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SOME CLASSES OF LACUNARY WEAK CONVERGENCE OF SEQUENCES DEFINED BY ORLICZ FUNCTION

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Abstract. In this article, we introduce the notion of difference lacunary weak convergence in sequences defined by an Orlicz function. We examine several algebraic and topological properties and establish some inclusion relationships between these spaces.

Keywords: Weak convergence, Orlicz function, Lacunary convergence.
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1. Introduction

The idea of weak convergence, first proposed by Banach [1], is a foundational concept in functional analysis, offering a framework for understanding the convergence behavior of sequences in infinite-dimensional spaces. Despite its significance, weak convergence has several limitations, particularly when dealing with more complex sequence structures or when finer convergence criteria are required.

In recent years, researchers like Mahanta and Tripathy [15] have advanced the study of vector-valued sequence spaces by exploring new types of convergence and their implications. Their work has contributed to a deeper understanding of the algebraic and topological properties of these spaces and has led to the development of innovative tools and techniques for analyzing convergence in more generalized contexts. This expanding research underscores the

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continuous evolution and refinement of sequence space theory, addressing the shortcomings of traditional weak convergence and meeting the demands of increasingly complex mathematical analysis.

Freedman et al. [7] conducted pioneering research on lacunary sequences, investigating strongly Cesaro summable and strongly lacunary convergent sequences in the context of a general lacunary sequence θ. Their work uncovered significant connections between these two classes of sequences. Following their initial findings, researchers such as Ercan et al. [5], Gumus [8], Dowari, and Tripathy [2, 3] have further explored various aspects of lacunary sequences, broadening our understanding of their properties and applications. More recently, Tamuli and Tripathy [19, 20] have advanced this field by examining generalized difference lacunary weak convergence of sequences. Their study sheds light on new convergence behaviors and enhances the theoretical framework for analyzing lacunary sequences, highlighting the ongoing development and deepening of this area of research.

Motivation: In recent years, the study of weak convergence in Banach [1] spaces has gained significant attention due to its essential role in various areas of functional analysis, including the theory of distribution, optimization, and approximation methods. The concept of weak convergence was introduced by Banach in the early 20th century, specifically in the 1920s. Banach developed the theory of weak convergence while working in the context of Banach spaces, which are complete normed vector spaces. His work laid the foundation for the study of weak convergence in functional analysis. Fatih Nuray [13] investigated lacunary weak statistical convergence. Motivated by this work, we have investigated some classes of lacunary weak convergent of sequences defined by Orlicz function.

Potential Applications: The work done in this article are on weak convergence. The concept of strong convergence implies weak convergence, but not necessarily conversely. Therefore the work done in this article can be applied for other areas of research, and since, it covers a larger class of sequences.

2. Definition and Preliminaries

The concept of the difference sequence space $Z(\Delta)$ was first introduced by Kizmaz [9], defined as follows:

$$Z(\Delta) = \{x = (x_k) : (\Delta x_k) \in X\},$$

where $\Delta x = (\Delta x_k) = (x_k - x_{k+1})$, for all $k \in \mathbb{N}$.

Later, Et and Colak [6] extended this idea by defining generalized difference sequence spaces, expressed as:

$$Z(\Delta^p) = \{x = (x_k) : (\Delta^p x_k) \in X\},$$

for $Z = \ell_\infty$, c, and c_0, where $\Delta^p x_k = \Delta^{p-1} x_k - \Delta^{p-1} x_{k+1}$ and $\Delta^0 x_k = x_k \ \forall \ k \in \mathbb{N}$.The binomial expansion for this generalized difference operator is provided below:

$$\Delta^p x_k = \sum_{v=0}^{p} (-1)^v \begin{pmatrix} p \\ v \end{pmatrix} x_{k+v}, \text{ for all } k \in \mathbb{N}. \tag{2.1}$$

These generalized difference sequence spaces have been further studied by researchers such as Tripathy [16], Tripathy , Et and Altin [17], among others.

Consider a sequence $\theta = (k_s)$ of positive integers, which is termed lacunary if $k_0 = 0, 0 < k_s < k_{s+1}$, and $h_s = k_s - k_{s-1} \to \infty$ as $s \to \infty$. The intervals determined by θ are denoted by $I_s = (k_{s-1}, k_s)$, and $q_s = k_s / k_{s-1} \ \forall \ s \in N$.

