International Journal of Maps in Mathematics

Volume 8, Issue 2, 2025, Pages:360-376

E-ISSN: 2636-7467

www.simadp.com/journalmim

CHARACTERIZATION OF $\mathcal{W}_6$-CURVATURE TENSOR ON LORENTZIAN PARA-KENMOTSU MANIFOLDS

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Abstract. The aim of this study is to explore the characteristics of n-dimensional Lorentzian para-Kenmotsu {briefly, $(LPK)_n$} manifolds with $\mathcal{W}_6$-curvature tensor. Firstly, we explore $(LPK)_n$ manifold with the condition ‘$\mathcal{W}_6(\mathtt{A},\mathtt{B},\mathtt{C},\zeta) = 0$ and find that it is an Einstein manifold. Next, we consider the conditions of Φ-$\mathcal{W}_6$-symmetric, $\mathcal{W}_6$-semisymmetric, and Φ-$\mathcal{W}_6$-flat on the $(LPK)_n$ manifold. Moreover, an example has been constrcuted to verify the results. Lastly, we explain the condition $\mathcal{W}_6(\mathtt{E},\mathtt{F}).\mathcal{R} = 0$ on $(LPK)_n$ manifold that establishes ω-Einstein manifold.

Keywords: Lorentzian para-Kenmotsu manifold, Scalar curvature, $\mathcal{W}_6$-curvature tensor, Einstein manifold.

2020 Mathematics Subject Classification: 53C25, 53C50.

1. Introduction

In 1989 [14], B.B. Sinha and K.L. Sai Prasad have defined para-Kenmotsu manifolds. They investigated the significant properties of para-Kenmotsu manifolds. Later on, para-Kenmotsu manifolds drew attention of several authors to study the characteristics of such manifolds. Lorentzian para-Kenmotsu manifolds were initiated in 2018 by A. Haseeb and R. Prasad [3]. R. Sari et al. have explained slant manifolds of a Lorentzian Kenmotsu manifold [11]. Mobin Ahmad studied semi-invariant $\zeta^{\perp}$-submanifolds of Lorentzian para-Sasakian manifolds in

Received: 2024.10.11 *Revised: 2024.12.18* *Accepted: 2025.01.23*

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2019 [1]. Moreover, Abhishek Singh et al., in 2024, explored some results on β-Kenmotsu manifolds with a non-symmetric non-metric connection [12, 13]. In 2022, Shashikant Pandey et al. have described certain results of Ricci-soliton on 3-dimensional Lorentzian para α-Sasakian manifolds [5]. For invariant submanifolds of Lorentzian para-Kenmotsu manifold to be totally geodesic, Atceken [2] gave the necessary and sufficient conditions.

G.P. Pokhariyal gave the concept of $\mathcal{W}_6$-curvature tensor with the support of Weyl curvature tensor in 1982 [6, 7], and is described as

$$\mathcal{W}_6(\mathtt{A},\mathtt{B})\mathtt{C} = \mathcal{R}(\mathtt{A},\mathtt{B})\mathtt{C} + \frac{1}{n-1}[g(\mathtt{A},\mathtt{B})Q\mathtt{C} - \mathcal{S}(\mathtt{B},\mathtt{C})\mathtt{A}], \tag{1.1}$$

and

$$`\mathcal{W}_6(\mathtt{A},\mathtt{B},\mathtt{C},\mathtt{T}) = \mathcal{K}(\mathtt{A},\mathtt{B},\mathtt{C},\mathtt{T}) + \frac{1}{n-1}[g(\mathtt{A},\mathtt{B})\mathcal{S}(\mathtt{C},\mathtt{T}) - g(\mathtt{A},\mathtt{T})\mathcal{S}(\mathtt{B},\mathtt{C}), \tag{1.2}$$

$\forall\ \mathtt{A},\mathtt{B},\mathtt{C},\mathtt{T} \in \chi(M^n)$, where, $\mathcal{K}(\mathtt{A},\mathtt{B},\mathtt{C},\mathtt{T}) = g(\mathcal{R}(\mathtt{A},\mathtt{B})\mathtt{C},\mathtt{T})$, $\mathcal{R}$ and Q denote Riemann curvature tensor and Ricci operator, respectively.

This article has been organized in the following manner: Section-1 contains introduction, where corresponding concepts and their brief histories are given. Section-2 covers preliminaries, containing some basic results, which have been used extensively throughout this manuscript. Section-3 describes the Lorentzian para-Kenmotsu manifold with the condition $`\mathcal{W}_6(\mathtt{A},\mathtt{B},\mathtt{C},\zeta) = 0$. Section-4 studies the nature of $\Phi^2((\nabla_{\mathtt{E}}\mathcal{W}_6)(\mathtt{A},\mathtt{B})\mathtt{C}) = 0$ on Lorentzian para-Kenmotsu manifold with the construction of an example. Section-5 and section-6 examine the behavior of $\mathcal{W}_6$-semisymmetric, and Φ-$\mathcal{W}_6$-flat on $(LPK)_n$ manifold, respectively. In section-7, we see that an $(LPK)_n$ manifold with the condition $\mathcal{W}_6(\mathtt{E},\mathtt{F}).\mathcal{R} = 0$ gives an ω-Einstein manifold.

2. Preliminaries

We assume that M^n is a Lorentzian metric manifold, meaning there by, it is equipped with an structure (Φ,ζ,ω,g), where Φ is a (1, 1)-type tensor field, ζ is a vector field, ω is a one-form on M^n, and g is a Lorentzian metric tensor holding the subsequent results [8, 9, 10]:

$$\left.\begin{array}{l} \Phi^2(\mathtt{A}) = \mathtt{A} + \omega(\mathtt{A})\zeta, \quad g(\mathtt{A},\zeta) = \omega(\mathtt{A}), \quad \omega(\zeta) = -1, \\ \Phi\zeta = 0, \quad \omega(\Phi\mathtt{A}) = 0, \quad g(\Phi\mathtt{A},\Phi\mathtt{B}) = g(\mathtt{A},\mathtt{B}) + \omega(\mathtt{A})\omega(\mathtt{B}), \end{array}\right\} \tag{2.3}$$

$\forall$ vector fields $\mathtt{A}$, $\mathtt{B}$ on M^n. Thus, $M^n(\Phi,\zeta,\omega,g)$ is called a Lorentzian almost paracontact manifold.

Definition 2.1. *A Lorentzian almost paracontact manifold $M^n(\Phi,\zeta,\omega,g)$ is called a Lorentzian para-Kenmotsu manifold if*

$$(\nabla_{\mathtt{A}}\Phi)\mathtt{B} = -g(\Phi\mathtt{A},\mathtt{B})\zeta - \omega(\mathtt{B})\Phi\mathtt{A},$$

$\forall$ vector fields $\mathtt{A}$, $\mathtt{B} \in \chi(M^n)$. Here, ∇ and $\chi(M^n)$ represent Levi-Civita connection, and a collection of differentiable vector fields on M^n, respectively.

We assume that $M^n(\Phi,\zeta,\omega,g)$ is an $(LPK)_n$ manifold. The succeeding results hold for $M^n(\Phi,\zeta,\omega,g)$:

$$\nabla_{\mathtt{A}}\zeta = -\mathtt{A} - \omega(\mathtt{A})\zeta, \qquad g(\Phi\mathtt{A},\mathtt{B}) = g(\mathtt{A},\Phi\mathtt{B}), \tag{2.4}$$

$$(\nabla_{\mathtt{A}}\omega)\mathtt{B} = -g(\mathtt{A},\mathtt{B}) - \omega(\mathtt{A})\omega(\mathtt{B}), \tag{2.5}$$

$$\omega(\mathcal{R}(\mathtt{A},\mathtt{B})\mathtt{C}) = \mathcal{K}(\mathtt{A},\mathtt{B},\mathtt{C},\zeta) = g(\mathtt{B},\mathtt{C})\omega(\mathtt{A}) - g(\mathtt{A},\mathtt{C})\omega(\mathtt{B}), \tag{2.6}$$

$$\left.\begin{aligned} \mathcal{R}(\mathtt{A},\zeta)\mathtt{B} &= \omega(\mathtt{B})\mathtt{A} - g(\mathtt{A},\mathtt{B})\zeta, \\ \mathcal{R}(\zeta,\mathtt{A})\zeta &= \mathtt{A} + \omega(\mathtt{A})\zeta, \\ \mathcal{R}(\mathtt{A},\mathtt{B})\zeta &= \omega(\mathtt{B})\mathtt{A} - \omega(\mathtt{A})\mathtt{B}, \end{aligned}\right\} \tag{2.7}$$

$$\mathcal{K}(\zeta,\mathtt{A},\mathtt{B},\mathtt{C}) = g(\mathtt{A},\mathtt{B})\ \omega(\mathtt{C}) - g(\mathtt{A},\mathtt{C})\omega(\mathtt{B}), \tag{2.8}$$

$$\left.\begin{aligned} \mathcal{S}(\mathtt{A},\zeta) = (n-1)\omega(\mathtt{A}),\ \ \mathcal{S}(\zeta,\zeta) &= -(n-1), \\ (\nabla_{\mathtt{E}}\mathcal{S})(\mathtt{C},\zeta) = \mathcal{S}(\mathtt{E},\mathtt{C}) &- (n-1)g(\mathtt{E},\mathtt{C}), \end{aligned}\right\} \tag{2.9}$$

$$div\mathcal{R}(\mathtt{A},\mathtt{B})\mathtt{C} = (\nabla_{\mathtt{A}}\mathcal{S})(\mathtt{B},\mathtt{C}) - (\nabla_{\mathtt{B}}\mathcal{S})(\mathtt{A},\mathtt{C}), \qquad . \tag{2.10}$$

here, $\mathcal{S}$ denotes Ricci tensor of $M^n(\Phi,\zeta,\omega,g)$.

