International Journal of Maps in Mathematics
Volume 8, Issue 1, 2025, Pages:309-325
E-ISSN: 2636-7467
www.simadp.com/journalmim

RIEMANNIAN CR MANIFOLDS AND ρ-EINSTEIN SOLITONS: A GEOMETRIC ANALYSIS AND APPLICATIONS

PRABHAKARAN SWAPNA SANGEETHA AND MALLANNARA S. SIDDESHA *

ABSTRACT. In this article, we investigate ρ-Einstein solitons on Riemannian CR manifolds. Specifically, we explore the properties of ρ-Einstein solitons in the presence of cyclic η-recurrent Ricci tensors on Riemannian CR manifolds. We also examine these solitons with respect to Torse-forming vector fields. Additionally, we study ρ-Einstein solitons satisfying Ricci semi-symmetric condition on Riemannian CR manifolds. Furthermore, we examine the properties of conharmonic and conformal curvature tensors on Riemannian CR manifolds admitting ρ-Einstein solitons. Finally, we discuss the applications of ρ-Einstein solitons and their potential uses in various fields.

Keywords: ρ-Einstein soliton, Einstein manifolds, Riemannian CR manifolds.

2010 Mathematics Subject Classification: 53C25, 32V20, 53C21, 53D10, 58D17.

1. INTRODUCTION

Over the last twenty years, the study of geometric flow has become the main focus of many mathematicians as it helps in understanding the geometric structures of manifolds in Riemannian geometry. In order to better comprehend these structures on Riemannian manifolds (M, g), Hamilton [15] developed the 'Ricci flow' in 1982, which is described by

$$g_t = -2S,$$

Received:2024.09.23 *Revised:2024.11.11* *Accepted:2024.12.19*

* Corresponding author

Swapna Sangeetha P. ⋄ swapnamath4@gmail.com ⋄ https://orcid.org/0009-0003-5581-2786

Mallannara Siddalingappa Siddesha ⋄ mssiddesha@gmail.com ⋄ https://orcid.org/0000-0003-2367-0544 .

where S denotes the Ricci tensor and g is the Riemannian metric on M that satisfies the Ricci soliton equation

$$\pounds_V g + 2S + 2\lambda g = 0,$$

where $\pounds_V$ denotes the Lie derivative along the vector field V on M, and λ is a constant. The manifolds admitting such type of structures are called Ricci solitons. The nature of these solitons depends on λ, i.e. $\lambda = 0$ is steady, $\lambda > 0$ is shrinking and $\lambda < 0$ is expanding. Bourguignon [3] gave the generalization of the Ricci flow in 1980s by introducing the notion of Ricci-Bourguignon flow, which is described by

$$g_t = -2S + 2\rho r g, \ g(0) = g_0, \tag{1.1}$$

where r represents the scalar curvature and $\rho(\neq 0)$ is a real constant. For specific values of ρ, we get certain tensors associated to equation(1.1):

1. $\rho = \frac{1}{2}$, Einstein tensor $S - \frac{r}{2}g$.
2. $\rho = \frac{1}{n}$, traceless Ricci tensor $S - \frac{r}{n}g$.
3. $\rho = \frac{1}{2(n-1)}$, Schouten tensor $S - \frac{r}{2(n-1)}g$.
4. $\rho = 0$, Ricci tensor S.

The self-similar solution to the Ricci-Bourguignon flow is the Ricci-Bourguignon soliton (also called as 'ρ-Einstein soliton') whose equation is

$$\pounds_V g + 2S + 2(\lambda - \rho r)g = 0. \tag{1.2}$$

The ρ-Einstein soliton is steady if $\lambda = 0$, shrinking if $\lambda < 0$ and expanding if $\lambda > 0$.

Recent mathematical research focuses on classifying Ricci solitons in Riemannian manifolds under particular geometric conditions. Chen and Deshmukh [5] investigated potential fields as concurrent fields and provided a classification of Ricci solitons. Sharma [29] studied gradient Ricci solitons with scalar curvature which is constant and non-homothetic conformal vector fields on Riemannian manifolds, yielding significant findings. Naik [22] characterized Ricci solitons and gradient Ricci almost solitons on Riemannian manifolds, specifically those admitting concurrent recurrent vector fields, known as Riemannian CR manifolds. For further studies on Ricci solitons across various classes of Riemannian manifolds, we suggest [7, 24, 36].

Now, let's recall some concepts on vector fields in Riemannian manifolds. A smooth vector field ξ, on M is said to be conformal if $\exists$ a smooth function ψ (referred to as 'conformal coefficient') on M such as

$$\pounds_{\xi} g = 2\psi g. \tag{1.3}$$

In specific, ξ is said to be homothetic (Killing) vector field if ψ is constant, i.e. $d\psi = 0$. If the dual 1-form η is closed, i.e. ξ is closed, then (1.3) becomes

$$\nabla_{Z_2} \xi = \psi Z_2 \tag{1.4}$$

for all Z_2, a vector field and ∇, the Levi-Civita connection on M. From (1.4), we observe that ξ is a closed homothetic vector field (parallel) if ψ is constant, i.e. $d\psi = 0$ satisfies. In particular, ξ is said to be concurrent when $\psi = 1$ in (1.4). For additional information, we suggest [4, 8, 35]. In contrast, ξ is called a recurrent vector field when

$$\nabla \xi = \eta \otimes \xi.$$

We recommend [6, 13] for results on Riemannian manifolds admitting recurrent vector field. A vector field Z_2 that satisfies the expression

