w}\widetilde{S_e}\widetilde{C_{pt}}$.*

Proof. Consider $\check{U} = \{\mathscr{U}_\alpha : \alpha \in \Lambda\}$ is a pairwise open cover of $(\mathscr{X}, \xi_1, \xi_2)$. Since ϱ is a $\widetilde{P_w}\widetilde{S_e}$-strongly function, there exists a $\widetilde{P_w}\widetilde{S_e}\widetilde{O_p}$ cover $\check{V} = \{\mathscr{V}_\gamma : \gamma \in \Gamma\}$ of $(\mathscr{Y}, \mu_1, \mu_2)$, such that each $\varrho^{-1}(\mathscr{V}_\gamma)$ has a finite subcover of $\check{U}$. But $(\mathscr{Y}, \mu_1, \mu_2)$ is $\widetilde{P_w}\widetilde{S_e}\widetilde{C_{pt}}$, so there exists a finite subset $\Gamma_1 \subseteq \Gamma$, such that $\mathscr{Y} = \bigcup_{\gamma \in \Gamma_1} \mathscr{V}_\gamma$. therefore, $\mathscr{X} = \bigcup_{\gamma \in \Gamma_1} \varrho^{-1}(\mathscr{V}_\gamma)$. Thus, each $\varrho^{-1}(\mathscr{V}_\gamma)$ has a finite subcover of $\check{U}$, implying that $\mathscr{X}$ is $\widetilde{P_w}\widetilde{S_e}\widetilde{C_{pt}}$.

□

Definition 4.7. *A bitopological space $(\mathscr{X}, \xi_1, \xi_2)$ is called $\widetilde{P_w}Z^* - \widetilde{S_e}$-paracompact if each $\widetilde{P_w}\widetilde{S_e}\widetilde{O}$ cover of $\mathscr{X}$ has a pairwise locally finite $\xi_1\xi_2$-open refinement.*

Definition 4.8. *A bitopological space $(\mathscr{X}, \xi_1, \xi_2)$ is called $\widetilde{P_w}\widetilde{S_e}$-paracompact if each $\widetilde{P_w}\widetilde{S_e}\widetilde{O}$ cover of $\mathscr{X}$ has a $\widetilde{P_w}$-locally finite pairwise open refinement.*

Theorem 4.17. *Given that $\varrho : (\mathscr{X}, \xi_1, \xi_2) \to (\mathscr{Y}, \mu_1, \mu_2)$ be a $\widetilde{P_w}\widetilde{S_e}\widetilde{P}$ function, and $(\mathscr{Y}, \mu_1, \mu_2)$ is a $\widetilde{P_w}\widetilde{S_e}$-paracompact. Then $(\mathscr{X}, \xi_1, \xi_2)$ is $\widetilde{P_w}\widetilde{S_e}$-paracompact.*