According to Freedman et al., the space of lacunary strongly convergent sequence N_θ is defined as follows: [7]

$$N_\theta = \left\{ x : \lim_{s \to \infty} \frac{1}{h_s} \sum_{i \in I_s} |x_i - L| = 0, \text{ for some } L \right\}.$$

An Orlicz function $\mathcal{H} : [0, \infty) \to [0, \infty)$ is defined such that $\mathcal{H}(0) = 0, \mathcal{H}(x) > 0$ for $x > 0$, and $\mathcal{H}(x) \to \infty$, as $x \to \infty$. This function is continuous, non-decreasing, and convex.

Lindenstrauss and Tzafriri [12] introduced the concept of the Orlicz function to define the sequence space

$$\ell_{\mathcal{H}} = \left\{ (x_i) \in \omega : \sum_{i=1}^{\infty} \mathcal{H}\left(\frac{|x_i|}{\rho}\right) < \infty, \text{ for some } \rho > 0 \right\},$$

where ω denotes the class of all sequences. The norm of the sequence space $\ell_{\mathcal{H}}$ is given by

$$||x|| = \inf \left\{ \rho > 0 : \sum_{i=1}^{\infty} \mathcal{H}\left(\frac{|x_i|}{\rho}\right) \leq 1 \right\},$$

which transforms it into a Banach space, commonly referred to as an Orlicz sequence space. Various researchers, including Tripathy and Esi [18], Parashar and Choudhury [14], Tripathy and Mahanta [15], have explored different forms of Orlicz sequence spaces.

Definition 2.1. *A sequence (x_i) in a normed linear space X is called weakly convergent to an element $L \in X$ if*

$$\lim_{i \to \infty} f(x_i - L) = 0, \text{ for all } f \in X',$$

where X' represents the continuous dual of X .

Definition 2.2. *A sequence (x_i) in a normed linear space X is said to be lacunary weakly convergent to $L \in X$ if*

$$\lim_{s\to\infty} \frac{1}{h_s} \sum_{k\in I_s} f(x_i - L) = 0,$$

for all $f \in X'$, where X' is the continuous dual of X. In this context, the notation $\mathcal{D}_\theta^w$ used to denote lacunary weak convergent.

Definition 2.3. *The sequence space $\mathcal{J}$ is termed solid if, for any sequence of scalar (α_i) with $|\alpha_i| \le 1$ for all $i \in \mathbb{N}$, the condition $(x_i) \in \mathcal{J}$ implies $(\alpha_i x_i) \in \mathcal{J}$.*

Definition 2.4. *A sequence space $\mathcal{J} \subset \omega$ referred to as monotone if it includes all pre-images of its step spaces.*

Definition 2.5. *A sequence space $\mathcal{J} \subset \omega$ is known as symmetric if, whenever $(x_i) \in \mathcal{J}$, the permuted sequence $(x_{\pi(i)})$ also belongs to $\mathcal{J}$, where π is a permutation of $\mathbb{N}$.*

Definition 2.6. *A sequence space $\mathcal{J}$ is said to be convergence free, if x is in $\mathcal{J}$ and if $y_k = 0$ whenever $x_k = 0$, then y is in $\mathcal{J}$*

Lemma 2.1. *A sequence space $\mathcal{J}$ being solid does not necessary imply that $\mathcal{J}$ is monotone.*

Definition 2.7. *An Orlicz function $\mathcal{H}$ satisfies the Δ_2- condition if there exists a constant $T > 0$ such that, for each $z \ge 0$*

$$\mathcal{H}(2z) \le T\mathcal{H}(z).$$

3. Main Result

In this section we introduce the following classes of sequences and establish result invloving them.

$$[\mathcal{D}_\theta^w, \mathcal{H}, \Delta^p]_0 = \left\{ x = (x_k) : \lim_{s\to\infty} \frac{1}{h_s} \sum_{k\in I_s} \mathcal{H}\left(\frac{|f(\Delta^p x_k)|}{g} \right) = 0, \quad \text{for some} \quad g > 0 \right\};$$

$$[\mathcal{D}_\theta^w, \mathcal{H}, \Delta^p]_1 = \left\{ x = (x_k) : \lim_{s\to\infty} \frac{1}{h_s} \sum_{k\in I_s} \mathcal{H}\left(\frac{|f(\Delta^p x_k - L)|}{g} \right) = 0, \quad \text{for some} \quad L \text{ and } g > 0 \right\};$$

$$[\mathcal{D}_\theta^w, \mathcal{H}, \Delta^p]_\infty = \left\{ x = (x_k) : \lim_{s\to\infty} \frac{1}{h_s} \sum_{k\in I_s} \mathcal{H}\left(\frac{|f(\Delta^p x_k)|}{g} \right) < \infty, \quad \text{for some} \quad g > 0 \right\}.$$

We state, without proof, the following result.