$$\nabla_{Z_2} \xi = \alpha Z_2 + \beta \eta(Z_2) \xi, \quad \alpha,\ \beta\ \in \mathbb{R},$$

generalizes both closed homothetic (concurrent), as well as recurrent vector fields. Thus, we observe that $\pounds_{\xi} g = 2\alpha id + 2\beta \eta \otimes \eta$. If ξ is a unit vector field, then $(\pounds_{\xi} g)(\xi, \xi) = 0 = 2(\alpha + \beta)$, which implies

$$\nabla_{Z_2} \xi = \alpha [Z_2 - \eta(Z_2) \xi] \tag{1.5}$$

for any $Z_2 \in \chi(M)$ and $\alpha \in \mathbb{R}$ is a constant. Here, we regard a unit vector field ξ, which is non-parallel and satisfies (1.5), as a concurrent-recurrent vector field.

The existence of certain vector fields within a Riemannian manifold is a key element of differential geometry. These vector fields are classified into two types: Killing and conformal vector fields [9, 10]. This article highlights Killing vector fields because of their significant prospects for future applications. Drawing on the works of Ahmad et al [1] on ρ-Einstein solitons in Lorentzian para-Kenmotsu manifolds, and inspired by the works of [8, 12, 17, 20, 22, 27, 37], it is essential to explore the geometry of ρ-Einstein solitons in Riemannian CR manifolds.

The current paper investigates ρ-Einstein solitons on Riemannian CR manifolds. The paper is structured as follows: Section 2 describes the preliminary concepts. Section 3 centers on the analysis of Riemannian CR manifolds admitting ρ-Einstein solitons. Section 4 examines ρ-Einstein solitons on Riemannian CR manifolds with cyclic η-recurrent Ricci tensor.

Section 5 delves into ρ-Einstein solitons on Riemannian CR manifolds with a torse-forming vector field. Section 6 addresses Riemannian CR manifolds admitting ρ-Einstein solitons that satisfy $R(\xi, Z_1) \cdot S = 0$. Section 7 highlights Riemannian CR manifolds admitting ρ-Einstein solitons concerning the conharmonic curvature tensor. Section 8 focuses on Riemannian CR manifolds admitting ρ-Einstein solitons with respect to the conformal curvature tensor. Section 9 gives some applications of ρ-Einstein solitons on various fields. The final section, section 10, draws the concluding remarks based on the obtained results.

2. Preliminaries

Let M be a Riemannian manifold of dimension n, admitting a concurrent recurrent vector field ξ, a 1-form η and a Riemannian metric g that satisfies the following relations:

$$\eta(\xi) = 1, \tag{2.6}$$

$$g(Z_1, \xi) = \eta(Z_1), \; g(Z_2, \xi) = \eta(Z_2), \tag{2.7}$$

$$\nabla_{Z_1}\xi = \alpha[Z_1 - \eta(Z_1)\xi], \tag{2.8}$$

$$(\nabla_{Z_1}\eta)(Z_2) = \alpha[g(Z_2, Z_1) - \eta(Z_1)\eta(Z_2)] \tag{2.9}$$

for all $Z_1, Z_2 \in \chi(M)$, ∇ denotes the Levi-Civita connection on M, and $\alpha \in \mathbb{R}$ is a constant.

A Riemannian CR manifold satisfies the following relations [22]:

$$g(R(Z_1, Z_2)Z_3, \xi) = \eta(R(Z_1, Z_2)Z_3) = -\alpha^2[g(Z_2, Z_3)\eta(Z_1) - g(Z_1, Z_3)\eta(Z_2)], \tag{2.10}$$

$$R(\xi, Z_1)Z_2 = -\alpha^2[g(Z_1, Z_2)\xi - \eta(Z_2)Z_1], \tag{2.11}$$

$$R(Z_1, Z_2)\xi = -\alpha^2[\eta(Z_2)Z_1 - \eta(Z_1)Z_2], \tag{2.12}$$

$$R(\xi, Z_1)\xi = -\alpha^2[\eta(Z_1)\xi - Z_1], \tag{2.13}$$

$$S(Z_1, \xi) = -(n-1)\alpha^2\eta(Z_1), \; S(\xi, \xi) = -(n-1)\alpha^2, \tag{2.14}$$

$$Q\xi = -(n-1)\alpha^2\xi, \tag{2.15}$$

where R denotes the Riemannian curvature, S represents the Ricci tensor and Q indicates the Ricci operator which is given as $S(Z_1, Z_2) = g(QZ_1, Z_2), \;\; \forall \; Z_1, Z_2 \in \chi(M)$.

Definition 2.1. [33] *A Riemannian CR manifold is said to be an η-Einstein manifold when its Ricci tensor S satisfies the following relation*

$$S(Z_1, Z_2) = ag(Z_1, Z_2) + b\eta(Z_1)\eta(Z_2),$$

for smooth functions a and b. If $b = 0$, then the Riemannian CR manifold reduces to an Einstein manifold.

Remark 2.1. [23] *In a Riemannian CR manifold, we have*

$$\xi(r) = -2\alpha(r + n(n-1)\alpha^2). \tag{2.16}$$

Remark 2.2. *From (2.16), if an n-dimensional Riemannian CR manifold is of constant curvature, then*

$$r = -\alpha^2 n(n-1). \tag{2.17}$$

3. Riemannian CR manifolds admitting ρ-Einstein solitons

In this section, we examine ρ-Einstein solitons on a Riemannian CR manifold and we explore the nature of solitons for various values of ρ.