Proof. Consider $\check{U} = \{\mathscr{U}_\alpha : \alpha \in \Lambda\}$ be a $\widetilde{P_w}\widetilde{S_e}\widetilde{O}$ cover of $(\mathscr{X}, \xi_1, \xi_2)$. Since ϱ is a $\widetilde{P_w}\widetilde{S_e}\widetilde{P}$ function, then for all $y \in \mathscr{Y}$, $\varrho^{-1}(y)$ is $\widetilde{P_w}\widetilde{S_e}\widetilde{C_{pt}}$. There exists a finite subset Λ_y, Λ_y^* of Λ such that $\varrho^{-1}(y) \subseteq \bigcup_{\alpha \in \Lambda_y} \{\mathscr{V}_\alpha : \alpha \in \Lambda_y\} \cup \bigcup_{\alpha \in \Lambda_y^*} \{W_\alpha : \alpha \in \Lambda_y^*\}$, where $\{\mathscr{V}_\alpha : \alpha \in \Lambda_y\}$ is $\widetilde{S_e}\widetilde{O}$ in $S(\xi_1)$, and $\{W_\alpha : \alpha \in \Lambda_y^*\}$ is $\widetilde{S_e}\widetilde{O}$ in $S(\xi_2)$. Let $O_y = \mathscr{Y} - \varrho\left(\mathscr{X} - \bigcup_{\alpha \in \Lambda_y} \mathscr{V}_\alpha\right)$ be a $\widetilde{S_e}\widetilde{O}$ set in $S(\mu_1)$ containing y, and $O_y^* = \mathscr{Y} - \varrho\left(\mathscr{X} - \bigcup_{\alpha \in \Lambda_y^*} W_\alpha\right)$ be a $\widetilde{S_e}\widetilde{O}$ set in $S(\mu_2)$ containing y, where $\varrho^{-1}(O_y) \subseteq \bigcup_{\alpha \in \Lambda_y} \mathscr{V}_\alpha$, and $\varrho^{-1}(O_y^*) \subseteq \bigcup_{\alpha \in \Lambda_y^*} W_\alpha$. Let $\check{O} = \{O_y : y \in \mathscr{Y}\} \cup \{O_y^* : y \in Y\}$ be a $\widetilde{P_w}\widetilde{S_e}\widetilde{O}$ cover of $\mathscr{Y}$. Since $(\mathscr{Y}, \mu_1, \mu_2)$ is $\widetilde{P_w}\widetilde{S_e}$-paracompact, $\check{O}$ has a $\widetilde{P_w}$-locally finite open refinement. Say $\check{H} = \{\mathscr{H}_\mathscr{B} : \mathscr{B} \in \Gamma_1\} \cup \{\mathscr{H}_\mathscr{B}^* : \mathscr{B} \in \Gamma_2\}$, where $\{H_\mathscr{B} : \mathscr{B} \in \Gamma_1\}$ is μ_1-locally finite paracompact of O_y, and $\{\mathscr{H}_\mathscr{B}^* : \mathscr{B} \in \Gamma_2\}$ is μ_2-locally finite paracompact of O_y^*, $\Gamma = \Gamma_1 \cup \Gamma_2$. Let $G_1 = \{\varrho^{-1}(\mathscr{H}_\mathscr{B}) \cap \mathscr{V}_{\alpha_\kappa} : \kappa = 1, 2, \ldots, n, \mathscr{B} \in \Gamma_1, \alpha \in \Lambda_y\}$ be a ξ_1-$\widetilde{S_e}$ open locally finite refinement of $\{\mathscr{V}_\alpha : \alpha \in \Lambda\}$, and let $G_2 = \{\varrho^{-1}(\mathscr{H}_\mathscr{B}^*) \cap W_{\alpha_\kappa} : \kappa = 1, 2, \ldots, n, \mathscr{B} \in \Gamma_2, \alpha \in \Lambda_\mathscr{Y}^*\}$ be a ξ_2-$\widetilde{S_e}\widetilde{O}$-locally finite refinement of $\{W_\alpha : \alpha \in \Lambda\}$. Let $\check{G} = G_1 \cup G_2$. Then $\check{G}$ is a $\widetilde{P_w}\widetilde{S_e}$-locally finite semi open refinement of $\check{U}$, so $(\mathscr{X}, \xi_1, \xi_2)$ is a $\widetilde{P_w}\widetilde{S_e}$-paracompact space.

□

Corollary 4.5. *Consider $\varrho : (\mathscr{X}, \xi_1, \xi_2) \to (\mathscr{Y}, \mu_1, \mu_2)$ be a $\widetilde{P_w}\widetilde{S_e}\widetilde{P}$ function, and $(\mathscr{Y}, \mu_1, \mu_2)$ is a $\widetilde{P_w}Z^* - \widetilde{S_e}$- paracompact,then $(\mathscr{X}, \xi_1, \xi_2)$ is $\widetilde{P_w}Z^* - \widetilde{S_e}$-paracompact.*

Corollary 4.6. *Consider* $\varrho : (\mathscr{X}, \xi_1, \xi_2) \to (\mathscr{Y}, \mu_1, \mu_2)$ *be a* $\widetilde{P_w}\widetilde{S_e}\widetilde{L}$ *perfect function, and* $(\mathscr{Y}, \mu_1, \mu_2)$ *is a* $\widetilde{P_w}Z^* - \widetilde{S_e}$*-paracompact, then* $(\mathscr{X}, \xi_1, \xi_2)$ *is* $\widetilde{P_w}Z^* - \widetilde{S_e}$*-paracompact.*

Theorem 4.18. *The* $\widetilde{P_w}\widetilde{S_e}$*-Hausdorff property is preserved under* $\widetilde{P_w}\widetilde{S_e}\widetilde{P}$*.*