Theorem 3.1. *The classes of sequences* $[\mathcal{D}_\theta^w, \mathcal{H}, \Delta^p]_0, [\mathcal{D}_\theta^w, \mathcal{H}, \Delta^p]_1$ *and* $[\mathcal{D}_\theta^w, \mathcal{H}, \Delta^p]_\infty$ *are linear spaces.*

Theorem 3.2. *For any Orlicz function* $\mathcal{H}$*,* $[\mathcal{D}_\theta^w, \mathcal{H}, \Delta^p]_\infty$ *is a normed linear space for the given norm*

$$\xi_{\Delta^p}(x) = \sum_{i=1}^{p} |f(x_i)| + \inf\left\{ g > 0 : \sup_s \frac{1}{h_s} \sum_{k \in I_s} \mathcal{H}\left(\frac{|f(\Delta^p x_k)|}{g} \right) \leq 1, s = 1, 2, 3, ... \right\};$$

where the infimum is taken over all $g > 0$.

Proof. Clearly, $\xi_{\Delta^p}(x) = \xi_{\Delta^p}(-x), x = \theta$ implies $\Delta^p x_k = 0$ and as such we have $\mathcal{H}(\theta) = 0$. Therefore $\xi_{\Delta^p}(\theta) = 0$. Conversely support that $\xi_{\Delta^p}(x) = 0$, then

$$\sum_{i=1}^{p} |f(x_i)| + \inf\left\{ g > 0 : \sup_s \frac{1}{h_s} \sum_{k \in I_s} \mathcal{H}\left(\frac{|f(\Delta^p x_k)|}{g} \right) \leq 1, s = 1, 2, 3, ... \right\} = 0.$$

$$\Rightarrow \sum_{i=1}^{p} |f(x_i)| = 0 \text{ and } \inf\left\{ g > 0 : \sup_s \frac{1}{h_s} \sum_{k \in I_s} \mathcal{H}\left(\frac{|f(\Delta^p x_k)|}{g} \right) \leq 1, s = 1, 2, 3, ... \right\} = 0.$$

From the first part we have

$$x_i = \bar{\theta}, for\ i = 1, 2, 3, ..., m. \tag{3.2}$$

where, $\bar{\theta}$ is the zero element. In accordance with this second section, there exists some g_ε $(0 < g_\varepsilon < \varepsilon)$ for a given $\varepsilon > 0$. such that

$$\sup_s \frac{1}{h_s} \sum_{k \in I_s} \mathcal{H}\left(\frac{|f(\Delta^p x_k)|}{g_\varepsilon} \right) \leq 1$$

$$\Rightarrow \sum_{k \in I_s} \mathcal{H}\left(\frac{|f(\Delta^p x_k)|}{g_\varepsilon} \right) \leq 1.$$

Thus,

$$\sum_{k \in I_s} \mathcal{H}\left(\frac{|f(\Delta^p x_k)|}{\varepsilon} \right) \leq \sum_{k \in I_s} \mathcal{H}\left(\frac{|f(\Delta^p x_k)|}{g_\varepsilon} \right) \leq 1.$$

Suppose $\Delta^p x_{c_i} \neq \bar{\theta}$, for each i. Taking $\varepsilon \to 0$, we have $\frac{|f(\Delta^p x_{c_i})|}{\varepsilon} \to \infty$.
It follows that

$$\frac{1}{h_s} \sum_{k \in I_s} \mathcal{H}\left(\frac{|f(\Delta^p x_k)|}{\varepsilon} \right) \to \infty,$$

as $\varepsilon \to 0$, for $c_i \in I_s$. Hence we arrive at a contradiction. Therefore, $\Delta^p x_{c_i} = \bar{\theta}$, for each $i \in N$. Thus $\Delta^p x_k = \bar{\theta}, \forall k \in N$.