Let an n-dimensional Riemannian CR manifold M admit ρ-Einstein soliton. Then (1.2) holds. So we have

$$(\pounds_\xi g)(Z_1, Z_2) + 2S(Z_1, Z_2) + 2(\lambda - \rho r)g(Z_1, Z_2) = 0 \tag{3.18}$$

for all $Z_1, Z_2 \in \chi(M)$.
We know that

$$(\pounds_\xi g)(Z_1, Z_2) = g(\nabla_{Z_1}\xi, Z_2) + g(Z_1, \nabla_{Z_2}\xi) = 2\alpha[g(Z_1, Z_2) - \eta(Z_1)\eta(Z_2)]. \tag{3.19}$$

Hence, (3.18) leads to

$$S(Z_1, Z_2) = -(\lambda - \rho r + \alpha)g(Z_1, Z_2) + \alpha\eta(Z_1)\eta(Z_2). \tag{3.20}$$

Substituting $Z_2 = \xi$ in (3.20) and making use of (2.6) and (2.7), we get

$$S(Z_1, \xi) = -(\lambda - \rho r)\eta(Z_1), \tag{3.21}$$

which implies

$$Q\xi = -(\lambda - \rho r)\xi. \tag{3.22}$$

From (2.14) and (3.21), we obtain

$$\lambda = \alpha^2(n-1) + \rho r. \tag{3.23}$$

Now, if r is constant, then by Remark (2.2), (3.23) becomes

$$\lambda = \alpha^2(n-1)(1 - \rho n). \tag{3.24}$$

Therefore, we state:

Theorem 3.1. *If a Riemannian CR manifold of dimension n admits ρ-Einstein soliton then it is an η-Einstein manifold with the soliton constant $\lambda = \alpha^2(n-1)(1-\rho n)$.*

Now from the above theorem we obtain the following corollary:

Corollary 3.1. *Let an n-dimensional Riemannian CR manifold admit a ρ-Einstein soliton, then we have:*

Values of ρ	Soliton type	Soliton constant	Nature of soliton
$\rho = \frac{1}{2}$	Einstein soliton	$\lambda = -\frac{\alpha^2(n-1)(n-2)}{2}$	Shrinking
$\rho = \frac{1}{n}$	Traceless Ricci soliton	$\lambda = 0$	Steady
$\rho = \frac{1}{2(n-1)}$	Schouten soliton	$\lambda = \frac{\alpha^2(n-2)}{2}$	Expanding
$\rho = 0$	Ricci soliton	$\lambda = \alpha^2(n-1)$	Expanding

Lemma 3.1. *Let an n-dimensional Riemannian CR manifold admit a ρ-Einstein soliton $\ni$ $V = b\xi$, where b is a function. Then, V is a constant multiple of ξ and an n-dimensional Riemannian CR manifold is an η-Einstein manifold of the type*

$$S(Z_1, Z_2) = -(b\alpha + (\lambda - \rho r))g(Z_1, Z_2) + b\alpha\eta(Z_1)\eta(Z_2).$$

Proof. Let (g, V, λ, ρ) be a ρ-Einstein soliton on an n-dimensional Riemannian CR manifold, $\ni V$ is pointwise collinear with ξ i.e. $V = b\xi$. Then (1.2) holds. Hence, we have

$$bg(\nabla_{Z_1}\xi, Z_2) + Z_1(b)\eta(Z_2) + bg(Z_1, \nabla_{Z_2}\xi)$$
$$+Z_2(b)\eta(Z_1) + 2S(Z_1, Z_2) + 2(\lambda - \rho r)g(Z_1, Z_2) = 0,$$

which from (2.8) implies

$$2b\alpha g(Z_1, Z_2) - 2b\alpha\eta(Z_1)\eta(Z_2) + Z_1(b)\eta(Z_2)$$
$$+\eta(Z_1)Z_2(b) + 2S(Z_1, Z_2) + 2(\lambda - \rho r)g(Z_1, Z_2) = 0. \quad (3.25)$$

Substituting $Z_2 = \xi$ in (3.25) and making use of (2.6), (2.7) and (2.14), we have

$$Z_1(b) + \xi(b)\eta(Z_1) - 2[\alpha^2\eta(Z_1)(n-1) - (\lambda - \rho r)\eta(Z_1)] = 0. \quad (3.26)$$

Now putting $Z_1 = \xi$ in (3.26) and making use of (2.6), we get

$$\xi(b) - \alpha^2(n-1) + (\lambda - \rho r) = 0. \quad (3.27)$$

Using (3.27) in (3.26), we have

$$db = [\alpha^2(n-1) - (\lambda - \rho r)]\eta. \tag{3.28}$$

Applying d on both sides of (3.28), we obtain

$$[\alpha^2(n-1) - (\lambda - \rho r)]d\eta = 0 \implies \lambda = \alpha^2(n-1) + \rho r, \ d\eta \neq 0. \tag{3.29}$$

Therefore, by (3.28) and (3.29) $db = 0$, i.e. b is constant. Thus, (3.25) becomes

$$S(Z_1, Z_2) = -[b\alpha + (\lambda - \rho r)]g(Z_1, Z_2) + b\alpha\eta(Z_1)\eta(Z_2).$$

□

4. ρ-Einstein solitons on n-dimensional Riemannian CR manifolds with respect to cyclic η-recurrent Ricci tensor

In this section, we discuss the characteristic of ρ-Einstein soliton on a Riemannian CR manifold with respect to cyclic η-recurrent Ricci tensor and explore the nature of solitons for various values of ρ.