Proof. Consider $(\mathscr{X}, \xi_1, \xi_2)$ be a $\widetilde{P_w}\widetilde{S_e}$-Hausdorff space, $\varrho : (\mathscr{X}, \xi_1, \xi_2) \to (\mathscr{Y}, \mu_1, \mu_2)$ be a $\widetilde{P_w}\widetilde{S_e}\widetilde{P}$ function, and $y_1 \neq y_2$ in $(\mathscr{Y}, \mu_1, \mu_2)$. Then $\varrho^{-1}(y_1)$ and $\varrho^{-1}(y_2)$ are disjoint and $\widetilde{P_w}\widetilde{S_e}\widetilde{C_{pt}}$ subsets of $(\mathscr{X}, \xi_1, \xi_2)$. Since $(\mathscr{X}, \xi_1, \xi_2)$ is a $\widetilde{P_w}\widetilde{S_e}$-Hausdorff space, there exists a ξ_1-$\widetilde{S_e}\widetilde{O}$ $\mathscr{U}$ of $\mathscr{X}$ and a ξ_2-$\widetilde{S_e}\widetilde{O}$ $\mathscr{V}$ such that $\varrho^{-1}(y_1) \subseteq \mathscr{U}$, $\varrho^{-1}(y_2) \subseteq \mathscr{V}$, and $\mathscr{U} \cap \mathscr{V} = \emptyset$. Let the sets $\mathscr{Y} - \varrho(\mathscr{X} - \mathscr{U})$ be a μ_1-$\widetilde{S_e}\widetilde{O}$ set in $(\mathscr{Y}, \mu_1, \mu_2)$ containing y_1, and $\mathscr{Y} - \varrho(\mathscr{X} - \mathscr{V})$ be a μ_2-$\widetilde{S_e}\widetilde{O}$ set in $(\mathscr{Y}, \mu_1, \mu_2)$ containing y_2, such that

$$
\begin{aligned}
&\mathscr{Y} - \varrho(\mathscr{X} - \mathscr{U}) \cap \mathscr{Y} - \varrho(\mathscr{X} - \mathscr{V}) \\
&= \mathscr{Y} - [\varrho(\mathscr{X} - \mathscr{U}) \cup \varrho(\mathscr{X} - \mathscr{V})] \\
&= \mathscr{Y} - \varrho(\mathscr{X} - (\mathscr{U} \cap \mathscr{V})) \\
&= \mathscr{Y} - \varrho(\mathscr{X}) \\
&= \emptyset.
\end{aligned}
$$

Therefore, $(\mathscr{Y}, \mu_1, \mu_2)$ is a $\widetilde{P_w}\widetilde{S_e}$-Hausdorff space.

□

Remark 4.3. *The* $\widetilde{P_w}\widetilde{S_e}$*-Hausdorff property preserved by the inverse images of* $\widetilde{P_w}\widetilde{S_e}\widetilde{P}$ *functions.*

Remark 4.4. *The* $\widetilde{P_w}\widetilde{S_e}$*-Hausdorff property is preserved under* $\widetilde{P_w}Z^* - \widetilde{S_e}\widetilde{P}$*.*

Remark 4.5. *The* $\widetilde{P_w}\widetilde{S_e}$*-Hausdorff property is preserved by the inverse images of* $\widetilde{P_w}Z^* - \widetilde{S_e}\widetilde{P}$ *functions*

Remark 4.6. *The* $\widetilde{P_w}\widetilde{S_e}$*-Hausdorff property is preserved by the inverse images of* $\widetilde{P_w}Z^* - \widetilde{S_e}\widetilde{L}\widetilde{P}$ *functions*

Lemma 4.1. *Consider* $\mathscr{X}$ *be a* $\widetilde{P_w}\widetilde{S_e}\widetilde{R_{eg}}$ *space, and* $\mathscr{D}$ *be a* ξ_κ-$\widetilde{S_e}\widetilde{C_{pt}}$ *subset of* $\mathscr{X}$*,* $\kappa = 1, 2$*. Then for each* ξ_κ-$\widetilde{S_e}$ *neighbourhood* $\mathscr{U}$ *of* $\mathscr{D}$*, there exists a* ξ_κ-$\widetilde{S_e}\widetilde{O}$ *set* W *such that* $\mathscr{D} \subset W \subset s\,Cl_{\xi_\iota}(W) \subset \mathscr{U}$*, where* $\kappa, \iota = 1, 2$ *and* $\kappa \neq \iota$*.*

Proof. For every $d \in \mathscr{D}$, there exists a ξ_κ-$\widetilde{S_e}\widetilde{O}$ set $\mathscr{V}(d)$ such that

$$\mathrm{sCl}_{\xi_\iota}\big(\mathscr{V}(d)\big) \subseteq \mathscr{U},$$

so,

$$\mathscr{D} \subseteq \bigcup_{t=1}^{n} \mathscr{V}(d_t) \subseteq \mathrm{sCl}_{\xi_\iota}\left(\bigcup_{t=1}^{n} \mathscr{V}(d_t)\right).$$