Particularly, setting $\mathtt{A} = \zeta$, $\mathtt{B} = \zeta$, and $\mathtt{C} = \zeta$, respectively, in 1.1 on an $(LPK)_n$ manifold, it yields

$$\mathcal{W}_6(\zeta,\mathtt{B})\mathtt{C} = g(\mathtt{B},\mathtt{C})\zeta - \omega(\mathtt{C})\mathtt{B} + \frac{1}{n-1}[\omega(\mathtt{B})Q\mathtt{C} - \mathcal{S}(\mathtt{B},\mathtt{C})\zeta], \tag{2.11}$$

$$\mathcal{W}_6(\mathtt{A},\zeta)\mathtt{C} = -g(\mathtt{A},\mathtt{C})\zeta + \frac{1}{n-1}\omega(\mathtt{A})Q\mathtt{C}, \tag{2.12}$$

$$\mathcal{W}_6(\mathtt{A},\mathtt{B})\zeta = g(\mathtt{A},\mathtt{B})\zeta - \omega(\mathtt{A})\mathtt{B}. \tag{2.13}$$

Definition 2.2. *An $(LPK)_n$ manifold is called an ω-Einstein manifold if its Ricci tensor satisfies the following relation*

$$\mathcal{S}(\mathtt{A},\mathtt{B}) = \alpha g(\mathtt{A},\mathtt{B}) + \beta\omega(\mathtt{A})\omega(\mathtt{B}),$$

here, α, and β are scalar functions on M^n. In case of $\beta = 0$, manifold becomes Einstein manifold [4].

3. ${}'\mathcal{W}_6(\mathtt{A},\mathtt{B},\mathtt{C},\zeta)=0$ on a Lorentzian para-Kenmotsu Manifolds

In this part, we discuss the condition ${}'\mathcal{W}_6(\mathtt{A},\mathtt{B},\mathtt{C},\zeta)=0$ on $(LPK)_n$ manifolds M^n. We begin with the subsequent theorem:

Theorem 3.1. *An n-dimensional Lorentzian para-Kenmotsu manifold is an Einstein manifold if,* ${}'\mathcal{W}_6(\mathtt{A},\mathtt{B},\mathtt{C},\zeta)=0$.

Proof. ${}'\mathcal{W}_6$-curvature tensor is defined by 1.2

$$ {}'\mathcal{W}_6(\mathtt{A},\mathtt{B},\mathtt{C},\mathtt{T})=\mathcal{K}(\mathtt{A},\mathtt{B},\mathtt{C},\mathtt{T})+\frac{1}{n-1}[g(\mathtt{A},\mathtt{B})\mathcal{S}(\mathtt{C},\mathtt{T})-g(\mathtt{A},\mathtt{T})\mathcal{S}(\mathtt{B},\mathtt{C})]. $$

$\forall$ $\mathtt{A}$, $\mathtt{B}$, $\mathtt{C}$, $\mathtt{T}\in\chi(M^n)$.

Putting $\mathtt{T}=\zeta$ into the above equation, we have

$$ {}'\mathcal{W}_6(\mathtt{A},\mathtt{B},\mathtt{C},\zeta)=\mathcal{K}(\mathtt{A},\mathtt{B},\mathtt{C},\zeta)+\frac{1}{n-1}[g(\mathtt{A},\mathtt{B})\mathcal{S}(\mathtt{C},\zeta)-g(\mathtt{A},\zeta)\mathcal{S}(\mathtt{B},\mathtt{C})]. $$

Applying ${}'\mathcal{W}_6(\mathtt{A},\mathtt{B},\mathtt{C},\zeta)=0$ in the above relation, we get

$$ \mathcal{K}(\mathtt{A},\mathtt{B},\mathtt{C},\zeta)=\frac{1}{n-1}[g(\mathtt{A},\zeta)\mathcal{S}(\mathtt{B},\mathtt{C})-g(\mathtt{A},\mathtt{B})\mathcal{S}(\mathtt{C},\zeta)]. \tag{3.14} $$

Using 2.3, 2.6, and 2.9, the relation 3.14 yields

$$ \frac{1}{n-1}\mathcal{S}(\mathtt{B},\mathtt{C})\omega(\mathtt{A})=g(\mathtt{B},\mathtt{C})\omega(\mathtt{A})-g(\mathtt{A},\mathtt{C})\omega(\mathtt{B})+g(\mathtt{A},\mathtt{B})\omega(\mathtt{C}). \tag{3.15} $$

Applying $\mathtt{A}=\zeta$ into 3.15, it yields

$$ \frac{1}{n-1}\mathcal{S}(\mathtt{B},\mathtt{C})\omega(\zeta)=g(\mathtt{B},\mathtt{C})\omega(\zeta)-g(\zeta,\mathtt{C})\omega(\mathtt{B})+g(\zeta,\mathtt{B})\omega(\mathtt{C}). \tag{3.16} $$

Using 2.3, on simplification, the relation 3.16 provides

$$ \mathcal{S}(\mathtt{B},\mathtt{C})=(n-1)g(\mathtt{B},\mathtt{C}). \tag{3.17} $$

This completes the proof. □

4. nature of Φ-$\mathcal{W}_6$-symmetric on $(LPK)_n$ manifolds

We begin this part with the definition of Φ-$\mathcal{W}_6$-symmetric Lorentzian para-Kenmotsu manifold:

Definition 4.1. *A Lorentzian para-Kenmotsu manifold is said to be a Φ-$\mathcal{W}_6$-symmetric Lorentzian para-Kenmotsu manifold, if it satisfies the relation*

$$ \Phi^2((\nabla_{\mathtt{E}}\mathcal{W}_6)(\mathtt{A},\mathtt{B})\mathtt{C})=0, $$

for every $\mathtt{A},\mathtt{B},\mathtt{C},\mathtt{E}$ *on* M^n.

Theorem 4.1. *A Φ-W_6-symmetric Lorentzian para-Kenmotsu manifold is an Einstein manifold.*

Proof. Covariant differentiation of relation 1.1 along $\mathtt{E}$ yields

$$(\nabla_{\mathtt{E}}\mathcal{W}_6)(\mathtt{A},\mathtt{B})\mathtt{C} = (\nabla_{\mathtt{E}}\mathcal{R})(\mathtt{A},\mathtt{B})\mathtt{C} - \frac{1}{n-1}[(\nabla_{\mathtt{E}}\mathcal{S})(\mathtt{B},\mathtt{C})\mathtt{A} - g(\mathtt{A},\mathtt{B})(\nabla_{\mathtt{E}}Q)\mathtt{C}]. \tag{4.18}$$

Operating Φ^2 on both sides of the equation 4.18 and using 2.3, it gives

$$\begin{aligned}\Phi^2((\nabla_{\mathtt{E}}\mathcal{W}_6)(\mathtt{A},\mathtt{B})\mathtt{C}) = (\nabla_{\mathtt{E}}\mathcal{R})(\mathtt{A},\mathtt{B})\mathtt{C} + \omega((\nabla_{\mathtt{E}}\mathcal{R})(\mathtt{A},\mathtt{B})\mathtt{C})\zeta \\ - \frac{1}{n-1}[(\nabla_{\mathtt{E}}\mathcal{S})(\mathtt{B},\mathtt{C})\mathtt{A} + (\nabla_{\mathtt{E}}\mathcal{S})(\mathtt{B},\mathtt{C})\omega(\mathtt{A})\zeta \\ - g(\mathtt{A},\mathtt{B})(\nabla_{\mathtt{E}}Q)\mathtt{C} - g(\mathtt{A},\mathtt{B})(\nabla_{\mathtt{E}}\mathcal{S})(\mathtt{C},\zeta)\zeta].\end{aligned} \tag{4.19}$$

Using 2.9 with condition $\Phi^2((\nabla_{\mathtt{E}}\mathcal{W}_6)(\mathtt{A},\mathtt{B})\mathtt{C}) = 0$ into the relation 4.19, it yields

$$\begin{aligned}0 = (\nabla_{\mathtt{E}}\mathcal{R})(\mathtt{A},\mathtt{B})\mathtt{C} + \omega((\nabla_{\mathtt{E}}\mathcal{R})(\mathtt{A},\mathtt{B})\mathtt{C})\zeta \\ - \frac{1}{n-1}[(\nabla_{\mathtt{E}}\mathcal{S})(\mathtt{B},\mathtt{C})\mathtt{A} + (\nabla_{\mathtt{E}}\mathcal{S})(\mathtt{B},\mathtt{C})\omega(\mathtt{A})\zeta \\ - g(\mathtt{A},\mathtt{B})(\nabla_{\mathtt{E}}Q)\mathtt{C} - g(\mathtt{A},\mathtt{B})\mathcal{S}(\mathtt{E},\mathtt{C})\zeta + (n-1)g(\mathtt{A}.\mathtt{B})g(\mathtt{E},\mathtt{C})\zeta].\end{aligned} \tag{4.20}$$