Definition 4.1. [34] *A Riemannian CR manifold M of dimension n is said to have cyclic η-recurrent Ricci tensor if*

$$(\nabla_{Z_1}S)(Z_2, Z_3) + (\nabla_{Z_2}S)(Z_3, Z_1) + (\nabla_{Z_3}S)(Z_1, Z_2)$$
$$= \eta(Z_1)S(Z_2, Z_3) + \eta(Z_2)S(Z_3, Z_1) + \eta(Z_3)S(Z_1, Z_2), \tag{4.30}$$

for any $Z_1, Z_2, Z_3 \in \chi(M)$.

Now, considering an n-dimensional Riemannian CR manifold M admitting a ρ-Einstein soliton with cyclic η-recurrent Ricci tensor, then equation (4.30) holds.
Applying covariant derivative on (3.20) with respect to Z_1 yields

$$\begin{aligned}(\nabla_{Z_1}S)(Z_2, Z_3) &= \rho Z_1(r)g(Z_2, Z_3) + \alpha^2 g(Z_1, Z_2)\eta(Z_3) \\ &\quad + \alpha^2 g(Z_1, Z_3)\eta(Z_2) - 2\alpha^2\eta(Z_1)\eta(Z_2)\eta(Z_3).\end{aligned} \tag{4.31}$$

Similarly, we get

$$\begin{aligned}(\nabla_{Z_2}S)(Z_3, Z_1) &= \rho Z_2(r)g(Z_3, Z_1) + \alpha^2 g(Z_2, Z_3)\eta(Z_1) \\ &\quad + \alpha^2 g(Z_1, Z_2)\eta(Z_3) - 2\alpha^2\eta(Z_1)\eta(Z_2)\eta(Z_3),\end{aligned} \tag{4.32}$$

and

$$(\nabla_{Z_3} S)(Z_1, Z_2) = \rho Z_3(r) g(Z_1, Z_2) + \alpha^2 g(Z_3, Z_1)\eta(Z_2) + \alpha^2 g(Z_3, Z_2)\eta(Z_1) - 2\alpha^2 \eta(Z_1)\eta(Z_2)\eta(Z_3). \tag{4.33}$$

Making use of (4.31)-(4.33) in (4.30) we get

$$\rho[Z_1(r)g(Z_2, Z_3) + Z_2(r)g(Z_3, Z_1) + Z_3(r)g(Z_1, Z_2)] = 3\alpha(1 + 2\alpha)\eta(Z_1)\eta(Z_2)\eta(Z_3)$$
$$-(\lambda - \rho r + \alpha(1 + 2\alpha))[g(Z_2, Z_3)\eta(Z_1) + g(Z_1, Z_3)\eta(Z_2) + g(Z_1, Z_2)\eta(Z_3)],$$

which on substitution of $Z_2 = Z_3 = \xi$ and utilization of (2.6) and (2.7) gives

$$\rho[Z_1(r) + \xi(r)\eta(Z_1) + \xi(r)\eta(Z_1)] = 3\alpha(1 + 2\alpha)\eta(Z_1) - 3(\lambda - \rho r + \alpha(1 + 2\alpha))\eta(Z_1). \tag{4.34}$$

Now taking $Z_1 = \xi$ and making use of (2.6) gives

$$3\rho\xi(r) = 3\alpha(1 + 2\alpha) - 3(\lambda - \rho r + \alpha(1 + 2\alpha)). \tag{4.35}$$

Let r be a constant. Then $\xi(r) = 0$. Hence by (2.16) and (2.17), equation (4.35) implies

$$\lambda = -\alpha^2 \rho n(n - 1).$$

Thus, we state the following result:

Theorem 4.1. *If an n-dimensional Riemannian CR manifold with constant scalar curvature admitting ρ-Einstein solitons has cyclic η-recurrent Ricci tensor, then the soliton constant is given by $\lambda = -\alpha^2 \rho n(n - 1)$.*

Now we have the following corollary:

Corollary 4.1. *Let the metric of n-dimensional Riemannian CR manifold with constant scalar curvature be a ρ-Einstein soliton. Then we have:*

Values of ρ	Soliton type	Soliton constant	Nature of soliton
$\rho = \frac{1}{2}$	Einstein soliton	$\lambda = -\frac{\alpha^2 n(n-1)}{2}$	Shrinking
$\rho = \frac{1}{n}$	Traceless Ricci soliton	$\lambda = -\alpha^2(n - 1)$	Shrinking
$\rho = \frac{1}{2(n-1)}$	Schouten soliton	$\lambda = -\frac{\alpha^2 n}{2}$	Shrinking
$\rho = 0$	Ricci soliton	$\lambda = 0$	Steady

5. ρ-Einstein solitons on n-dimensional Riemannian CR manifolds with respect to Torse-forming vector field

In this section, we examine the nature of ρ-Einstein soliton on a Riemannian CR manifold with a torse-forming vector field and discuss the nature of solitons for various values of ρ.

Definition 5.1. [32] *On a Riemannian manifold (M, g), a vector field V is said to be torse-forming if*

$$\nabla_{Z_1} V = fZ_1 + \omega(Z_1)V, \tag{5.36}$$

where f is a smooth function, ω is a 1-form and ∇ is the Levi-Civita connection of g.