Let $W = \bigcup_{t=1}^{n} \mathscr{V}(d_t)$. Then, W is ξ_κ-$\widetilde{S_e}\widetilde{O}$. But,

$$\mathrm{sCl}_{\xi_\iota}(W) = \mathrm{sCl}_{\xi_\iota}\left(\bigcup_{t=1}^{n} \mathscr{V}(d_t)\right) = \bigcup_{t=1}^{n} \mathrm{sCl}_{\xi_\iota}\big(\mathscr{V}(d_t)\big) \subseteq \mathscr{U}.$$

Hence,

$$\mathscr{D} \subseteq W \subseteq \mathrm{sCl}_{\xi_\iota}(W) \subseteq \mathscr{U}, \quad \kappa, \iota \in \{1, 2\},\ \kappa \neq \iota.$$

□

Lemma 4.2. *Consider $\mathscr{X}$ be a $\widetilde{P_w}\widetilde{S_e}\widetilde{R_{eg}}$ space, and $\mathscr{D}$ be a ξ_κ-$\widetilde{S_e}\widetilde{L}$ subset of $\mathscr{X}$, $\kappa = 1, 2$. Then for each ξ_κ-$\widetilde{S_e}$ neighbourhood $\mathscr{U}$ of $\mathscr{D}$, there exists a ξ_κ-$\widetilde{S_e}\widetilde{O}$ set W such that $\mathscr{D} \subset W \subset s\ Cl_{\xi_\iota}(W) \subset \mathscr{U}$, where $\kappa, \iota = 1, 2$ and $\kappa \neq \iota$.*

Theorem 4.19. *Consider $\varrho : (\mathscr{X}, \xi_1, \xi_2) \to (\mathscr{Y}, \mu_1, \mu_2)$ be a $\widetilde{P_w}\widetilde{S_e}\widetilde{P}$ function, and $(\mathscr{X}, \xi_1, \xi_2)$ is a $\widetilde{P_w}\widetilde{S_e}\widetilde{R_{eg}}$, then $(\mathscr{Y}, \mu_1, \mu_2)$ is $\widetilde{P_w}\widetilde{S_e}\widetilde{R_{eg}}$.*

Proof. Given μ_κ-$\widetilde{S_e}\widetilde{O}$ set $\mathscr{V}$, $y \in \mathscr{Y}$, $\kappa, \iota = 1, 2$, $\varrho^{-1}(y) \in \varrho^{-1}(\mathscr{V})$ in $\mathscr{X}$. Since $\mathscr{X}$ is $\widetilde{P_w}\widetilde{S_e}\widetilde{R_{eg}}$, there exists ξ_κ-$\widetilde{S_e}\widetilde{O}$ set $\mathscr{U}$ (by using Lemma 4.1), such that $\varrho^{-1}(y) \subseteq W \subseteq \mathrm{sCl}_{\xi_\iota}(W) \subseteq \varrho^{-1}(\mathscr{V})$, $\kappa \neq \iota$. Since ϱ is ξ_κ-$\widetilde{S_e}\widetilde{C_{nt}}$, then there exists μ_κ-$\widetilde{S_e}$ neighborhood W of y, such that $\varrho^{-1}(y) \subseteq W$, but $W \subset \varrho(\mathrm{s\ Cl}_{\xi_\iota}\mathscr{U}) \subset \mathscr{V}$. Since $\varrho(\mathrm{s\ Cl}_{\xi_\iota}\mathscr{U})$ is μ_ι-$\widetilde{S_e}$ closed, $y \in W \subset \mathrm{s\ Cl}_{\mu_\iota}(W) \subset \varrho(\mathrm{s\ Cl}_{\xi_\iota}\mathscr{U}) \subset \mathscr{V}$.
Hence, $\mathscr{Y}$ is $\widetilde{P_w}\widetilde{S_e}\widetilde{R_{eg}}$.