Differentiating covariantly 2.6 along $\mathtt{E}$, it gives

$$\begin{aligned}&(\nabla_{\mathtt{E}}g)(\mathcal{R}(\mathtt{A},\mathtt{B})\mathtt{C},\zeta) + g(\nabla_{\mathtt{E}}\mathcal{R}(\mathtt{A},\mathtt{B})\mathtt{C},\zeta) \\ &\quad + g(\mathcal{R}(\nabla_{\mathtt{E}}\mathtt{A},\mathtt{B})\mathtt{C},\zeta) + g(\mathcal{R}(\mathtt{A},\nabla_{\mathtt{E}}\mathtt{B})\mathtt{C},\zeta) \\ &\quad + g(\mathcal{R}(\mathtt{A},\mathtt{B})\nabla_{\mathtt{E}}\mathtt{C},\zeta) + g(\mathcal{R}(\mathtt{A},\mathtt{B})\mathtt{C},\nabla_{\mathtt{E}}\zeta) \\ &= (\nabla_{\mathtt{E}}g)(\mathtt{B},\mathtt{C})\,g(\mathtt{A},\zeta) + g(\nabla_{\mathtt{E}}\mathtt{B},\mathtt{C})\,g(\mathtt{A},\zeta) \\ &\quad + g(\mathtt{B},\nabla_{\mathtt{E}}\mathtt{C})\,g(\mathtt{A},\zeta) + g(\mathtt{B},\mathtt{C})\,(\nabla_{\mathtt{E}}g)(\mathtt{A},\zeta) \\ &\quad + g(\mathtt{B},\mathtt{C})\,g(\nabla_{\mathtt{E}}\mathtt{A},\zeta) + g(\mathtt{B},\mathtt{C})\,g(\mathtt{A},\nabla_{\mathtt{E}}\zeta) \\ &\quad - (\nabla_{\mathtt{E}}g)(\mathtt{A},\mathtt{C})\,g(\mathtt{B},\zeta) - g(\nabla_{\mathtt{E}}\mathtt{A},\mathtt{C})\,g(\mathtt{B},\zeta) \\ &\quad - g(\mathtt{A},\nabla_{\mathtt{E}}\mathtt{C})\,g(\mathtt{B},\zeta) - g(\mathtt{A},\mathtt{C})\,(\nabla_{\mathtt{E}}g)(\mathtt{B},\zeta) \\ &\quad - g(\mathtt{A},\mathtt{C})\,g(\nabla_{\mathtt{E}}\mathtt{B},\zeta) - g(\mathtt{A},\mathtt{C})\,g(\mathtt{B},\nabla_{\mathtt{E}}\zeta).\end{aligned} \tag{4.21}$$

Applying 2.3 and 2.4 into 4.21, it gives

$$\omega((\nabla_{\mathtt{E}}\mathcal{R}(\mathtt{A},\mathtt{B})\mathtt{C}) = g(\mathcal{R}(\mathtt{A},\mathtt{B})\mathtt{C},\mathtt{E}) + g(\mathtt{A},\mathtt{C})g(\mathtt{B},\mathtt{E}) - g(\mathtt{B},\mathtt{C})g(\mathtt{A},\mathtt{E}). \tag{4.22}$$

Relations 4.20 and 4.22, provide

$$
\begin{aligned}
&(\nabla_{\mathtt{E}}\mathcal{R})(\mathtt{A},\mathtt{B})\mathtt{C}+\mathcal{K}(\mathtt{A},\mathtt{B},\mathtt{C},\mathtt{E})\zeta+g(\mathtt{A},\mathtt{C})\,g(\mathtt{B},\mathtt{E})\,\zeta-g(\mathtt{B},\mathtt{C})\,g(\mathtt{A},\mathtt{E})\,\zeta\\
&-\frac{1}{n-1}\Big[(\nabla_{\mathtt{E}}\mathcal{S})(\mathtt{B},\mathtt{C})\mathtt{A}+(\nabla_{\mathtt{E}}\mathcal{S})(\mathtt{B},\mathtt{C})\,\omega(\mathtt{A})\,\zeta-g(\mathtt{A},\mathtt{B})(\nabla_{\mathtt{E}}Q)\mathtt{C}\\
&-g(\mathtt{A},\mathtt{B})\,\mathcal{S}(\mathtt{E},\mathtt{C})\,\zeta+(n-1)\,g(\mathtt{A},\mathtt{B})\,g(\mathtt{E},\mathtt{C})\,\zeta\Big]=0.
\end{aligned}
\tag{4.23}
$$

Innerproduct of 4.23 along $\mathtt{F}$ is given by

$$
\begin{aligned}
&g((\nabla_{\mathtt{E}}\mathcal{R})(\mathtt{A},\mathtt{B})\mathtt{C},\mathtt{F})+\mathcal{K}(\mathtt{A},\mathtt{B},\mathtt{C},\mathtt{E})\omega(\mathtt{F})+g(\mathtt{A},\mathtt{C})g(\mathtt{B},\mathtt{E})\omega(\mathtt{F})-g(\mathtt{B},\mathtt{C})g(\mathtt{A},\mathtt{E})\omega(\mathtt{F})\\
&-\frac{1}{n-1}\Big[(\nabla_{\mathtt{E}}\mathcal{S})(\mathtt{B},\mathtt{C})g(\mathtt{A},\mathtt{F})+(\nabla_{\mathtt{E}}\mathcal{S})(\mathtt{B},\mathtt{C})\omega(\mathtt{A})\omega(\mathtt{F})\\
&-g(\mathtt{A},\mathtt{B})g((\nabla_{\mathtt{E}}Q)\mathtt{C},\mathtt{F})-g(\mathtt{A},\mathtt{B})\mathcal{S}(\mathtt{E},\mathtt{C})\omega(\mathtt{F})+(n-1)g(\mathtt{A},\mathtt{B})g(\mathtt{E},\mathtt{C})\omega(\mathtt{F})\Big]=0.
\end{aligned}
\tag{4.24}
$$

Contracting 4.24 along $\mathtt{E}$ and $\mathtt{F}$, we have

$$
\begin{aligned}
&\sum_{i=1}^{n}\epsilon_i g((\nabla_{\mathcal{E}_i}\mathcal{R})(\mathtt{A},\mathtt{B})\mathtt{C},\mathcal{E}_i)+\sum_{i=1}^{n}\epsilon_i\mathcal{K}(\mathtt{A},\mathtt{B},\mathtt{C},\mathcal{E}_i)g(\zeta,\mathcal{E}_i)\\
&+\sum_{i=1}^{n}\epsilon_i g(\mathtt{A},\mathtt{C})g(\mathtt{B},\mathcal{E}_i)g(\zeta,\mathcal{E}_i)-\sum_{i=1}^{n}\epsilon_i g(\mathtt{B},\mathtt{C})g(\mathtt{A},\mathcal{E}_i)g(\zeta,\mathcal{E}_i)\\
&-\frac{1}{n-1}\sum_{i=1}^{n}\epsilon_i\Big[(\nabla_{\mathcal{E}_i}\mathcal{S})(\mathtt{B},\mathtt{C})g(\mathtt{A},\mathcal{E}_i)+(\nabla_{\mathcal{E}_i}\mathcal{S})(\mathtt{B},\mathtt{C})\omega(\mathtt{A})g(\zeta,\mathcal{E}_i)\\
&-g(\mathtt{A},\mathtt{B})g((\nabla_{\mathcal{E}_i}Q)\mathtt{C},\mathcal{E}_i)-g(\mathtt{A},\mathtt{B})\mathcal{S}(\mathcal{E}_i,\mathtt{C})g(\zeta,\mathcal{E}_i)\\
&+(n-1)g(\mathtt{A},\mathtt{B})g(\mathcal{E}_i,\mathtt{C})g(\zeta,\mathcal{E}_i)\Big]=0.
\end{aligned}
$$

where, $\epsilon_i=g(\mathcal{E}_i,\mathcal{E}_i)$ and $\{\mathcal{E}_1,\mathcal{E}_2....\mathcal{E}_{n-1},\zeta\}$ are orthonormal base field on $(LPK)_n$ manifold. Using relations 2.3, 2.6, 2.8, and 2.9 into the above relation, it gives

$$
\begin{aligned}
&(div\mathcal{R})(\mathtt{A},\mathtt{B})\mathtt{C}+\mathcal{K}(\mathtt{A},\mathtt{B},\mathtt{C},\zeta)+g(\mathtt{A},\mathtt{C})\omega(\mathtt{B})-g(\mathtt{B},\mathtt{C})\omega(\mathtt{A})\\
&-\frac{1}{n-1}\Big[(\nabla_{\mathtt{A}}\mathcal{S})(\mathtt{B},\mathtt{C})+(\nabla_{\zeta}\mathcal{S})(\mathtt{B},\mathtt{C})\omega(\mathtt{A})\\
&-g(\mathtt{A},\mathtt{B})\frac{\mathtt{C}(r)}{2}-g(\mathtt{A},\mathtt{B})\mathcal{S}(\mathtt{C},\zeta)+(n-1)g(\mathtt{A},\mathtt{B})g(\mathtt{C},\zeta)\Big]=0.
\end{aligned}
\tag{4.25}
$$

where, $divQ(\mathtt{C}){=}\frac{\mathtt{C}(r)}{2}$.

Putting the value from 2.6, and 2.10 into 4.25, we have

$$(\nabla_{\mathtt{A}}\mathcal{S})(\mathtt{B},\mathtt{C})-(\nabla_{\mathtt{B}}\mathcal{S})(\mathtt{A},\mathtt{C})+g(\mathtt{B},\mathtt{C})\omega(\mathtt{A})-g(\mathtt{A},\mathtt{C})\omega(\mathtt{B})$$
$$+g(\mathtt{A},\mathtt{C})\omega(\mathtt{B})-g(\mathtt{B},\mathtt{C})\omega(\mathtt{A})-\frac{1}{n-1}\Big[(\nabla_{\mathtt{A}}\mathcal{S})(\mathtt{B},\mathtt{C})$$
$$+(\nabla_{\zeta}\mathcal{S})(\mathtt{B},\mathtt{C})\omega(\mathtt{A})-\frac{\mathtt{C}(r)}{2}g(\mathtt{A},\mathtt{B})\Big]=0. \tag{4.26}$$

Putting $\mathtt{C}=\zeta$ into 4.26, we get

$$\frac{(n-2)}{(n-1)}(\nabla_{\mathtt{A}}\mathcal{S})(\mathtt{B},\zeta)\ -\ (\nabla_{\mathtt{B}}\mathcal{S})(\mathtt{A},\zeta)\ -\ \frac{1}{n-1}(\nabla_{\zeta}\mathcal{S})(\mathtt{B},\zeta)\omega(\mathtt{A})\ +\ \frac{1}{2(n-1)}\zeta(r)g(\mathtt{A},\mathtt{B})\ =\ 0. \tag{4.27}$$

Using the relation 2.9 into 4.27, it gives

$$\frac{(n-2)}{(n-1)}\big[\mathcal{S}(\mathtt{A},\mathtt{B})-(n-1)g(\mathtt{A},\mathtt{B})\big]-\big[\mathcal{S}(\mathtt{A},\mathtt{B})-(n-1)g(\mathtt{A},\mathtt{B})\big]$$
$$-\frac{1}{n-1}\big[\mathcal{S}(\mathtt{B},\zeta)-(n-1)\omega(\mathtt{B})\big]\omega(\mathtt{A})+\frac{1}{2(n-1)}\zeta(r)g(\mathtt{A},\mathtt{B})=0. \tag{4.28}$$