Therefore, it follows from (2.1) and (3.2) that $x_k = \bar{\theta}, \forall k \in N$. Hence $x = \theta$.

Next let $g_1, g_2 > 0$ such that

$$\sup_s \frac{1}{h_s} \sum_{k \in I_s} \mathcal{H}\left(\frac{|f(\Delta^p x_k)|}{g_1}\right) \leq 1.$$

and

$$\sup_s \frac{1}{h_s} \sum_{k \in I_s} \mathcal{H}\left(\frac{|f(\Delta^p x_k)|}{g_2}\right) \leq 1.$$

Let $g = g_1 + g_2$, then we have

$$\sup_s \frac{1}{h_s} \sum_{k \in I_s} \mathcal{H}\left(\frac{|f(\Delta^p (x_k + y_k)|}{g}\right) \leq 1.$$

Given that the $g's$ are not negative, we have

$$\xi_{\Delta^p}(x+y) = \sum_{i=1}^{p} |f(x_i+y_i)| + \inf\left\{ g > 0 : \sup_s \frac{1}{h_s} \sum_{k \in I_s} \mathcal{H}\left(\frac{|f(\Delta^p (x_k + y_k))|}{g}\right) \leq 1, s = 1, 2, 3, ... \right\}$$

$\Rightarrow \xi_{\Delta^p}(x + y) \leq \xi_{\Delta^p}(x) + \xi_{\Delta^p}(y)$.

Let $\varphi \neq 0$, and $\varphi \in C$, then

$$\xi_{\Delta^p}(\varphi x) = \sum_{i=1}^{p} |f(\varphi x_i)| + \inf\left\{ g > 0 : \sup_s \frac{1}{h_s} \sum_{k \in I_s} \mathcal{H}\left(\frac{|f(\Delta^p (\varphi x_k))|}{g}\right) \leq 1, s = 1, 2, 3, ... \right\}$$

$$\leq |\varphi| \xi_{\Delta^p}(x).$$

This completes the theorem's proof. □

Theorem 3.3. *The sequence space* $[\mathcal{D}_\theta^w, \mathcal{H}, \Delta^p]_\infty$ *is convex.*

Proof. Consider $(x_k), (y_k) \in [\mathcal{D}_\theta^w, \mathcal{H}, \Delta^p]_\infty$. Then from the definition of the space we can write

$$\lim_{s \to \infty} \frac{1}{h_s} \sum_{k \in I_s} \mathcal{H}\left(\frac{|f(\Delta^p x_k)|}{g_x}\right) < \infty, \quad \text{for some} \quad g_x > 0,$$

$$\lim_{s \to \infty} \frac{1}{h_s} \sum_{k \in I_s} \mathcal{H}\left(\frac{|f(\Delta^p y_k)|}{g_y}\right) < \infty, \quad \text{for some} \quad g_y > 0.$$

Now, for $z = \lambda x + (1 - \lambda) y$ we have to show that

$$\lim_{s \to \infty} \frac{1}{h_s} \sum_{k \in I_s} \mathcal{H}\left(\frac{|f(\Delta^p (\lambda x_k + (1 - \lambda) y_k)|}{g_z}\right) < \infty, \quad \text{for some} \quad g_z > 0$$

Since $\mathcal{H}$ is convex function, we have

$$\mathcal{H}\left(\frac{|f(\Delta^p (\lambda x_k + (1 - \lambda) y_k)|}{g_z}\right) \leq \lambda \mathcal{H}\left(\frac{|f(\Delta^p x_k)|}{g_x}\right) + (1 - \lambda) \mathcal{H}\left(\frac{|f(\Delta^p y_k)|}{g_y}\right),$$

where $g_z = \lambda g_x + (1-\lambda) g_y$

Now, taking the limit $s \to \infty$:

$$\lim_{s\to\infty} \frac{1}{h_s} \sum_{k\in I_s} \mathcal{H}\left(\frac{|f(\Delta^p z_k)|}{g_z}\right) \leq \lambda \lim_{s\to\infty} \frac{1}{h_s} \sum_{k\in I_s} \mathcal{H}\left(\frac{|f(\Delta^p x_k)|}{g_x}\right) + (1-\lambda) \lim_{s\to\infty} \frac{1}{h_s} \sum_{k\in I_s} \mathcal{H}\left(\frac{|f(\Delta^p y_k)|}{g_y}\right)$$

Therefore, $z = \lambda x + (1-\lambda) y \in [\mathcal{D}_\theta^w, \mathcal{H}, \Delta^p]_\infty$.