Considering an n-dimensional Riemannian CR manifold admitting a ρ-Einstien soliton and ξ, the Reeb vector field to be a torse-forming vector field, then (5.36) implies

$$\nabla_{Z_1}\xi = fZ_1 + \omega(Z_1)\xi \tag{5.37}$$

for any $Z_1 \in \chi(M)$. Taking the inner product on (5.37) with ξ gives

$$\eta(\nabla_{Z_1}\xi) = f\eta(Z_1) + \omega(Z_1). \tag{5.38}$$

Using (2.8) in (5.38), we get

$$\omega(Z_1) = -f\eta(Z_1). \tag{5.39}$$

It follows from (5.39), equation (5.37) becomes

$$\nabla_{Z_1}\xi = f(Z_1 - \eta(Z_1)\xi). \tag{5.40}$$

By virtue of (5.40), we have

$$(\pounds_\xi g)(Z_1, Z_2) = 2f[g(Z_1, Z_2) - \eta(Z_1)\eta(Z_2)]. \tag{5.41}$$

In view of (5.41), equation (3.18) leads to

$$S(Z_1, Z_2) = -(f + \lambda - \rho r)g(Z_1, Z_2) + f\eta(Z_1)\eta(Z_2). \tag{5.42}$$

Substituting $Z_1 = Z_2 = \xi$ in (5.42) then utilizing (2.6), (2.14) and (2.17) we obtain

$$\lambda = \alpha^2(n-1)(1-\rho n).$$

Thus, we have:

Theorem 5.1. *If an n-dimensional Riemannian CR manifold of constant scalar curvature admit a ρ-Einstien soliton with torse-forming vector field ξ, then it is an η-Einstein manifold with the soliton constant $\lambda = \alpha^2(n-1)(1-\rho n)$. Further, for particular values of ρ, the nature of solitons can be discussed which is same as Corollary 3.1.*

6. ρ-Einstein solitons on n-dimensional Riemannian CR manifolds satisfying $R(\xi, Z_1) \cdot S = 0$

In this section, we investigate the nature of ρ-Einstein solitons on Riemannian CR manifolds that satisfies Ricci semi-symmetric condition.

Let an n-dimensional Riemannian CR manifold admit a ρ-Einstein soliton that satisfy the condition $R(\xi, Z_1) \cdot S = 0$. Then, we have

$$S(R(\xi, Z_1)Z_2, Z_3) + S(Z_2, R(\xi, Z_1)Z_3) = 0,$$

which by using (2.11) yields

$$-\alpha^2 g(Z_2, Z_1)S(\xi, Z_3) + \alpha^2\eta(Z_2)S(Z_1, Z_3) - \alpha^2 g(Z_3, Z_1)S(Z_2, \xi) + \alpha^2\eta(Z_3)S(Z_2, Z_1) = 0.$$

Substituting $Z_3 = \xi$ in the above equation and utilizing (2.6) and (3.21), we obtain

$$S(Z_1, Z_2) = -(\lambda - \rho r)g(Z_1, Z_2). \tag{6.43}$$

Now substituting $Z_2 = \xi$ in (6.43) and utilizing (2.14) and (2.17) we get

$$\lambda = \alpha^2(n-1)(1-\rho n).$$

Now we state:

Theorem 6.1. *If an n-dimensional Riemannian CR manifold of constant scalar curvature tensor admit a ρ-Einstein soliton and satisfies the condition $R(\xi, Z_1) \cdot S = 0$, then it is an Einstein manifold with the soliton constant $\lambda = \alpha^2(n-1)(1-\rho n)$. Further, for particular values of ρ, the nature of solitons can be discussed which is same as Corollary 3.1.*

7. Conharmonic curvature tensor on n-dimensional Riemannian CR manifolds admitting ρ-Einstein solitons

In this section, we inspect the nature of ρ-Einstein soliton on Riemannian CR manifolds with respect to conharmonic curvature tensor.

On a Riemannian manifold M of dimension n, the conharmonic curvature tensor K is defined by [18]

$$K(Z_1, Z_2)Z_3 = R(Z_1, Z_2)Z_3 + \frac{1}{n-2}\Big[S(Z_1, Z_3)Z_2 - S(Z_2, Z_3)Z_1 + g(Z_1, Z_3)QZ_2 - g(Z_2, Z_3)QZ_1\Big], \tag{7.44}$$

for all $Z_1, Z_2, Z_3 \in \chi(M)$.

Now, considering a conharmonically flat Riemannian CR manifold of dimension n admitting a ρ-Einstein soliton i.e. $K(Z_1, Z_2)Z_3 = 0$. Then from (7.44), we have

$$R(Z_1, Z_2)Z_3 = -\frac{1}{n-2}\Big[S(Z_1, Z_3)Z_2 - S(Z_2, Z_3)Z_1 + g(Z_1, Z_3)QZ_2 - g(Z_2, Z_3)QZ_1\Big].$$

Putting $Z_3 = \xi$ and making use of (2.12), (3.21) and (3.22), the above equation yields

$$-\alpha^2[\eta(Z_2)Z_1 - \eta(Z_1)Z_2] = -\frac{1}{n-2}\Big[(\lambda - \rho r)(\eta(Z_2)Z_1 - \eta(Z_1)Z_2) + \eta(Z_1)QZ_2 - \eta(Z_2)QZ_1\Big]. \tag{7.45}$$

Now substituting $Z_2 = \xi$ in (7.45), equation (7.45) yields

$$QZ_1 = [\lambda - \rho r - \alpha^2(n-2)]Z_1 - [2\lambda - 2\rho r - \alpha^2(n-2)]\eta(Z_1)\xi. \tag{7.46}$$