□

Corollary 4.7. *Consider $\varrho : (\mathscr{X}, \xi_1, \xi_2) \to (\mathscr{Y}, \mu_1, \mu_2)$ be a$\widetilde{P_w}\widetilde{S_e}\widetilde{L}\widetilde{P}$ function, and $(\mathscr{X}, \xi_1, \xi_2)$ is a $\widetilde{P_w}\widetilde{S_e}\widetilde{R_{eg}}$, then $(\mathscr{Y}, \mu_1, \mu_2)$ is $\widetilde{P_w}\widetilde{S_e}\widetilde{R_{eg}}$.*

Remark 4.7. *The $\widetilde{P_w}\widetilde{S_e}\widetilde{R_{eg}}$ space is preserved under the inverse image of $\widetilde{P_w}\widetilde{S_e}\widetilde{P}$.*

Remark 4.8. *The $\widetilde{P_w}\widetilde{S_e}\widetilde{R_{eg}}$ space is preserved under $\widetilde{P_w}Z^* - \widetilde{S_e}\widetilde{P}$.*

Corollary 4.8. *The $\widetilde{P_w}\widetilde{S_e}\widetilde{R_{eg}}$ space is preserved under the inverse image of $\widetilde{P_w}Z^* - \widetilde{S_e}\widetilde{P}$.*

Corollary 4.9. *The $\widetilde{P_w}\widetilde{S_e}\widetilde{R_{eg}}$ space is preserved under the inverse image of $\widetilde{P_w}Z^* - \widetilde{S_e}\widetilde{L}\widetilde{P}$.*

Definition 4.9. *A bitopological space $(\mathscr{X}, \xi_1, \xi_2)$ is called $\widetilde{P_w}\widetilde{S_e}$-normal if for each ξ_κ-$\widetilde{S_e}\widetilde{C}$ set $\mathscr{D}$ and ξ_ι-$\widetilde{S_e}\widetilde{C}$ set $\mathscr{B}$, there exist ξ_ι-$\widetilde{S_e}\widetilde{O}$ set $\mathscr{U}$ and ξ_κ-$\widetilde{S_e}\widetilde{O}$ set $\mathscr{V}$, such that $\mathscr{D} \subset \mathscr{U}$, $\mathscr{B} \subset \mathscr{V}$, and $\mathscr{U} \cap \mathscr{V} = \emptyset$, where $\kappa, \iota = 1, 2$ and $\kappa \neq \iota$.*

Theorem 4.20. *Consider $\varrho : (\mathscr{X}, \xi_1, \xi_2) \to (\mathscr{Y}, \mu_1, \mu_2)$ be a $\widetilde{P_w}\widetilde{S_e}\widetilde{P}$ function, and if $(\mathscr{X}, \xi_1, \xi_2)$ is $\widetilde{P_w}\widetilde{S_e}$-normal, then $(\mathscr{Y}, \mu_1, \mu_2)$ is also $\widetilde{P_w}\widetilde{S_e}$-normal.*

Proof. It follows by applying Lemma 4.1 and theorem 4.19. □

Theorem 4.21. *Consider $(\mathscr{X}, \xi_1, \xi_2)$ and $(\mathscr{Y}, \mu_1, \mu_2)$ be any bitopological spaces. If $(\mathscr{X}, \xi_1, \xi_2)$ is $\widetilde{P_w}Z^* - \widetilde{S_e}\widetilde{C_{pt}}$, then the projection function: $\vartheta : (\mathscr{X} \times \mathscr{Y}, \xi_1 \times \mu_1, \xi_2 \times \mu_2) \to (\mathscr{Y}, \mu_1, \mu_2)$ is $\widetilde{P_w}\widetilde{S_e}\widetilde{C}$.*

Proof. If $(\mathscr{X}, \xi_1, \xi_2)$ is $\widetilde{P_w}Z^* - \widetilde{S_e}\widetilde{C_{pt}}$, then $(\mathscr{X}, \xi_1)$ is $Z^* - \widetilde{S_e}\widetilde{C_{pt}}$, and $(\mathscr{X}, \xi_2)$ is $Z^* - \widetilde{S_e}\widetilde{C_{pt}}$. Thus, the projection functions $\vartheta_1 : (\mathscr{X} \times \mathscr{Y}, \xi_1 \times \mu_1) \to (\mathscr{Y}, \mu_1)$ and $\vartheta_2 : (\mathscr{X} \times \mathscr{Y}, \xi_2 \times \mu_2) \to (\mathscr{Y}, \mu_2)$ are semi closed. Therefore, ϑ is $\widetilde{P_w}\widetilde{S_e}\widetilde{C}$.