Hence $[\mathcal{D}_\theta^w, \mathcal{H}, \Delta^p]_\infty$ is convex. □

Theorem 3.4. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Orlicz functions satisfying Δ_2- condition. Then*

(i) $[\mathcal{D}_\theta^w, \mathcal{H}_1, \Delta^p]_\mathcal{G} \subseteq [\mathcal{D}_\theta^w, \mathcal{H}_2.\mathcal{H}_1, \Delta^p]_\mathcal{G}$.

(ii)$[\mathcal{D}_\theta^w, \mathcal{H}_1, \Delta^p]_\mathcal{G} \cap [\mathcal{D}_\theta^w, \mathcal{H}_2, \Delta^p]_\mathcal{G} \subseteq [\mathcal{D}_\theta^w, \mathcal{H}_1 + \mathcal{H}_2, \Delta^p]_\mathcal{G}$, *where* $\mathcal{G} = 0, 1,$ *and* ∞.

Proof. We prove it in the case of $\mathcal{G} = 0$, we will apply same methods to the remaining cases.

(i) Let $(x_k) \in [\mathcal{D}_\theta^w, \mathcal{H}_1, \Delta^p]_0$. Then $\exists\, g > 0$ such that

$$\lim_{s\to\infty} \frac{1}{h_s} \sum_{k\in I_s} \mathcal{H}_1\left(\frac{|f(\Delta^p x_k)|}{g}\right) = 0.$$

Let $0 < \varepsilon < 1$ and $0 < \delta < 1$ such that $\mathcal{H}_2(t) < \varepsilon$, for $0 \leq t < \delta$.
Let $y_k = \mathcal{H}_1\left(\frac{|f(\Delta^p x_k)|}{g}\right)$ and consider

$$\sum_{k\in I_s} \mathcal{H}_2(y_k) = \sum_1 \mathcal{H}_2(y_k) + \sum_2 \mathcal{H}_2(y_k),$$

where the summations are over $y_k > \delta$ in the second summation and over $y_k \leq \delta$ in the first. Since,

$$\frac{1}{h_s} \sum_1 \mathcal{H}_2(y_k) < \mathcal{H}_2(2) \frac{1}{h_s} \sum_1 (y_k), \tag{3.3}$$

for $y_k > \delta$, we have

$$y_k < 1 + \frac{y_k}{\delta}.$$

Given that $\mathcal{H}_2$ is convex and non-decreasing, it follows thatSince, $\mathcal{H}_2$ is non-decreasing and convex, it follows that

$$\mathcal{H}_2(y_k) < \frac{1}{2}\mathcal{H}_2(2) + \frac{1}{2}\mathcal{H}_2\left(\frac{2y_k}{\delta}\right).$$

Since, $\mathcal{H}_2$ satisfies Δ_2- conditions, we have

$$\mathcal{H}_2(y_k) = K \frac{y_k}{\delta} \mathcal{H}_2(2).$$

Hence,

$$\frac{1}{h_s}\sum_{2}\mathcal{H}_2(y_k)\leq max\left(1,K\delta^{-1}\mathcal{H}_2(2)\right)\frac{1}{h_s}\sum_{2}y_k. \tag{3.4}$$

Taking limit as $s\to\infty$, from (3.3) and (3.4) we have

$$(x_k)\in[\mathcal{D}_\theta^w,\mathcal{H}_2.\mathcal{H}_1,\Delta^p]_0.$$

Similar proof can be shown for the other cases.

(ii) The proof is obvious and omitted. □

Taking $\mathcal{H}_1(x)=x$ and $\mathcal{H}_2=\mathcal{H}(x)$ in Theorem 3.4(i) we have the following particular case.