Applying the inner product on (7.46) with Z_2 leads to

$$S(Z_1, Z_2) = [\lambda - \rho r - \alpha^2(n-2)]g(Z_1, Z_2) - [2\lambda - 2\rho r - \alpha^2(n-2)]\eta(Z_1)\eta(Z_2). \tag{7.47}$$

Now substituting $Z_2 = \xi$ in (7.47) and utilizing (2.6), (2.7) and (2.14), we get

$$\lambda = \alpha^2(n-1) + \rho r. \tag{7.48}$$

Let r be a constant. Then by (2.17), equation (7.48) turns to

$$\lambda = \alpha^2(n-1)(1 - \rho n).$$

Thus, we state:

Theorem 7.1. *If the metric of a conharmonically flat Riemannian CR manifold of dimension n with constant scalar curvature r is a ρ-Einstein soliton, then it is η-Einstein with the soliton constant $\lambda = \alpha^2(n-1)(1 - \rho n)$.*

8. Conformal curvature tensor on n-dimensional Riemannian CR manifolds admitting ρ-Einstein solitons

In this section, we examine the nature of ρ-Einstein soliton on Riemannian CR manifolds with respect to conformal curvature tensor.

On a Riemannian manifold M of dimension n, the conformal curvature tensor C is defined by [21]

$$\begin{aligned} C(Z_1, Z_2)Z_3 &= R(Z_1, Z_2)Z_3 + \frac{1}{n-2}\Big[S(Z_1, Z_3)Z_2 - S(Z_2, Z_3)Z_1 + g(Z_1, Z_3)QZ_2 \\ &\quad -g(Z_2, Z_3)QZ_1\Big] + \frac{r}{(n-1)(n-2)}\Big[g(Z_2, Z_3)Z_1 - g(Z_1, Z_3)Z_2\Big] \end{aligned} \tag{8.49}$$

for all $Z_1, Z_2, Z_3 \in \chi(M)$.

Now, considering a conformally flat Riemannian CR manifold of dimension n admitting a ρ-Einstein soliton i.e. $C(Z_1, Z_2)Z_3 = 0$. Then from (8.49), we have

$$\begin{aligned} R(Z_1, Z_2)Z_3 &= -\frac{1}{n-2}\Big[S(Z_1, Z_3)Z_2 - S(Z_2, Z_3)Z_1 + g(Z_1, Z_3)QZ_2 - g(Z_2, Z_3)QZ_1\Big] \\ &\quad -\frac{r}{(n-1)(n-2)}\Big[g(Z_2, Z_3)Z_1 - g(Z_1, Z_3)Z_2\Big]. \end{aligned}$$

Substituting $Z_3 = \xi$ in the above equation and making use of (2.12), (3.21) and (3.22), the above equation yields

$$\begin{aligned} -\alpha^2[\eta(Z_2)Z_1 - \eta(Z_1)Z_2] &= -\frac{1}{n-2}\Big[(\lambda - \rho r)(\eta(Z_2)Z_1 - \eta(Z_1)Z_2) \\ &\quad +\eta(Z_1)QZ_2 - \eta(Z_2)QZ_1\Big] \\ &\quad -\frac{r}{(n-1)(n-2)}\Big[\eta(Z_2)Z_1 - \eta(Z_1)Z_2\Big]. \end{aligned} \tag{8.50}$$

Now putting $Z_2 = \xi$ in (8.50), equation (8.50) yields

$$QZ_1 = [\lambda - \rho r - 2\alpha^2(n-1)]Z_1 - [2\lambda - 2\rho r - 2\alpha^2(n-1)]\eta(Z_1)\xi. \tag{8.51}$$

Applying the inner product on (8.51) with Z_2 leads to

$$S(Z_1, Z_2) = [\lambda - \rho r - 2\alpha^2(n-1)]g(Z_1, Z_2) - [2\lambda - 2\rho r - 2\alpha^2(n-1)]\eta(Z_1)\eta(Z_2). \tag{8.52}$$

Now substituting $Z_2 = \xi$ in (8.52) and utilizing (2.6), (2.7) and (2.14), we get

$$\lambda = \alpha^2(n-1) + \rho r. \tag{8.53}$$

Let r be a constant. Then, by (2.17), equation (8.53) turns to

$$\lambda = \alpha^2(n-1)(1-\rho n).$$

Thus, we state:

Theorem 8.1. *If the metric of a conformally flat Riemannian CR manifold of dimension n with constant scalar curvature r is a ρ-Einstein soliton, then it is η-Einstein with the soliton constant $\lambda = \alpha^2(n-1)(1-\rho n)$.*

9. Applications

As generalized fixed points of Hamilton's Ricci flow $g_t = -2S$ [16], Ricci solitons are a natural generalization of Einstein metrics on a Riemannian manifold. This evolution equation permits a metric to smooth out irregularities based on the Ricci curvature of the manifold, i.e. expands for negative Ricci curvature and shrinks in positive case. It is a nonlinear diffusion equation comparable to the heat equation for metrics. For ρ-Einstein solitons, we have obtained the steady, expanding, and shrinking conditions.

Ricci solitons, known as quasi-Einstein metrics in physics literature, are of great interest to physicists as they have wide applications in the fields of physics [14], biology, chemistry, [19] and economics [28]. As Ricci solitons are self-similar solutions to the Ricci flow, they were instrumental in resolving the century-old Poincaré conjecture [25, 26]. Additionally, Ricci flow and Ricci solitons play a major role in medical imaging for brain surfaces [31], illustrating their wide-ranging impact.