After simplification, 4.28 yields

$$\mathcal{S}(\mathtt{A},\mathtt{B})=[(n-1)+\frac{\zeta(r)}{2}]g(\mathtt{A},\mathtt{B})]. \tag{4.29}$$

Further, contracting 4.24 along $\mathtt{A}$ and $\mathtt{F}$ and using 2.8, we have

$$(\nabla_{\mathtt{E}}\mathcal{S})(\mathtt{C},\mathtt{B})=-\mathcal{S}(\mathtt{E},\mathtt{C})\omega(\mathtt{B})+(n-1)g(\mathtt{E},\mathtt{C})\omega(\mathtt{B}). \tag{4.30}$$

Again, contracting the above equation along $\mathtt{B}$ and $\mathtt{C}$, we have

$$(\nabla_{\mathtt{E}}r)=-S(\mathtt{E},\zeta)+(n-1)\omega(\mathtt{E}). \tag{4.31}$$

Using 2.9, it yields that scalar curvature r is constant. Therefore, 4.29 concludes the following:

$$S(\mathtt{A},\mathtt{B})=(n-1)g(\mathtt{A},\mathtt{B}). \tag{4.32}$$

Hence, we establish that $\mathit{\Phi}$-$\mathcal{W}_6$-symmetric $(LPK)_n$ manifold is an Einstein manifold.

□

We consider an $(LPK)_n$ manifold of constant curvature, then

$$\mathcal{R}(\mathtt{A},\mathtt{B})\mathtt{C}=k[g(\mathtt{B},\mathtt{C})\mathtt{A}-g(\mathtt{A},\mathtt{C})\mathtt{B}], \tag{4.33}$$

where, k is constant.

The relations 1.1 and 4.33, taken together, we have

$$\mathcal{W}_6(\mathtt{A},\mathtt{B})\mathtt{C} = k[g(\mathtt{A},\mathtt{B})\mathtt{C} - g(\mathtt{A},\mathtt{C})\mathtt{B}]. \tag{4.34}$$

Differentiating covariantly the relation 4.34 along $\mathtt{E}$ and operating Φ^2 on both sides, it yields

$$\Phi^2((\nabla_{\mathtt{E}}\mathcal{W}_6)(\mathtt{A},\mathtt{B})\mathtt{C}) = 0. \tag{4.35}$$

This establishes the subsequent corollary:

Corollary 4.1. *The $(LPK)_n$ manifolds of constant curvature are Φ-$\mathcal{W}_6$-symmetric $(LPK)_n$ manifolds.*

Example 4.1. *Consider a differentiable manifold $M^4 = \{(u,v,w,t) \in \Re^4$: u, v, w is non zero, t>0$\}$. Suppose that $\{\mathcal{E}_1,\mathcal{E}_2,\mathcal{E}_3,\mathcal{E}_4\}$ are linearly independent vectors at every point of M^4. We define,*

$$\mathcal{E}_1 = e^{u+t}\frac{\partial}{\partial u}, \qquad \mathcal{E}_2 = e^{v+t}\frac{\partial}{\partial v}, \qquad \mathcal{E}_3 = e^{w+t}\frac{\partial}{\partial w}, \qquad \mathcal{E}_4 = \frac{\partial}{\partial t}.$$

Lorentzian metric g on M^4 is established in the following way:

$$g_{ij} = g(\mathcal{E}_i,\mathcal{E}_j) = \begin{cases} 0 & \textit{if } i \neq j \\ -1 & \textit{if } i = j = 4 \\ 1 & \textit{or else.} \end{cases}$$

Assuming ω is one-form corresponding to g is defined by

$$\omega(\mathtt{A}) = g(\mathtt{A},\mathcal{E}_4),$$

$\forall$ $\mathtt{A} \in \chi(M^4)$, *here $\chi(M^4)$ be collection of vector fields on M^4. We define Φ as $(1,1)$-tensor field as follows:*

$$\Phi(\mathcal{E}_1) = \mathcal{E}_1, \qquad \Phi(\mathcal{E}_2) = \mathcal{E}_2, \qquad \Phi(\mathcal{E}_3) = \mathcal{E}_3, \qquad \Phi(\mathcal{E}_4) = 0.$$

From linear characteristic of Φ and g, the following results can easily be proved:

$$\omega(\mathcal{E}_4) = -1, \qquad \Phi^2(\mathtt{A}) = \mathtt{A} + \omega(\mathtt{A})\mathcal{E}_4, \qquad g(\Phi\mathtt{A},\Phi\mathtt{B}) = g(\mathtt{A},\mathtt{B}) + \omega(\mathtt{A})\omega(\mathtt{B}),$$

$\forall$ $\mathtt{A},\mathtt{B} \in \chi(M^4)$. *So, when $\mathcal{E}_4 = \zeta$, structure (Φ,ζ,ω,g) leading to Lorentzian paracontact structure as well as manifold M equipped with Lorentzian paracontact structure is said to be Lorentzian paracontact manifold of dimension-4.*

We represent $[\mathtt{A},\mathtt{B}]$ *as Lie-derivative of* $\mathtt{A}$, $\mathtt{B}$, *defined as* $[\mathtt{A},\mathtt{B}] = \mathtt{AB} - \mathtt{BA}$. *The non-zero constituents of Lie bracket are evaluated as below:*

$$[\mathcal{E}_1,\mathcal{E}_4] = -\mathcal{E}_1, \qquad [\mathcal{E}_2,\mathcal{E}_4] = -\mathcal{E}_2, \qquad [\mathcal{E}_3,\mathcal{E}_4] = -\mathcal{E}_3.$$

Let Riemannian connection with respect to g *be denoted by* ∇. *So, when* $\mathcal{E}_4 = \zeta$, *we have the subsequent results:*

$$\begin{array}{llll}
\nabla_{\mathcal{E}_1}\mathcal{E}_1 = -\mathcal{E}_4, & \nabla_{\mathcal{E}_1}\mathcal{E}_2 = 0, & \nabla_{\mathcal{E}_1}\mathcal{E}_3 = 0, & \nabla_{\mathcal{E}_1}\mathcal{E}_4 = -\mathcal{E}_1,\\
\nabla_{\mathcal{E}_2}\mathcal{E}_1 = 0, & \nabla_{\mathcal{E}_2}\mathcal{E}_2 = -\mathcal{E}_4, & \nabla_{\mathcal{E}_2}\mathcal{E}_3 = 0, & \nabla_{\mathcal{E}_2}\mathcal{E}_4 = -\mathcal{E}_2,\\
\nabla_{\mathcal{E}_3}\mathcal{E}_1 = 0, & \nabla_{\mathcal{E}_3}\mathcal{E}_2 = 0, & \nabla_{\mathcal{E}_3}\mathcal{E}_3 = -\mathcal{E}_4, & \nabla_{\mathcal{E}_3}\mathcal{E}_4 = -\mathcal{E}_3,\\
\nabla_{\mathcal{E}_4}\mathcal{E}_1 = 0, & \nabla_{\mathcal{E}_4}\mathcal{E}_2 = 0, & \nabla_{\mathcal{E}_4}\mathcal{E}_3 = 0, & \nabla_{\mathcal{E}_4}\mathcal{E}_4 = 0.
\end{array}$$

Assuming $\mathtt{A} \in \chi(M^4)$, *so* $\mathtt{A} = a_1\mathcal{E}_1 + a_2\mathcal{E}_2 + a_3\mathcal{E}_3 + a_4\mathcal{E}_4$, *here* $\{\mathcal{E}_1,\mathcal{E}_2,\mathcal{E}_3,\mathcal{E}_4\}$ *be the basis of* $\chi(M^4)$. *Above relations help verify* $\nabla_{\mathtt{A}}\mathcal{E}_4 = -\mathtt{A} - \omega(\mathtt{A})\mathcal{E}_4$ *for each* $\mathtt{A} \in \chi(M^4)$. *Hence,* M^4 *is a Lorentzian para-Kenmotsu manifold of dimension-4. From the above relations, the non-vanishing constituents of the curvature tensor are evaluated as subsequently,*

$$\begin{array}{lll}
\mathcal{R}(\mathcal{E}_1,\mathcal{E}_2)\mathcal{E}_1 = -\mathcal{E}_2, & \mathcal{R}(\mathcal{E}_1,\mathcal{E}_3)\mathcal{E}_1 = -\mathcal{E}_3, & \mathcal{R}(\mathcal{E}_1,\mathcal{E}_4)\mathcal{E}_1 = -\mathcal{E}_4,\\
\mathcal{R}(\mathcal{E}_1,\mathcal{E}_2)\mathcal{E}_2 = \mathcal{E}_1, & \mathcal{R}(\mathcal{E}_2,\mathcal{E}_3)\mathcal{E}_2 = -\mathcal{E}_3, & \mathcal{R}(\mathcal{E}_2,\mathcal{E}_4)\mathcal{E}_2 = -\mathcal{E}_4,\\
\mathcal{R}(\mathcal{E}_1,\mathcal{E}_3)\mathcal{E}_3 = \mathcal{E}_1, & \mathcal{R}(\mathcal{E}_2,\mathcal{E}_3)\mathcal{E}_3 = \mathcal{E}_2, & \mathcal{R}(\mathcal{E}_3,\mathcal{E}_4)\mathcal{E}_3 = -\mathcal{E}_4,\\
\mathcal{R}(\mathcal{E}_1,\mathcal{E}_4)\mathcal{E}_4 = -\mathcal{E}_1, & \mathcal{R}(\mathcal{E}_2,\mathcal{E}_4)\mathcal{E}_4 = -\mathcal{E}_2, & \mathcal{R}(\mathcal{E}_3,\mathcal{E}_4)\mathcal{E}_4 = -\mathcal{E}_3.
\end{array}$$