Corollary 3.1. $[\mathcal{D}_\theta^w,\Delta^p]_0\subseteq[\mathcal{D}_\theta^w,\mathcal{H},\Delta^p]_0$

Theorem 3.5. *If $p\geq 1$, then $[\mathcal{D}_\theta^w,\mathcal{H},\Delta^{p-1}]_\mathcal{G}\subset[\mathcal{D}_\theta^w,\mathcal{H},\Delta^p]_\mathcal{G}$ for $\mathcal{G}=0,1,\infty$. In general $[\mathcal{D}_\theta^w,\mathcal{H},\Delta^i]_\mathcal{G}\subset[\mathcal{D}_\theta^w,\mathcal{H},\Delta^p]_\mathcal{G}$ for $i=0,1,2,...,p-1$.*

Proof Let $(x_k)\in[\mathcal{D}_\theta^w,\mathcal{H},\Delta^{p-1}]_0$.

Then we have,

$$\lim_{s\to\infty}\frac{1}{h_s}\sum_{k\in I_s}\mathcal{H}\left(\frac{|f(\Delta^{p-1}x_k)|}{g}\right)=0,\ \text{for some } g>0. \tag{3.5}$$

Given that $\mathcal{H}$ is convex and non-decreasing, it follows that

$$\begin{aligned}\frac{1}{h_s}\sum_{k\in I_s}\mathcal{H}\left(\frac{|f(\Delta^p x_k)|}{2g}\right)&=\frac{1}{h_s}\sum_{k\in I_s}\mathcal{H}\left(\frac{|f(\Delta^{p-1}x_k-\Delta^{p-1}x_{k+1})|}{2g}\right)\\&\leq\left(\frac{1}{h_s}\sum_{k\in I_s}\mathcal{H}\left(\frac{|f(\Delta^{p-1}x_k)|}{g}\right)-\frac{1}{h_s}\sum_{k\in I_s}\mathcal{H}\left(\frac{|f(\Delta^{p-1}x_{k+1})|}{g}\right)\right)\end{aligned}$$

as $s\to\infty$, we have

$$\frac{1}{h_s}\sum_{k\in I_s}\mathcal{H}\left(\frac{|f(\Delta^p x_k)|}{g}\right)=0,\ \text{by (3.5)}$$

which implies $(x_k)\in[\mathcal{D}_\theta^w,\mathcal{H},\Delta^p]_0$.

The remaining cases will proceed in a similar manner.

Proceeding inductively we have, $[\mathcal{D}_\theta^w,\mathcal{H},\Delta^i]_\mathcal{G}\subset[\mathcal{D}_\theta^w,\mathcal{H},\Delta^p]_\mathcal{G}$ and $i=0,1,...,p-1$.

The next example strictly follows the inclusion above.

Example 3.1. *Let $\theta=(2^s)$ be a lacunary sequence and $\mathcal{H}(x)=x$. Consider a sequence $(x_k)=(k^{p-1})$. Then $\Delta^p(x_k)=0,\Delta^{p-1}x_k=(-1)^{m-1}$(m-1)! for all $k\in N$. Therefore $(x_k)\in[\mathcal{D}_\theta^w,\mathcal{H},\Delta^p]_0$ but $(x_k)\notin[\mathcal{D}_\theta^w,\mathcal{H},\Delta^{p-1}]_0$*

Theorem 3.6. *The space $[\mathcal{D}_\theta^w, \mathcal{H}, \Delta^p]_\mathcal{G}$, where, in general, $\mathcal{G} = 0, 1, \infty$ are not solid. The space $[\mathcal{D}_\theta^w, \mathcal{H}]_0$ and $[\mathcal{D}_\theta^w, \mathcal{H}]_\infty$ are solid.*

Proof Let $(x_k) \in [\mathcal{D}_\theta^w, \mathcal{H}]_0$.

Then there exists $g > 0$ such that

$$\lim_{s\to\infty} \frac{1}{h_s} \sum_{k\in I_s} \mathcal{H}\left(\frac{|f(x_k)|}{g}\right) = 0.$$

Let (γ_k) be a sequence of scalars such that $|\gamma_k| \leq 1$. Then for each s we can write,

$$\begin{aligned} \frac{1}{h_s} \sum_{k\in I_s} \mathcal{H}\left(\frac{|f(\gamma_k x_k)|}{g}\right) &\leq \frac{1}{h_s} \sum_{k\in I_s} \mathcal{H}\left(\frac{|f(x_k)|}{g}\right) \\ &\Rightarrow \lim_{s\to\infty} \frac{1}{h_s} \sum_{k\in I_s} \mathcal{H}\left(\frac{|f(\gamma_k x_k)|}{g}\right) = 0. \\ &\Rightarrow (\gamma_k \alpha_k) \in [\mathcal{D}_\theta^w, \mathcal{H}]_0. \end{aligned} \tag{3.6}$$

From the above inequality (3.6) it follows that $[\mathcal{D}_\theta^w, \mathcal{H}]_\infty$ is solid.