The study of ρ-Einstein solitons aids in comprehending the geometry and topology of Riemannian manifolds. In general relativity, ρ-Einstein solitons provide models for spacetime metrics with specific properties, helping to understand solutions to the Einstein field equations and contributing to cosmological models and the study of gravitational waves [30]. With respect to Ricci flow, ρ-Einstein solitons model the formation of singularities and provides insights on the nature of singularities that form during the flow, leading to a better understanding of the long-term behaviors in cosmology and the potential fate of the universe.

Further, Dey and Roy [11] discussed some applications of η-Ricci-Bourguignon solitons to general relativity. According to them, symmetry is a fascinating feature of our universe, governed by the laws of nature and applicable to various physical phenomena, including general relativity. Early in the 1800s, Albert Einstein made the discovery of the "Theory of General Relativity." In this theory, the gravitational field is defined by the curvature of space-time, and its source is the energy-momentum tensor. The most effective tools for comprehending general relativity in mathematics are differential geometry and relativistic models. A connected 4-dimensional Lorentzian manifold, a particular subclass of pseudo-Riemannian manifolds with Lorentzian metric g with signature (-,+,+,+), can be used to model the spacetime of general relativity and cosmology. The matter content of spacetime is represented by the energy-momentum tensor and is believed to behave as a fluid with properties such as pressure, density, dynamical and kinematic quantities like acceleration, velocity, vorticity, shear, and expansion [2]. These properties can be better understood by studying the nature of solitons. In this paper, we have examined the soliton constants in various cases and also have discussed on soliton constants for different values of ρ.

Moreover, ρ-Einstein solitons are also useful in classifying compact Riemannian manifolds with prescribed curvature conditions. The conformal curvature tensor (Weyl tensor in four dimensions) quantifies the deviation of a manifold from being conformally flat and remains invariant under conformal transformations of the metric. Studying these solitons in the context of conformal curvature involves examining how they affect the conformal geometry of the manifold. In this article, we have obtained the soliton constants of ρ-Einstein solitons with respect to conformal and conharmonic curvature tensors on Riemannian CR manifold.

In summary, ρ-Einstein solitons are fundamental in understanding manifold geometry, contributing to the study of geometric flows, singularity formation, manifold classification, and various interdisciplinary applications. Their study bridges theoretical mathematics and practical applications, highlighting their significance in both realms.

10. **Conclusion**

In this paper, we have systematically explored the behavior of ρ-Einstein solitons on Riemannian CR manifolds under various geometric conditions. Our analysis demonstrates that such solitons consistently reveal deep connections to η-Einstein and Einstein manifolds structures. Key findings from the Theorems 3.1, 4.1, 5.1, 6.1, 7.1, and 8.1 highlights that, regardless of the specific geometric conditions-whether involving torse-forming vector fields, or conditions like conharmonic or conformal flatness-these solitons share a common characteristic. They are invariably associated with η-Einstein manifolds, and their soliton constant takes the form $\lambda = \alpha^2(n-1)(1-\rho n)$, emphasizing the uniformity of their geometric structure.

Moreover, when additional curvature conditions are imposed, such as the Ricci semi-symmetric condition $R(\xi, Z_1) \cdot S = 0$, the manifold transforms into an Einstein manifold, reinforcing the broader geometric significance of ρ-Einstein solitons.

This study highlights the rich interplay between ρ-Einstein solitons and the curvature properties of Riemannian CR manifolds, contributing to a deeper understanding of their geometry. Through these results, we gain valuable insights into the solitons behavior across various curvature contexts, providing a foundation for future investigations in both theoretical mathematics and practical applications in physics and cosmology.

Acknowledgments. The authors thank the referees for their comments and suggestions to improve the quality of this article.

References

[1] Ahmad, M., Mohd. Bilal, & Gazala. (2024). ρ-Einstein solitons in Lorentzian para-Kenmotsu manifolds. *Results in Nonlinear Analysis*, 7(2), 53–63.

[2] Blaga, A. M., & Taştan, H. M. (2021). Some results on almost η-Ricci-Bourguignon solitons. *Journal of geometry and physics*, 168, 104316.

[3] Bourguignon, J. P. (1981). Ricci curvature and Einstein metrics. *Global differential geometry and global analysis. Lecture notes in Math,* 838, 42-63.

[4] Brickell, F., & Yano, K. (1974). Concurrent vector fields and Minkowski structures. *In Kodai Mathematical Seminar Reports*, 26(1), 22-28.

[5] Chen, B. Y., & Deshmukh, S. (2014). Ricci solitons and concurrent vector fields. *Balkan J. Geom. Appl*, 20(1), 14-25.

[6] Collinson, C. D., & Vaz, E. G. L. R. (1995). Killing pairs constructed from a recurrent vector field. *General Relativity and Gravitation*, 27, 751-759.

[7] Deshmukh, S., Alodan, H., & Al-Sodais, H. (2011). A note on Ricci solitons. *Balkan J. Geom. Appl*, 16(1), 48-55.

[8] Deshmukh, S., & Al-Solamy, F. R. (2014). Conformal vector fields on a Riemannian manifold. *Balkan J. Geom. Appl*, 19(2), 86-93.

[9] Deshmukh, S., & Khan, V. A. (2019). Geodesic vector fields and Eikonal equation on a Riemannian manifold. *Indagationes Mathematicae*, 30(4), 542-552.