□

Theorem 4.22. *Consider $(\mathscr{X}, \xi_1, \xi_2)$ and $(\mathscr{Y}, \mu_1, \mu_2)$ be any bitopological spaces. If $(\mathscr{X}, \xi_1, \xi_2)$ is $\widetilde{P_w}Z^* - \widetilde{S_e}\widetilde{L}$, then the projection function: $\vartheta : (\mathscr{X} \times \mathscr{Y}, \xi_1 \times \mu_1, \xi_2 \times \mu_2) \to (\mathscr{Y}, \mu_1, \mu_2)$ is $\widetilde{P_w}\widetilde{S_e}\widetilde{C}$.*

Proof. Given that $(\mathscr{X}, \xi_1, \xi_2)$ is $\widetilde{P_w}Z^* - \widetilde{S_e}\widetilde{C_{pt}}$, then $(\mathscr{X}, \xi_1)$ is $Z^* - \widetilde{S_e}\widetilde{L}$, and $(\mathscr{X}, \xi_2)$ is $Z^* - \widetilde{S_e}\widetilde{L}$. Thus, the projection functions $\vartheta_1 : (\mathscr{X} \times \mathscr{Y}, \xi_1 \times \mu_1) \to (\mathscr{Y}, \mu_1)$ and $\vartheta_2 : (\mathscr{X} \times \mathscr{Y}, \xi_2 \times \mu_2) \to (\mathscr{Y}, \mu_2)$ are semi closed. Therefore, ϑ is $\widetilde{P_w}\widetilde{S_e}\widetilde{C}$. □

Corollary 4.10. *Let $(\mathscr{X}, \xi_1, \xi_2)$ and $(\mathscr{Y}, \mu_1, \mu_2)$ be $\widetilde{P_w}Z^* - \widetilde{S_e}\widetilde{C_{pt}}$ spaces, then $(\mathscr{X} \times \mathscr{Y}, \xi_1 \times \mu_1, \xi_2 \times \mu_2)$ is $\widetilde{P_w}Z^* - \widetilde{S_e}\widetilde{C_{pt}}$.*

Corollary 4.11. *Let $(\mathscr{X}, \xi_1, \xi_2)$ and $(\mathscr{Y}, \mu_1, \mu_2)$ be $\widetilde{P_w}Z^* - \widetilde{S_e}\widetilde{L}$ spaces, then $(\mathscr{X} \times \mathscr{Y}, \xi_1 \times \mu_1, \xi_2 \times \mu_2)$ is $\widetilde{P_w}Z^* - \widetilde{S_e}\widetilde{L}$.*

5. Prospective Applications

This section investigates the future possibilities of new aggregation approaches, specifically the applications of semi-Lindelöf perfect functions and semi-perfect functions within bitopological spaces. We next look at how these notions might be used to promote both theoretical research and practical applications in a variety of fields.

In neutrosophic topology, where any point can have degrees of truth, indeterminacy, and falsehood, pairwise perfect functions could provide a glimpse of how different topologies interact when some truth-values are preserved across mappings. This could find use in fuzzy systems, decision-making, or uncertainty modeling.

Having numerous topology types in mathematical modeling of economics and decision theory might be effective in representing several utility functions or preference structures, particularly in multicriteria decision-making. When it comes to optimization or equilibrium models, a pairwise perfect function can be thought of as a mapping that precisely maintains certain fundamental characteristics across various assessment criteria.

Additionally might enhance AI models. Artificial intelligence models may learn more quickly and achieve superior performance in tasks such as language processing and reinforcement learning if they simultaneously use pairwise semi perfect functions in bitopological spaces. Moreover, Semi-Lindelöf perfect functions may increase prediction models' accuracy by allowing better handling of complicated data topologies.

As well the semi-Lindelöf perfect function and semi perfect function in bitopological spaces may lead to the development of novel quantum algorithms that take use of quantum systems dual nature.

6. Conclusion

The paper investigated the relationships between pairwise semi-Lindelöf perfect functions, pairwise perfect spaces, pairwise Lindelöf perfect functions, and pairwise semi Lindelöf perfect functions in the bitopological spaces from which those functions come from. Using the conceptual framework of pairwise semi perfect functions provided currently, the research illustrated the requirements for connecting pairwise semi sets and continuous closed functions. We assessed the relationships between these ideas and declared them with many different functions. The other objective of the examination was to highlight specific details of the intricate features of pairwise semi-perfect functions as well as some peculiarities in the Cartesian multiplication of these functions under specific conditions. Additionally, some illustrative scenarios and the main features of these notions were thoroughly investigated. We outlined their essential traits in their entirety and clarified what was needed for getting them into parity. The research also highlighted these functions characteristics and offered numerous examples of them. These responsibilities will serve as a basis for additional research into the prospective outcomes of each of these roles. Further research could explore additional iterations of these features. Pairwise semi-perfect functions can be defined and studied in various contexts, such as: (1) Neutrosophic semi-perfect function in bitopological spaces; (2) Pairwise semi-perfect functions and domain theory; (3) Pairwise fuzzy semi-perfect functions; and (4) Finding applications for our newly discovered pairwise semi-perfect function results in Artificial Intelligence, Multi-Dimensional Optimization, Quantum Computing, Risk Management,and broad continuity.