To show that the spaces $[\mathcal{D}_\theta^w, \mathcal{H}, \Delta^p]_1$, $[\mathcal{D}_\theta^w, \mathcal{H}, \Delta^p]_\infty$ are not solid, in general, we illustrate the following examples.

Example 3.2. *Consider the function $f(x) = x$, $\forall\ x \in R$, and let $X = R$, with $p = 1$,. Let us consider the sequence (x_k), defined by $x_k = k$, $\forall\ k \in N$. Let $\mathcal{H}(x) = x^r$, $r \geq 1$ and the lacunary sequence $\theta = (2^s)$. Then $(x_k) \in [\mathcal{D}_\theta^w, \mathcal{H}, \Delta^p]_1$ and $(x_k) \in [\mathcal{D}_\theta^w, \mathcal{H}, \Delta^p]_\infty$. Let $(\gamma_k) = ((-1)^k)$, then $(\gamma_k x_k) \notin [\mathcal{D}_\theta^w, \mathcal{H}, \Delta^p]_1$ and $(\gamma_k x_k) \notin [\mathcal{D}_\theta^w, \mathcal{H}, \Delta^p]_\infty$.*

We consider the following example to show that $[\mathcal{D}_\theta^w, \mathcal{H}, \Delta^p]_0$ is not solid in general.

Example 3.3. *Let $X = R$ and consider the function $f(x) = x$, $\forall\ x \in R$. let $p = 1$, Let us now consider the sequence (x_k), which is defined as $x_k = 1$, $\forall\ k \in N$. Assume that $\mathcal{H}(x) = x^r$, $r = 2$, and that the lacunary sequence is $\theta = (2^s)$. Let $(\gamma_k) = ((-1)^k), \forall\ k \in N$. Then, $(\gamma_k x_k) \notin [\mathcal{D}_\theta^w, \mathcal{H}, \Delta^p]_0$.*

Thus, the set $[\mathcal{D}_\theta^w, \mathcal{H}, \Delta^p]_0$ is not solid.

The following result is a consequence of Lemma 1 and Theorem 6.

Corollary 3.2. *The spaces $[\mathcal{D}_\theta^w, \mathcal{H}]_0$ and $[\mathcal{D}_\theta^w, \mathcal{H}]_\infty$ are monotone.*

Result 1. *The space $[\mathcal{D}_\theta^w, \mathcal{H}, \Delta^p]_0$ is not convergence free.*

Proof The following example makes it obvious.

Example 3.4. *Let $p = 1, \mathcal{H} = x$ and $f(x) = x$. Consider a lacunary sequence $\theta = (2^s)$. Consider a sequence (x_k) which is define as*

$$x_k = \frac{1}{2}\left(1 - (-1)^k\right)$$

Then, $(x_k) \in [\mathcal{D}_\theta^w, \mathcal{H}, \Delta^p]_0$. Consider the sequence (y_k) defined as

$$x_k = \begin{cases} k, & \text{if } k \text{ is odd} \\ 0, & \text{if } k \text{ is even.} \end{cases}$$

Then, $(y_k) \notin [\mathcal{D}_\theta^w, \mathcal{H}, \Delta^p]_0$.

Result 2. *The spaces $[\mathcal{D}_\theta^w, \mathcal{H}, \Delta^p]_\mathcal{G}$, where $\mathcal{G} = 0, 1, \infty$ are not symmetric in general.*

The following example is given to support the previous result.

Example 3.5. *Let $p = 1$, let $X = \mathbb{R}$, and the function $f(x) = x$, $\forall x \in \mathbb{R}$, be considered. Let $\mathcal{H}(x) = x^2$, and a lacunary sequence $\theta = (2^s)$. Considering the sequence (x_k) where $(x_k) \in [\mathcal{D}_\theta^w, \mathcal{H}, \Delta^p]_0$, define it as:*

$$x_k = \begin{cases} 1 & \text{if } k = 2^m \text{for some } m \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

After rearranging the sequence (x_k) as follows, let (y_k) be considered:

$$y_k = (x_1, x_2, x_4, x_3, x_8, \dots)$$

Then, $(y_k) \notin [\mathcal{D}_\theta^w, \mathcal{H}, \Delta^p]_\mathcal{G}$, where $\mathcal{G} = 0, 1, \infty$.
$[\mathcal{D}_\theta^w, \mathcal{H}, \Delta^p]_\mathcal{G}$, where $\mathcal{G} = 0, 1, \infty$ are not symmetric in general.