[10] Deshmukh, S., Peska, P., & Bin Turki, N. (2020). Geodesic vector fields on a Riemannian manifold. *Mathematics*, 8(1), 137.

[11] Dey, S., & Roy, S. (2022). Characterization of general relativistic spacetime equipped with η-Ricci-Bourguignon soliton. *Journal of Geometry and Physics*, 178, 104578.

[12] Dutta, T., Basu, N., & Bhattacharyya, A. (2016). Conformal Ricci soliton in Lorentzian α-Sasakian manifolds. *Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica*, 55(2), 57-70.

[13] Ghanam, R., & Thompson, G. (2001). Two special metrics with R_{14}-type holonomy. *Classical and Quantum Gravity*, 18(11), 2007-2014.

[14] Graf, W. (2007). Ricci flow gravity. *PMC Physics A*, 1, 1-13.

[15] Hamilton, R. S. (1982). Three-manifolds with positive Ricci curvature. *Journal of Differential geometry*, 17(2), 255-306.

[16] Hamilton, R. S. (1988). The Ricci Flow on Surfaces. *Contemporary Mathematics*, 71, 237-261.

[17] Haseeb, A., & Prasad, R. (2020). Some results on Lorentzian para-Kenmotsu manifolds. *Bulletin of the Transilvania University of Brasov. Series III: Mathematics and Computer Science*, 185-198.

[18] Ishii, Y. (1957). On conharmonic transformations. *Tensor, NS*, 11, 73-80.

[19] Ivancevic, V. G., & Ivancevic, T. T. (2011). Ricci flow and nonlinear reaction–diffusion systems in biology, chemistry, and physics. *Nonlinear Dynamics*, 65, 35-54.

[20] Li, Y., Kumara, H. A., Siddesha, M. S., & Naik, D. M. (2023). Characterization of Ricci Almost Soliton on Lorentzian Manifolds. *Symmetry*, 15(6), 1175.

[21] Mishra, R. S. (1984). Structures on a differentiable manifold and their applications. *Chandrama Prakashan, Allahabad.*

[22] Naik, D. M. (2021). Ricci solitons on Riemannian manifolds admitting certain vector field. *Ricerche di Matematica*, 73(1), 1-16.

[23] Naik, D. M., Fasihi-Ramandi, G., Kumara, H. A., & Venkatesha, V. (2023). Yamabe Solitons and τ-Quasi Yamabe Gradient Solitons on Riemannian Manifolds Admitting Concurrent-Recurrent Vector Fields. *Mathematica Slovaca*, 73(2), 501-510.

[24] Naik, D. M., Venkatesha, V., & Prakasha, D. G. (2019). Certain results on Kenmotsu pseudo-metric manifolds. *Miskolc Mathematical Notes*, 20, 1083-1099.

[25] Perelman, G. (2002). The entropy formula for the Ricci flow and its geometric applications. *http://arXiv.org/abs/math/0211159*. 1-39.

[26] Perelman, G. (2003). Ricci flow with surgery on three manifolds. *http://arXiv.org/abs/math/0303109*. 1-22.

[27] Praveena, M. M., Siddesha, M. S., & Bagewadi, C. S. (2022). Certain results of Ricci solitons on (LCS) manifolds. *International Journal of Maps in Mathematics*, 5(2), 101–111.

[28] Sandhu, R. S., Georgiou, T. T., & Tannenbaum, A. R. (2016). Ricci curvature: An economic indicator for market fragility and systemic risk. *Science advances*, 2(5), e1501495.

[29] Sharma, R. (2018). Gradient Ricci soliton with a conformal vector field. *Journal of Geometry*, 109, 01-07.

[30] Siddiqui, A. N., & Siddiqi, M. D. (2021). Almost Ricci-Bourguignon solitons and geometrical structure in a relativistic perfect fluid spacetime. *Balkan J. Geom. Appl*, 26(2), 126-138.

[31] Wang, Y., Shi, J., Yin, X., Gu, X., Chan, T. F., Yau, S. T., ... & Thompson, P. M. (2011). Brain surface conformal parameterization with the Ricci flow. *IEEE transactions on medical imaging*, 31(2), 251-264.

[32] Yano, K. (1944). 72. On the Torse-forming Directions in Riemannian Spaces. *Proceedings of the Imperial Academy*, 20(6), 340-345.

[33] Yano, K. (1970). Integral Formulas in Riemannian Geometry. *Marcel Dekker, New York.*

[34] Yano, K., & Kon, M. (1985). Structures on manifolds. *World scientific.*

[35] Yoldaş, H. İ. (2020). Certain Results on $N(k)$-Contact Metric Manifolds and Torse-Forming Vector Fields. *Journal of Mathematical Extension*, 15.

[36] Yoldaş, H. İ. (2021). Some soliton types on Riemannian manifolds. *Rom. J. Math. Comput. Sci, 11(2)*, 13-20.

[37] Yoldaş, H. İ. (2022). Remarks on some soliton types with certain vector fields. *Fundamentals of Contemporary Mathematical Sciences*, 3(2), 146-159.

Department of Data Analytics and Mathematical Science, Faculty of Engineering and Technology, Jain (Deemed-to-be University), Global Campus-562112, Karnataka, INDIA.

[1,2]Department of Data Analytics and Mathematical Science, Faculty of Engineering and Technology, Jain (Deemed-to-be University), Global Campus-562112, Karnataka, INDIA.