Acknowledgments. The researchers express their gratitude to the reviewer for providing valuable feedback and constructive recommendations that have enhanced the overall quality of their manuscript.

Conflict of interest. The authors declare no potential conflict of interests.

References

[1] Acharjee, S., Tripathy, B.C., & Papadopoulos, K. (2017). Two Forms of Pairwise Lindelöfness and Some Results Related to Hereditary Class in a Bigeneralized Topological Space. *New Math. Nat. Comput.*, 13, 181-193.

[2] Acharjee, S., Papadopoulos, K., & Tripathy, B.C. (2018). Note on p_1-Lindelof Spaces which are not Contra Second Countable Spaces in Bitopology.

[3] Arici, F., & Mesland, B. (2019). Toeplitz Extensions in Noncommutative Topology and Mathematical Physics. *Trends in Mathematics.*

[4] Atoom, A. A., & Abdelrahman, M. A. B (2025). Difference Compactness in Bitopological Spaces: Foundations from Difference Sets and Dual Views with Applications. Boletim da Sociedade Paranaense de Matematica, 43.

[5] Atoom, A., Qoqazeh, H., Abdelrahman, M. A. B., Hussein, E., Mahmoud, D. A., & Owledat, A. A. (2025). A Spectrum of Semi-Perfect Functions in Topology: Classification and Implications. WSEAS Transactions on Mathematics, 24, 347-357.

[6] Atoom, A. A., Qoqazeh, H., & Abu-Alkishik, N. A. B. E. E. L. A. (2024). Study the structure of difference Lindelöf topological spaces and their properties. *Journal of Applied Mathematics & Informatics*, 42(3), 471-481.

[7] Atoom, A. A., & Bani-Ahmad, F. E. R. A. S. (2024). Between Pairwise-α-Perfect Functions and Pairwise-t-α-Perfect Functions. Journal of applied mathematics & informatics, 42(1), 15-29.

[8] Atoom, A., Qoqazeh, H., Alholi, M. M., ALmuhur, E., Hussein, E., Owledat, A. A., & Al-Nana, A. A. (2024). A New Outlook on Omega Closed Functions in Bitopological Spaces and Related Aspects. European Journal of Pure and Applied Mathematics, 17(4), 2574-2585.

[9] Balasubramanian, S. (2009). Pairwise Semi Compact and Pairwise Semi Lindeloff Spaces. *Department of Mathematics Northwest University*, 5(3), 62-71.

[10] Bose, S. (1981). Semi Open Sets, Semi-Continuity and Semi-Open Mappings in Bitopological Spaces. *Bulletin of the Calcutta Mathematical Society*, 73, 237-246.

[11] Bouseliana, H., & Kılıçman, A. (2015). Mappings on Pairwise Para-Lindelöf Bitopological Spaces. *Journal of the Egyptian Mathematical Society*, 23, 377-381.

[12] Bubenik, P. (2012). Statistical Topological Data Analysis Using Persistence Landscapes. *J. Mach. Learn. Res.*, 16, 77-102.

[13] Ding, K., Fang, C., & Ma, G. (2022). Non-Hermitian Topology and Exceptional-Point Geometries. *Nature Reviews Physics*, 4, 745-760.

[14] Dorsett, C. (1981). Semi Compactness, Semi Separation Axioms, and Product Spaces. *Bulletin of the Malaysian Mathematical Society*, 4(2), 21-28.

[15] Dorsett, C. (1982). Semi Convergence and Semi Compactness. *Indian Journal of Mechanics and Mathematics*, 19(1), 11-17.

[16] Fletcher, P., Hoyle, H. B., III, & Patty, C. W. (1969). The comparison of topologies.

[17] Ganster, M. (1987). Some Remarks on Strongly Compact Spaces and Semi Compact Spaces. *Bulletin of the Malaysian Mathematical Society*, 10(2), 67-81.

[18] Ganster, M. (1990). On Covering Properties and Generalized Open Sets in Topological Spaces. *Mathematical Chronicle*, 19, 27-33.