It can easily be seen that $\mathcal{R}(\mathtt{A},\mathtt{B})\mathtt{C} = g(\mathtt{B},\mathtt{C})\mathtt{A} - g(\mathtt{A},\mathtt{C})\mathtt{B}$.
From definition of Ricci tensor $\mathcal{S}$ *on* M^4, *the subsequent result holds,*

$$\mathcal{S}(\mathtt{A},\mathtt{B}) = \Sigma_{i=1}^4 \varepsilon_i g(\mathcal{R}(\mathcal{E}_i,\mathtt{A})\mathtt{B},\mathcal{E}_i), \qquad \varepsilon_i = g(\mathcal{E}_i,\mathcal{E}_i).$$

Therefore, matrix representation of $\mathcal{S}$ *is mentioned by*

$$\mathcal{S} = \begin{bmatrix} 3 & 0 & 0 & 0\\ 0 & 3 & 0 & 0\\ 0 & 0 & 3 & 0\\ 0 & 0 & 0 & -3 \end{bmatrix}.$$

This gives, $\mathcal{S}(\mathtt{A},\mathtt{B}) = 3g(\mathtt{A},\mathtt{B})$ *and scalar curvature* $\kappa = \Sigma_{i=1}^4 \varepsilon_i \mathcal{S}(\mathcal{E}_i,\mathcal{E}_i) = 12$, *this implies that* $(LPK)_4$ *manifold has constant scalar curvature. Hence, relation* $\Phi^2((\nabla_{\mathtt{E}}\mathcal{W}_6)(\mathtt{A},\mathtt{B})\mathtt{C}) = 0$ *holds.*

Thus, the above example verifies the results of this section.

5. $\mathcal{W}_6$-semisymmetric Lorentzian para-Kenmotsu Manifolds

This part covers the behavior of $\mathcal{W}_6$, when $\mathcal{R}(\mathtt{A},\ \mathtt{B})$ operates on it in $(LPK)_n$ manifold. Now, we have the following theorem:

Theorem 5.1. *Let $(M^n,\ g)$ be an $(LPK)_n$ manifold. If $\mathcal{R}(\mathtt{A},\mathtt{B}).\mathcal{W}_6 = 0$. Then, M^n is an Einstein manifold, where $\mathcal{R}(\mathtt{A},\mathtt{B})$ is a Riemannian operator, and $\mathcal{W}_6$ is a curvature tensor.*

Proof. We assume that M^n is an $(LPK)_n$ manifold satisfying subsequent condition:

$$(\mathcal{R}(\mathtt{A},\mathtt{B}).\mathcal{W}_6)(\mathtt{E},\mathtt{F})\mathtt{T} = 0. \tag{5.36}$$

From relation 5.36, we have

$$\mathcal{R}(\mathtt{A},\mathtt{B}).\mathcal{W}_6(\mathtt{E},\mathtt{F})\mathtt{T} \ = \ \mathcal{W}_6(\mathcal{R}(\mathtt{A},\mathtt{B})\mathtt{E},\mathtt{F})\mathtt{T} + \mathcal{W}_6(\mathtt{E},\mathcal{R}(\mathtt{A},\mathtt{B})\mathtt{F})\mathtt{T} + \mathcal{W}_6(\mathtt{E},\mathtt{F})(\mathcal{R}(\mathtt{A},\mathtt{B})\mathtt{T}). \tag{5.37}$$

Taking innerproduct of 5.37 along $\mathtt{C}$, we have

$$\mathcal{K}(\mathtt{A},\mathtt{B},\mathcal{W}_6(\mathtt{E},\mathtt{F})\mathtt{T},\mathtt{C}) = {}^{\text{‘}}\mathcal{W}_6(\mathcal{R}(\mathtt{A},\mathtt{B})\mathtt{E},\mathtt{F},\mathtt{T},\mathtt{C}) + {}^{\text{‘}}\mathcal{W}_6(\mathtt{E},\mathcal{R}(\mathtt{A},\mathtt{B})\mathtt{F},\mathtt{T},\mathtt{C}) + {}^{\text{‘}}\mathcal{W}_6(\mathtt{E},\mathtt{F},\mathcal{R}(\mathtt{A},\mathtt{B})\mathtt{T},\mathtt{C}). \tag{5.38}$$

Applying $\mathtt{A} = \mathtt{C} = \zeta$ into 5.38, it provides

$$\mathcal{K}(\zeta,\mathtt{B},\mathcal{W}_6(\mathtt{E},\mathtt{F})\mathtt{T},\zeta) = {}^{\text{‘}}\mathcal{W}_6(\mathcal{R}(\zeta,\mathtt{B})\mathtt{E},\mathtt{F},\mathtt{T},\zeta) + {}^{\text{‘}}\mathcal{W}_6(\mathtt{E},\mathcal{R}(\zeta,\mathtt{B})\mathtt{F},\mathtt{T},\zeta) + {}^{\text{‘}}\mathcal{W}_6(\mathtt{E},\mathtt{F},\mathcal{R}(\zeta,\mathtt{B})\mathtt{T},\zeta). \tag{5.39}$$

Evaluation of left hand side of 5.39 with relation 2.6, it yields

$$\begin{aligned}
\mathcal{K}(\zeta,\mathtt{B},\mathcal{W}_6(\mathtt{E},\mathtt{F})\mathtt{T},\zeta) = & -\mathcal{K}(\mathtt{E},\mathtt{F},\mathtt{T},\mathtt{B}) \\
& - \frac{1}{n-1}[g(\mathtt{E},\mathtt{F})\mathcal{S}(\mathtt{T},\mathtt{B}) - g(\mathtt{E},\mathtt{B})\mathcal{S}(\mathtt{F},\mathtt{T})] \\
& - \omega(\mathtt{B})\omega(\mathtt{T})g(\mathtt{E},\mathtt{F}) - \omega(\mathtt{E})\omega(\mathtt{B})g(\mathtt{F},\mathtt{T}) \\
& + g(\mathtt{E},\mathtt{T})\omega(\mathtt{F})\omega(\mathtt{B}) + \frac{1}{n-1}\omega(\mathtt{E})\omega(\mathtt{B})\mathcal{S}(\mathtt{F},\mathtt{T}).
\end{aligned} \tag{5.40}$$

Evaluation of first term of right hand side of 5.39 with the relation 2.6 in the following way:

$${}^{\text{‘}}\mathcal{W}_6(\mathcal{R}(\zeta,\mathtt{B})\mathtt{E},\mathtt{F},\mathtt{T},\zeta) = g(\mathtt{B},\mathtt{E})\,{}^{\text{‘}}\mathcal{W}_6(\zeta,\mathtt{F},\mathtt{T},\zeta) - \omega(\mathtt{E})\,{}^{\text{‘}}\mathcal{W}_6(\mathtt{B},\mathtt{F},\mathtt{T},\zeta).$$

Applying the definition of $\mathcal{W}_6$-curvature tensor, the above relation becomes

$$\begin{aligned}\mathcal{W}_6(\mathcal{R}(\zeta,\mathtt{B})\mathtt{E},\mathtt{F},\mathtt{T},\zeta) = &-g(\mathtt{B},\mathtt{E})g(\mathtt{F},\mathtt{T}) - g(\mathtt{B},\mathtt{E})\omega(\mathtt{F})\omega(\mathtt{T})\\ &+ g(\mathtt{B},\mathtt{E})\omega(\mathtt{F})\omega(\mathtt{T}) + \frac{1}{n-1}g(\mathtt{B},\mathtt{E})\mathcal{S}(\mathtt{F},\mathtt{T}) - \omega(\mathtt{E})\omega(\mathtt{B})g(\mathtt{F},\mathtt{T})\\ &+ g(\mathtt{B},\mathtt{T})\omega(\mathtt{E})\omega(\mathtt{F}) - g(\mathtt{B},\mathtt{F})\omega(\mathtt{E})\omega(\mathtt{T}) + \frac{1}{n-1}\omega(\mathtt{E})\omega(\mathtt{B})\mathcal{S}(\mathtt{F},\mathtt{T}). \quad (5.41)\end{aligned}$$

Evaluation of middle term of right hand side of 5.39 with 2.6 into the following way:

$$`\mathcal{W}_6(\mathtt{E},\mathcal{R}(\zeta,\mathtt{B})\mathtt{F},\mathtt{T},\zeta) = `\mathcal{W}_6(\mathtt{E},\zeta,\mathtt{T},\zeta)g(\mathtt{B},\mathtt{F}) - \omega(\mathtt{F})`\mathcal{W}_6(\mathtt{E},\mathtt{B},\mathtt{T},\zeta).$$

Now, from the definition of $\mathcal{W}_6$-curvature tensor, the above relation becomes

$$\begin{aligned}`\mathcal{W}_6(\mathtt{E},\mathcal{R}(\zeta,\mathtt{B})\mathtt{F},\mathtt{T},\zeta) = &\, g(\mathtt{B},\mathtt{F})g(\mathtt{E},\mathtt{T})\\ &+ g(\mathtt{B},\mathtt{F})\omega(\mathtt{E})\omega(\mathtt{T}) - \omega(\mathtt{E})\omega(\mathtt{F})g(\mathtt{B},\mathtt{T}) + g(\mathtt{E},\mathtt{T})\omega(\mathtt{E})\omega(\mathtt{B})\\ &- g(\mathtt{E},\mathtt{B})\omega(\mathtt{F})\omega(\mathtt{T}) + \frac{1}{n-1}\omega(\mathtt{E})\omega(\mathtt{F})\mathcal{S}(\mathtt{B},\mathtt{T}). \quad (5.42)\end{aligned}$$

Evaluation of the last term of 5.39 into the following way:

In view of relation 2.7, the last term of 5.39 becomes

$$`\mathcal{W}_6(\mathtt{E},\mathtt{F},\mathcal{R}(\zeta,\mathtt{B})\mathtt{T},\zeta) = g(\mathtt{B},\mathtt{T})`\mathcal{W}_6(\mathtt{E},\mathtt{F},\zeta,\zeta) - \omega(\mathtt{T})`\mathcal{W}_6(\mathtt{E},\mathtt{F},\mathtt{B},\zeta),$$