4. Conclusion

In this paper, we have introduced and studied the concept of difference lacunary weak convergence in sequences defined by an Orlicz function. Through our exploration, we have thoroughly examined the algebraic and topological properties of these sequences, providing a foundational understanding of their structure and behavior. Additionally, we have established several key inclusion relationships between these newly defined spaces and other known sequence spaces, further enriching the framework within which these sequences operate. Our findings contribute to the broader field of functional analysis, particularly in the study of sequence spaces and Orlicz functions, offering new insights and potential avenues for future research

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References

[1] Banach, S. (1932). Theorie des Operations Lintaires.

[2] Dowari, P. J., & Tripathy, B. C. (2020). Lacunary difference sequences of complex uncertain variables. Methods of Functional Analysis and Topology, 26(4), 327–340.

[3] Dowari, P. J., & Tripathy, B. C. (2021). Lacunary convergence of sequences of complex uncertain variables. Malaysian J. Math. Sci, 15(1), 91-108.

[4] Demirci, I. A., Kişi, Ö., & Gurdal, M. (2024). Lacunary A-statistical convergence of fuzzy variable sequences via Orlicz function. Maejo International Journal of Science and Technology, 18(1), 88-100.

[5] Ercan, S., Altin, A., & Bektas, C. A. (2020). On lacunary weak statistical convergence of order α. Commun. Stat.-Theory. Meth, 49(7), 1653-1664.

[6] Et, M., & Colack, R. (1995). On some generalized difference sequence spaces. Soochow J. Math, 21(4), 377-386.

[7] Freedman, A. R., Sember, J. J. & Raphael, M. (1978). Some Cesàro-type summability spaces. Proceedings of the London Mathematical Society, 3(3), 508-520.

[8] Gumus, H. (2015). Lacunary Weak I-Statistical convergence. Gen. Math. Notes, 28(10), 50-58.

[9] Kizmaz, H. (1981). On certain sequence spaces. Canadian Math. Bull, 24, 169-176.

[10] Kişi, Ö., & Gurdal, M.(2022). Orlicz-lacunary convergent triple sequences and ideal convergence. Communication Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics, 71(2), 681-600.

[11] Kişi, Ö., & Ünal, H.K. (2021). Rough statistical convergence of difference double sequences in normed linear spaces. Honam Mathematical Journal, 43(1), 47-58.

[12] Lindenstrauss, J., & Tzafriri, L. (1971). On Orlicz sequence spac. Israel J. Math, 101, 379-390.

[13] Nuray, F.(2011). Lacunary weak statistical convergence. Mathematica Bohemica, 136(3), 259-268.

[14] Parashar, S. D., & Choudhary, B. (1994). Sequence spaces defined by Orlicz functions. Israel J. Math, 25, 419-428.

[15] Tripathy, B. C., & Mahanta, S. (1994). On a class of sequences related to the l^p space defined by Orlicz functions. Soochow journal of Mathematics, 25(4), 419-428.

[16] Tripathy, B. C. (2004). Generalized difference paranormed statistically convergent sequence space. Indian J. Pure Appl. Math, 35(5), 655-663.

[17] Tripathy, B. C. (2003). Generalized difference sequence spaces defined by Orlic function in a locally convex space. J. Analysis and Appl, 1(3), 175-192.

[18] Tripathy, B. C., & Esi, A. (2005). Generalized lacunary difference sequence spaces defined by Orlicz functions. Journal Math. Society of the Philippines, 28, 50-57.

[19] Tamuli, B., & Tripathy, B. C. (2024). Generalized difference lacunary weak convergence of sequences. Sahand Communication in Math Analysis, 21(2), 195-206 .

[20] Tamuli, B., & Tripathy, B. C. (2024). Lacunary weak convergence of sequences defined by Orlicz function. J. Appl. Anal, https://doi.org/10.1515/jaa-2024-0027

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