[19] Goubault-Larrecq, J. (2013). Non-Hausdorff Topology and Domain Theory.

[20] Hanna, F., & Dorsett, C. (1984). Semi Compactness. *Questions and Answers in General Topology*, 2, 38-47.

[21] Kelly, J. C. (1963). Bitopological Spaces. *Proceedings of the London Mathematical Society*, 13, 71-89.

[22] Keiding, H. (2009). Topological Vector Spaces Admissible in Economic Equilibrium Theory. *Journal of Mathematical Analysis and Applications*, 351(2), 675-681.

[23] Khan, M. A., & Uyanık, M. (2021). Topological Connectedness and Behavioral Assumptions on Preferences: A Two-Way Relationship. *Economic Theory*, 71(2), 411-460.

[24] Khedr, F. H., Al-Areefi, S. M., & Noiri, T. (1992). Precontinuity and semi-precontinuity in bitopological spaces. *Indian Journal of Pure and Applied Mathematics*, 23, 625-625.

[25] Kılıçman, A., & Salleh, Z. (2008). Pairwise Weakly Regular-Lindelöf Spaces. *Abstract and Applied Analysis*, 2008, 1-13.

[26] Levine, N. (1963). Semi-Open Sets and Semi-Continuity in Topological Spaces. *American Mathematical Monthly*, 70, 36-41.

[27] Maheshwari, S. N., & Prasad, R. (1977). Semi-Open Sets and Semi-Continuous Function in Bitopological Spaces. *Mathematical Notes*, 26, 29-37.

[28] Mashhour, A. S., Abd El-Monsef, M. E., Hasanein, I. A., & Noiri, T. (1984). Strongly Compact Spaces. *Delta Journal of Science*, 8(1), 30-46.

[29] Megaritis, A. C. (2022). Covering Dimension and Ideal Topological Spaces. *Quaestiones Mathematicae*, 45(2), 197-212.

[30] Michel, B. (2015). A Statistical Approach to Topological Data Analysis.

[31] Mohsin, R. R. (2019). Pairwise Locally Compact and Compactification in Double Topological Spaces. *Journal of Kufa for Mathematics.*

[32] Mukharjee, A., & Bose, M. K. (2016). Some Results on Nearly Pairwise Compact Spaces. *Bulletin of the Malaysian Mathematical Sciences Society*, 39, 933-940.

[33] Mustafa, J. M. (2004). Semi Compactness and Semi Lindelofness in Bitopological Spaces. *Abhath Al-Yarmouk*, 13(2), 309-314.

[34] Mustafa, J. M. (2007). Pairwise Semi Paracompact Spaces. *International Mathematical Forum*, 2(49), 2447-2452.

[35] Qoqazeh, H., Atoom, A., Alholi, M., ALmuhur, E., Hussein, E., Owledat, A., & Al-Nana, A. (2024). KC-bitopological spaces. AIMS Mathematics, 9(11), 32182-32199.

[36] Reilly, I. L. (1973). Pairwise Lindelof Bitopological Spaces. *Kyungpook Mathematical Journal*, 13(1), 1-4.

[37] Rivera-Castro, R., Pletnev, A., Pilyugina, P., Diaz, G.U., Nazarov, I., Zhu, W., & Burnaev, E. V. (2019). Topology-Based Clusterwise Regression for User Segmentation and Demand Forecasting. *2019 IEEE International Conference on Data Science and Advanced Analytics (DSAA)*, 326-336.

[38] Salama, A. A., Hanafy, I. M., & Dabash, M. S. (2018).Semi-Compact and Semi-Lindelöf Spaces via Neutrosophic Crisp Set Theory. *asia mathematika*, 2(2), 41-48.

[39] Salleh, Z., Nurillaev, M., & Che Taib, C.M. (2023). Product Properties for Generalized Pairwise Lindelöf Spaces. *Mathematics and Statistics.*

[40] Singh, H. N., & Mishra, S. (2020). Some Results on Pairwise Locally Compact Bitopological Spaces.

[41] Turner, K., & Spreemann,G. (2019). Same But Different: Distance Correlations Between Topological Summaries. *Topological Data Analysis*.

(A. A. Atoom) Department of Mathematics, Faculty of Science, Ajloun National University, P.O. Box 43, Ajloun 26810, JORDAN

(M. A. Bani Abdelrahman) Department of Mathematics, Faculty of Science, Ajloun National University, P.O. Box 43, Ajloun 26810, JORDAN