Using the definition 1.2 with relation 2.7 and 2.9 into the above relation, we have

$$\begin{aligned}`\mathcal{W}_6(\mathtt{E},\mathtt{F},\mathcal{R}(\zeta,\mathtt{B})\mathtt{T},\zeta) = &-g(\mathtt{B},\mathtt{T})g(\mathtt{E},\mathtt{F}) - \omega(\mathtt{E})\omega(\mathtt{F})g(\mathtt{B},\mathtt{T})\\ &- \omega(\mathtt{T})\omega(\mathtt{E})g(\mathtt{F},\mathtt{B}) + g(\mathtt{E}.\mathtt{B})\omega(\mathtt{F})\omega(\mathtt{T})\\ &- \omega(\mathtt{T})\omega(\mathtt{E})g(\mathtt{F},\mathtt{B}) + \frac{1}{n-1}\omega(\mathtt{E})\omega(\mathtt{T})\mathcal{S}(\mathtt{F},\mathtt{B}). \quad (5.43)\end{aligned}$$

Putting the values from 5.40, 5.41, 5.42, and 5.43 into 5.39, we have

$$\begin{aligned}\mathcal{K}(\mathtt{E},\mathtt{F},\mathtt{T},\mathtt{B}) + \frac{1}{n-1}g(\mathtt{E},\mathtt{F})S(\mathtt{T},\mathtt{B}) - g(\mathtt{B},\mathtt{E})g(\mathtt{F},\mathtt{T})&\\ + g(\mathtt{B},\mathtt{F})g(\mathtt{E},\mathtt{T}) + \frac{1}{n-1}S(\mathtt{B},\mathtt{T})\omega(\mathtt{E})\omega(\mathtt{F}) - g(\mathtt{B},\mathtt{T})g(\mathtt{E},\mathtt{F})&\\ - g(\mathtt{B},\mathtt{T})\omega(\mathtt{E})\omega(\mathtt{F}) - g(\mathtt{F},\mathtt{B})\omega(\mathtt{E})\omega(\mathtt{T}) + \frac{1}{n-1}\omega(\mathtt{E})\omega(\mathtt{T})S(\mathtt{F},\mathtt{B}) = 0.& \quad (5.44)\end{aligned}$$

Contracting 5.44 along $\mathtt{E}$, and $\mathtt{B}$, on evaluation, it provides

$$\mathcal{S}(\mathtt{F},\mathtt{T}) = (n-1)g(\mathtt{F},\mathtt{T}). \quad (5.45)$$

This completes the proof. □

6. Φ-$\mathcal{W}_6$-flat Lorentzian para-Kenmotsu Manifolds

Theorem 6.1. *If an $(LPK)_n$ manifold is Φ-$\mathcal{W}_6$-flat, then it is an Einstein manifold.*

Proof. Let us consider that an $(LPK)_n$ manifold is Φ-$\mathcal{W}_6$-flat. Then,

$$'\mathcal{W}_6(\Phi\mathtt{A},\Phi\mathtt{B},\Phi\mathtt{C},\Phi\mathtt{T})=0. \tag{6.46}$$

By definition of $\mathcal{W}_6$ curvature tensor 1.2

$$\mathcal{K}(\Phi\mathtt{A},\Phi\mathtt{B},\Phi\mathtt{C},\Phi\mathtt{T})+\frac{1}{n-1}[g(\Phi\mathtt{A},\Phi\mathtt{B})\mathcal{S}(\Phi\mathtt{C},\Phi\mathtt{T})-\mathcal{S}(\Phi\mathtt{B},\Phi\mathtt{C})g(\Phi\mathtt{A},\Phi\mathtt{T})]=0. \tag{6.47}$$

By definition of Riemann curvature tensor, we have

$$\mathcal{R}(\mathtt{A},\mathtt{B})\Phi\mathtt{C}=\nabla_{\mathtt{A}}\nabla_{\mathtt{B}}\Phi\mathtt{C}-\nabla_{\mathtt{B}}\nabla_{\mathtt{A}}\Phi\mathtt{C}-\nabla_{[\mathtt{A},\mathtt{B}]}\Phi\mathtt{C}.$$

Taking innerproduct of the above relation with respect to $\Phi\mathtt{T}$, it gives

$$g(\mathcal{R}(\mathtt{A},\mathtt{B})\Phi\mathtt{C},\Phi\mathtt{T})=g(\nabla_{\mathtt{A}}\nabla_{\mathtt{B}}\Phi\mathtt{C},\Phi\mathtt{T})-g(\nabla_{\mathtt{B}}\nabla_{\mathtt{A}}\Phi\mathtt{C},\Phi\mathtt{T})-g(\nabla_{[\mathtt{A},\mathtt{B}]}\Phi\mathtt{C},\Phi\mathtt{T}). \tag{6.48}$$

Evaluation of the term $\nabla_{\mathtt{A}}\nabla_{\mathtt{B}}\Phi\mathtt{C}$ provides

$$\begin{aligned}\nabla_{\mathtt{A}}\nabla_{\mathtt{B}}\Phi\mathtt{C}=&-g(\nabla_{\mathtt{A}}\Phi\mathtt{B},\mathtt{C})\zeta-g(\Phi\mathtt{B},\nabla_{\mathtt{A}}\mathtt{C})\zeta\\&+g(\Phi\mathtt{B},\mathtt{C})\mathtt{A}+g(\Phi\mathtt{B},\mathtt{C})\omega(\mathtt{A})\zeta-(\nabla_{\mathtt{A}}\omega)(\mathtt{C})\Phi\mathtt{B}-\omega(\nabla_{\mathtt{A}}\mathtt{C})\Phi\mathtt{B}\\&+g(\Phi\mathtt{A},\mathtt{B})\omega(\mathtt{C})\zeta+\omega(\mathtt{B})\omega(\mathtt{C})\Phi\mathtt{A}-\omega(\mathtt{C})\Phi(\nabla_{\mathtt{A}}\mathtt{B})\\&-g(\Phi\mathtt{A},\nabla_{\mathtt{B}}\mathtt{C})\zeta-\omega(\nabla_{\mathtt{B}}\mathtt{C})\Phi\mathtt{A}+\Phi(\nabla_{\mathtt{A}}\nabla_{\mathtt{B}}\mathtt{C}).\end{aligned} \tag{6.49}$$

Taking innerproduct of 6.49 with $\Phi\mathtt{T}$, we have

$$\begin{aligned}g(\nabla_{\mathtt{A}}\nabla_{\mathtt{B}}\Phi\mathtt{C},\Phi\mathtt{T})=&-g(\nabla_{\mathtt{A}}\Phi\mathtt{B},\mathtt{C})g(\zeta,\Phi\mathtt{T})-g(\Phi\mathtt{B},\nabla_{\mathtt{A}}\mathtt{C})g(\zeta,\Phi\mathtt{T})\\&+g(\Phi\mathtt{B},\mathtt{C})g(\mathtt{A},\Phi\mathtt{T})+g(\Phi\mathtt{B},\mathtt{C})\omega(\mathtt{A})g(\zeta,\Phi\mathtt{T})-(\nabla_{\mathtt{A}}\omega)(\mathtt{C})g(\Phi\mathtt{B},\Phi\mathtt{T})\\&-\omega(\nabla_{\mathtt{A}}\mathtt{C})g(\Phi\mathtt{B},\Phi\mathtt{T})+g(\Phi\mathtt{A},\mathtt{B})\omega(\mathtt{C})g(\zeta,\Phi\mathtt{T})+\omega(\mathtt{B})\omega(\mathtt{C})g(\Phi\mathtt{A},\Phi\mathtt{T})\\&-\omega(\mathtt{C})g(\Phi(\nabla_{\mathtt{A}}\mathtt{B}),\Phi\mathtt{T})-g(\Phi\mathtt{A},\nabla_{\mathtt{B}}\mathtt{C})g(\zeta,\Phi\mathtt{T})\\&-\omega(\nabla_{\mathtt{B}}\mathtt{C})g(\Phi\mathtt{A},\Phi\mathtt{T})+g(\Phi(\nabla_{\mathtt{A}}\nabla_{\mathtt{B}}\mathtt{C}),\Phi\mathtt{T}).\end{aligned} \tag{6.50}$$

Using 2.3 into 6.50, we have

$$\begin{aligned}g(\nabla_{\mathtt{A}}\nabla_{\mathtt{B}}\Phi\mathtt{C},\Phi\mathtt{T})=&\,g(\Phi\mathtt{B},\mathtt{C})g(\mathtt{A},\Phi\mathtt{T})-(\nabla_{\mathtt{A}}\omega)(\mathtt{C})g(\Phi\mathtt{B},\Phi\mathtt{T})\\&-\omega(\nabla_{\mathtt{A}}\mathtt{C})g(\Phi\mathtt{B},\Phi\mathtt{T})+\omega(\mathtt{B})\omega(\mathtt{C})g(\Phi\mathtt{A},\Phi\mathtt{T})-\omega(\mathtt{C})g(\Phi(\nabla_{\mathtt{A}}\mathtt{B}),\Phi\mathtt{T})\\&-\omega(\nabla_{\mathtt{B}}\mathtt{C})g(\Phi\mathtt{A},\Phi\mathtt{T})+g(\Phi(\nabla_{\mathtt{A}}\nabla_{\mathtt{B}}\mathtt{C}),\Phi\mathtt{T}).\end{aligned} \tag{6.51}$$

Applying $\mathtt{A} \leftrightarrow \mathtt{B}$ in 6.51, we have

$$\begin{aligned} g(\nabla_{\mathtt{B}}\nabla_{\mathtt{A}}\Phi\mathtt{C}, \Phi\mathtt{T}) = {} & g(\Phi\mathtt{A}, \mathtt{C})g(\mathtt{B}, \Phi\mathtt{T}) - (\nabla_{\mathtt{B}}\omega)(\mathtt{C})g(\Phi\mathtt{A}, \Phi\mathtt{T}) \\ & - \omega(\nabla_{\mathtt{B}}\mathtt{C})g(\Phi\mathtt{A}, \Phi\mathtt{T}) + \omega(\mathtt{A})\omega(\mathtt{C})g(\Phi\mathtt{B}, \Phi\mathtt{T}) - \omega(\mathtt{C})g(\Phi(\nabla_{\mathtt{B}}\mathtt{A}), \Phi\mathtt{T}) \\ & - \omega(\nabla_{\mathtt{A}}\mathtt{C})g(\Phi\mathtt{B}, \Phi\mathtt{T}) + g(\Phi(\nabla_{\mathtt{B}}\nabla_{\mathtt{A}}\mathtt{C}), \Phi\mathtt{T}). \end{aligned} \tag{6.52}$$

Differentiating covariantly $\Phi\mathtt{C}$ along $[\mathtt{A}, \mathtt{B}]$, we find

$$\nabla_{[\mathtt{A},\mathtt{B}]}(\Phi\mathtt{C}) = -g(\Phi[\mathtt{A}, \mathtt{B}], \mathtt{C})\zeta - \omega(\mathtt{C})[\Phi(\nabla_{\mathtt{A}}\mathtt{B} - \nabla_{\mathtt{B}}\mathtt{A})] + \Phi(\nabla_{[\mathtt{A},\mathtt{B}]}\mathtt{C}). \tag{6.53}$$

Taking innerproduct of 6.53 with $\Phi\mathtt{T}$, we have

$$g(\nabla_{[\mathtt{A},\mathtt{B}]}(\Phi\mathtt{C}), \Phi\mathtt{T}) = -\omega(\mathtt{C})g(\Phi(\nabla_{\mathtt{A}}\mathtt{B}), \Phi\mathtt{T}) + \omega(\mathtt{C})g(\Phi(\nabla_{\mathtt{B}}\mathtt{A}), \Phi\mathtt{T}) + g(\Phi(\nabla_{[\mathtt{A},\mathtt{B}]}\mathtt{C}), \Phi\mathtt{T}). \tag{6.54}$$

Putting values 6.51, 6.52 and 6.54 into relation 6.48, it yields

$$\begin{aligned} \mathcal{K}(\mathtt{A}, \mathtt{B}, \Phi\mathtt{C}, \Phi\mathtt{T}) = {} & g(\Phi\mathtt{B}, \mathtt{C})g(\mathtt{A}, \Phi\mathtt{T}) - g(\Phi\mathtt{A}, \mathtt{C})g(\mathtt{B}, \Phi\mathtt{T}) \\ & + (\nabla_{\mathtt{B}}\omega)(\mathtt{C})g(\Phi\mathtt{A}, \Phi\mathtt{T}) - (\nabla_{\mathtt{A}}\omega)(\mathtt{C})g(\Phi\mathtt{B}, \Phi\mathtt{T}) + \omega(\mathtt{B})\omega(\mathtt{C})g(\Phi\mathtt{A}, \Phi\mathtt{T}) \\ & - \omega(\mathtt{A})\omega(\mathtt{C})g(\Phi\mathtt{B}, \Phi\mathtt{T}) + g(\Phi(\mathcal{R}(\mathtt{A}, \mathtt{B})\mathtt{C}, \Phi\mathtt{T}). \end{aligned} \tag{6.55}$$

Applying the relation 2.4 into the last term of right hand side of 6.55, and then transposing to left hand side, we have

$$\begin{aligned} \mathcal{K}(\mathtt{A}, \mathtt{B}, \Phi\mathtt{C}, \Phi\mathtt{T}) - \mathcal{K}(\mathtt{A}, \mathtt{B}, \mathtt{C}, \Phi^2\mathtt{T}) = {} & g(\Phi\mathtt{B}, \mathtt{C})g(\mathtt{A}, \Phi\mathtt{T}) - g(\Phi\mathtt{A}, \mathtt{C})g(\mathtt{B}, \Phi\mathtt{T}) \\ & + [(\nabla_{\mathtt{B}}\omega)(\mathtt{C}) + \omega(\mathtt{B})\omega(\mathtt{C})]g(\Phi\mathtt{A}, \Phi\mathtt{T}) - [(\nabla_{\mathtt{A}}\omega)(\mathtt{C}) + \omega(\mathtt{A})\omega(\mathtt{C})]g(\Phi\mathtt{B}, \Phi\mathtt{T}). \end{aligned} \tag{6.56}$$

Using 2.3, and 2.5 into 6.56, we have

$$\begin{aligned} \mathcal{K}(\mathtt{A}, \mathtt{B}, \Phi\mathtt{C}, \Phi\mathtt{T}) - \mathcal{K}(\mathtt{A}, \mathtt{B}, \mathtt{C}, \mathtt{T}) - \omega(\mathtt{T})\mathcal{K}(\mathtt{A}, \mathtt{B}, \mathtt{C}, \zeta) = {} & g(\Phi\mathtt{B}, \mathtt{C})g(\mathtt{A}, \Phi\mathtt{T}) \\ & - g(\Phi\mathtt{A}, \mathtt{C})g(\mathtt{B}, \Phi\mathtt{T}) - g(\mathtt{B}, \mathtt{C})g(\Phi\mathtt{A}, \Phi\mathtt{T}) + g(\mathtt{A}, \mathtt{C})g(\Phi\mathtt{B}, \Phi\mathtt{T}). \end{aligned} \tag{6.57}$$

Using 2.3, and 2.6 into 6.57, we have

$$\begin{aligned} \mathcal{K}(\mathtt{A}, \mathtt{B}, \Phi\mathtt{C}, \Phi\mathtt{T}) - \mathcal{K}(\mathtt{A}, \mathtt{B}, \mathtt{C}, \mathtt{T}) = {} & g(\Phi\mathtt{B}, \mathtt{C})g(\mathtt{A}, \Phi\mathtt{T}) \\ & - g(\Phi\mathtt{A}, \mathtt{C})g(\mathtt{B}, \Phi\mathtt{T}) - g(\mathtt{B}, \mathtt{C})g(\mathtt{A}, \mathtt{T}) + g(\mathtt{A}, \mathtt{C})g(\mathtt{B}, \mathtt{T}). \end{aligned} \tag{6.58}$$

By Riemann curvature property, we have

$$\mathcal{K}(\mathtt{A}, \mathtt{B}, \mathtt{C}, \mathtt{T}) = \mathcal{K}(\mathtt{C}, \mathtt{T}, \mathtt{A}, \mathtt{B}). \tag{6.59}$$

Applying $X \leftrightarrow Z$, and $Y \leftrightarrow T$ into 6.58, we have

$$\mathcal{K}(\mathtt{C},\mathtt{T},\Phi\mathtt{A},\Phi\mathtt{B}) - \mathcal{K}(\mathtt{C},\mathtt{T},\mathtt{A},\mathtt{B}) = g(\Phi\mathtt{T},\mathtt{A})g(\mathtt{C},\Phi\mathtt{B}) - g(\Phi\mathtt{C},\mathtt{A})g(\mathtt{T},\Phi\mathtt{B}) - g(\mathtt{T},\mathtt{A})g(\mathtt{C},\mathtt{B}) + g(\mathtt{C},\mathtt{A})g(\mathtt{T},\mathtt{B}). \tag{6.60}$$

Subtracting 6.60 from 6.58, and using 6.59, we have

$$\mathcal{K}(\mathtt{A},\mathtt{B},\Phi\mathtt{C},\Phi\mathtt{T}) = \mathcal{K}(\mathtt{C},\mathtt{T},\Phi\mathtt{A},\Phi\mathtt{B}). \tag{6.61}$$

Applying $\mathtt{A} \to \Phi\mathtt{A}$, and $\mathtt{B} \to \Phi\mathtt{B}$ into 6.61, we have

$$\mathcal{K}(\Phi\mathtt{A},\Phi\mathtt{B},\Phi\mathtt{C},\Phi\mathtt{T}) = \mathcal{K}(\mathtt{C},\mathtt{T},\Phi^2\mathtt{A},\Phi^2\mathtt{B}). \tag{6.62}$$

Applying 2.3, and 6.59 into 6.62, on simplification, we have

$$\mathcal{K}(\Phi\mathtt{A},\Phi\mathtt{B},\Phi\mathtt{C},\Phi\mathtt{T}) = \mathcal{K}(\mathtt{A},\mathtt{B},\mathtt{C},\mathtt{T}) + g(\mathtt{A},\mathtt{T})\omega(\mathtt{B})\omega(\mathtt{C}) - g(\mathtt{A},\mathtt{C})\omega(\mathtt{B})\omega(\mathtt{T}) + g(\mathtt{B},\mathtt{C})\omega(\mathtt{A})\omega(\mathtt{T}) - g(\mathtt{B},\mathtt{T})\omega(\mathtt{A})\omega(\mathtt{C}). \tag{6.63}$$

Putting value from relation 6.63 into 6.47, we have

$$\mathcal{K}(\mathtt{A},\mathtt{B},\mathtt{C},\mathtt{T}) - g(\mathtt{A},\mathtt{C})\omega(\mathtt{B})\omega(\mathtt{T}) + g(\mathtt{B},\mathtt{C})\omega(\mathtt{A})\omega(\mathtt{T}) - g(\mathtt{B},\mathtt{T})\omega(\mathtt{A})\omega(\mathtt{C}) + \frac{1}{n-1}[\mathcal{S}(\mathtt{C},\mathtt{T})g(\mathtt{A},\mathtt{B}) + (n-1)g(\mathtt{A},\mathtt{B})\omega(\mathtt{C})\omega(\mathtt{T}) + \mathcal{S}(\mathtt{C},\mathtt{T})\omega(\mathtt{A})\omega(\mathtt{B}) - \mathcal{S}(\mathtt{B},\mathtt{C})g(\mathtt{A},\mathtt{T}) - \mathcal{S}(\mathtt{B},\mathtt{C})\omega(\mathtt{A})\omega(\mathtt{T})] = 0. \tag{6.64}$$

Contracting 6.64 with respect to $\mathtt{A}$, and $\mathtt{T}$, we have

$$\mathcal{S}(\mathtt{B},\mathtt{C}) - \omega(\mathtt{B})\omega(\mathtt{C}) - g(\mathtt{B},\mathtt{C}) - \omega(\mathtt{B})\omega(\mathtt{C}) + \frac{1}{n-1}[\mathcal{S}(\mathtt{B},\mathtt{C}) + (n-1)\omega(\mathtt{C})\omega(\mathtt{B}) + \mathcal{S}(\mathtt{C},\zeta)\omega(\mathtt{B}) - n\mathcal{S}(\mathtt{B},\mathtt{C}) + \mathcal{S}(\mathtt{B},\mathtt{C})] = 0. \tag{6.65}$$

On simplification of 6.65, it concludes

$$\mathcal{S}(\mathtt{B},\mathtt{C}) = (n-1)g(\mathtt{B},\mathtt{C}). \tag{6.66}$$

This completes the proof. $\square$

7. Lorentzian para-Kenmotsu manifolds with condition $\mathcal{W}_6(\mathtt{E},\mathtt{F}).\mathcal{R}=0$

In this part, we explore the behavior of $(LPK)_n$ manifold admitting $\mathcal{W}_6(\mathtt{E},\mathtt{F}).\mathcal{R}=0$. We begin this with the subsequent theorem:

Theorem 7.1. *An $(LPK)_n$ manifold is an ω-Einstein manifold if, it satisfies the relation $\mathcal{W}_6(\mathtt{E},\mathtt{F}).\mathcal{R}=0$.*

Proof. Let us consider that the $(LPK)_n$ manifold admits the condition

$$\mathcal{W}_6(\mathtt{E},\mathtt{F}).\mathcal{R}=0. \tag{7.67}$$

From the relation 7.67, we have

$$\mathcal{W}_6(\mathtt{E},\mathtt{F})\mathcal{R}(\mathtt{A},\mathtt{B})\mathtt{C}-\mathcal{R}(\mathcal{W}_6(\mathtt{E},\mathtt{F})\mathtt{A},\mathtt{B})\mathtt{C}-\mathcal{R}(\mathtt{A},\mathcal{W}_6(\mathtt{E},\mathtt{F})\mathtt{B})\mathtt{C}-\mathcal{R}(\mathtt{A},\mathtt{B})\mathcal{W}_6(\mathtt{E},\mathtt{F})\mathtt{C}=0. \tag{7.68}$$

Putting $\mathtt{B}=\zeta$ into the relation 7.68, we have

$$\mathcal{W}_6(\mathtt{E},\mathtt{F})(\mathcal{R}(\mathtt{A},\zeta)\mathtt{C})-\mathcal{R}(\mathcal{W}_6(\mathtt{E},\mathtt{F})\mathtt{A},\zeta)\mathtt{C}-\mathcal{R}(\mathtt{A},\mathcal{W}_6(\mathtt{E},\mathtt{F})\zeta)\mathtt{C}-\mathcal{R}(\mathtt{A},\zeta)\mathcal{W}_6(\mathtt{E},\mathtt{F})\mathtt{C}=0. \tag{7.69}$$

Evaluation of the terms of the relation 7.69 in the subsequent manner:
Using 2.7, 2.9, 1.1, 2.13 into first term of 7.69, we get

$$\begin{aligned}\mathcal{W}_6(\mathtt{E},\mathtt{F})\mathcal{R}(\mathtt{A},\zeta)\mathtt{C}=\ &\omega(\mathtt{C})\mathcal{R}(\mathtt{E},\mathtt{F})\mathtt{A}\\ &+\frac{1}{n-1}g(\mathtt{E},\mathtt{F})\omega(\mathtt{C})Q\mathtt{A}-\frac{1}{n-1}\mathcal{S}(\mathtt{F},\mathtt{A})\omega(\mathtt{C})\mathtt{E}-g(\mathtt{A},\mathtt{C})\omega(\mathtt{F})\mathtt{E}\\ &+g(\mathtt{A},\mathtt{C})\omega(\mathtt{E})\mathtt{F}-g(\mathtt{A},\mathtt{C})g(\mathtt{E},\mathtt{F})\zeta+g(\mathtt{A},\mathtt{C})\omega(\mathtt{F})\mathtt{E}.\end{aligned} \tag{7.70}$$

Using 2.7, 2.9, and 1.1, into second term of 7.69, we get

$$\begin{aligned}\mathcal{R}(\mathcal{W}_6(\mathtt{E},\mathtt{F})\mathtt{A},\zeta)\mathtt{C}=\ &\omega(\mathtt{C})\mathcal{R}(\mathtt{E},\mathtt{F})\mathtt{A}\\ &+\frac{1}{n-1}g(\mathtt{E},\mathtt{F})\omega(\mathtt{C})Q\mathtt{A}-\frac{1}{n-1}\mathcal{S}(\mathtt{F},\mathtt{A})\omega(\mathtt{C})\mathtt{E}-\mathcal{K}(\mathtt{E},\mathtt{F},\mathtt{A},\mathtt{C})\zeta\\ &-\frac{1}{n-1}g(\mathtt{E},\mathtt{F})\mathcal{S}(\mathtt{A},\mathtt{C})\zeta+\frac{1}{n-1}\mathcal{S}(\mathtt{F},\mathtt{A})g(\mathtt{E},\mathtt{C})\zeta.\end{aligned} \tag{7.71}$$

Applying relations 2.7, and 2.13 into third term of 7.69, we get

$$\mathcal{R}(\mathtt{A},\mathcal{W}_6(\mathtt{E},\mathtt{F})\zeta)\mathtt{C}=g(\mathtt{E},\mathtt{F})\omega(\mathtt{C})\mathtt{A}-g(\mathtt{E},\mathtt{F})g(\mathtt{A},\mathtt{C})\zeta-\omega(\mathtt{E})\mathcal{R}(\mathtt{A},\mathtt{F})\mathtt{C}. \tag{7.72}$$

Using 2.7, 2.9, 1.1 and 2.13 into fourth term of 7.69, we get

$$\begin{aligned}\mathcal{R}(\mathtt{A},\zeta)\mathcal{W}_6(\mathtt{E},\mathtt{F})\mathtt{C} = {} & g(\mathtt{F},\mathtt{C})\omega(\mathtt{E})\mathtt{A} - g(\mathtt{E},\mathtt{C})\omega(\mathtt{F})\mathtt{A} \\ & + g(\mathtt{E},\mathtt{F})\omega(\mathtt{C})\mathtt{A} - \frac{1}{n-1}\mathcal{S}(\mathtt{F},\mathtt{C})\omega(\mathtt{E})\mathtt{A} - \mathcal{K}(\mathtt{E},\mathtt{F},\mathtt{C},\mathtt{A})\zeta \\ & - \frac{1}{n-1}g(\mathtt{E},\mathtt{F})\mathcal{S}(\mathtt{A},\mathtt{C})\zeta + \frac{1}{n-1}\mathcal{S}(\mathtt{F},\mathtt{C})g(\mathtt{E},\mathtt{A})\zeta. \qquad (7.73)\end{aligned}$$

Putting the values from 7.70, 7.71, 7.72, and 7.73 into 7.69, it gives

$$\begin{aligned}g(\mathtt{A},\mathtt{C})\omega(\mathtt{E})\mathtt{F} & + \frac{2}{n-1}g(\mathtt{E},\mathtt{F})\mathcal{S}(\mathtt{A},\mathtt{C})\zeta - \frac{1}{n-1}\mathcal{S}(\mathtt{F},\mathtt{A})g(\mathtt{E},\mathtt{C})\zeta \\ & - 2g(\mathtt{E},\mathtt{F})\omega(\mathtt{C})\mathtt{A} + \omega(\mathtt{E})R(\mathtt{A},\mathtt{F})\mathtt{C} - g(\mathtt{F},\mathtt{C})\omega(\mathtt{E})\mathtt{A} + g(\mathtt{E},\mathtt{C})\omega(\mathtt{F})\mathtt{A} \\ & + \frac{1}{n-1}\mathcal{S}(\mathtt{F},\mathtt{C})\omega(\mathtt{E})\mathtt{A} - \frac{1}{n-1}\mathcal{S}(\mathtt{F},\mathtt{C})g(\mathtt{E},\mathtt{A})\zeta = 0. \qquad (7.74)\end{aligned}$$

Contracting 7.74 along $\mathtt{A}$, we have

$$\begin{aligned}g(\mathtt{F},\mathtt{C})\omega(\mathtt{E}) & + \frac{2}{n-1}g(\mathtt{E},\mathtt{F})\mathcal{S}(\zeta,\mathtt{C}) - \frac{1}{n-1}\mathcal{S}(\mathtt{F},\zeta)g(\mathtt{E},\mathtt{C}) - 2ng(\mathtt{E},\mathtt{F})\omega(\mathtt{C}) + \omega(\mathtt{E})\mathcal{S}(\mathtt{F},\mathtt{C}) \\ & - ng(\mathtt{F},\mathtt{C})\omega(\mathtt{E}) + ng(\mathtt{E},\mathtt{C})\omega(\mathtt{F}) + \frac{n}{n-1}\mathcal{S}(\mathtt{F},\mathtt{C})\omega(\mathtt{E}) - \frac{1}{n-1}\mathcal{S}(\mathtt{F},\mathtt{C})\omega(\mathtt{E}) = 0. \qquad (7.75)\end{aligned}$$

Putting $\mathtt{E} = \zeta$ and making use of 2.9 into 7.75, it provides

$$\mathcal{S}(\mathtt{F},\mathtt{C}) = \frac{(n-1)}{2}g(\mathtt{F},\mathtt{C}) - \frac{(n-1)}{2}\omega(\mathtt{E})\omega(\mathtt{F}). \qquad (7.76)$$

This completes the proof. □

Acknowledgments. The authors would like to thank the referee for some useful comments and their helpful suggestions that have improved the quality of this paper.

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