International Journal of Maps in Mathematics

*Volume **8**, Issue **2**, 2025, Pages:**681-701***

E-ISSN: 2636-7467

www.simadp.com/journalmim

GE-ALGEBRAS WITH NORMS

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Abstract. In this paper, we introduce and study the concept of normed GE-algebras, an extension of GE-algebras equipped with a GE-norm, which provides a framework to measure the magnitude of algebraic elements. We define the magnitude function and explore its properties in the context of GE-algebras. Through theorems and propositions, we examine the behavior of sequences in these normed structures, demonstrating convergence properties, quasi-metrics, and the relationship between norms and algebraic operations. We also establish the connection between normed GE-algebras and their product spaces, as well as the implications for convergent sequences and limit uniqueness. Finally, we generalize these results to mappings between normed GE-algebras and investigate the implications of GE-morphisms in preserving convergence behavior.

Keywords: GE-norm, Normed GE-algebra, Magnitude, Convergent, Limit.

2020 Mathematics Subject Classification: 03G25, 06F35.

1. Introduction

In the 1950s, Hilbert algebras were introduced by L. Henkin and T. Skolem as a means to investigate non-classical logics, particularly intuitionistic logic. As demonstrated by A. Diego, these algebras belong to the category of locally finite varieties, a fact highlighted in [6]. Over time, a community of scholars developed the theory of Hilbert algebras, as evidenced by works such as [4, 5, 7]. In the broader scope of algebraic structures, the process of generalization is of utmost importance. Y. B. Jun et al. introduced the concept of

Received: 2024.09.11 *Revised: 2025.05.09* *Accepted: 2025.07.24*

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BH-algebras as a generalization of BCH/BCI/BCK-algebras and investigated its important properties in [9]. R. H. Abass introduced the notions of norm and distance in BH-algebras and given some basic properties in normed BH-algebras in [1].

The introduction of GE-algebras, proposed by R. K. Bandaru et al. as an extension of Hilbert algebras, marked a significant step in this direction. This advancement led to the examination of various properties, as explored in [2]. The evolution of GE-algebras was greatly influenced by filter theory. In light of this, R. K. Bandaru et al. introduced the concept of belligerent GE-filters in GE-algebras, closely investigating its attributes as documented in [3]. Generalized algebraic structures, such as GE-algebras, offer a broad framework to study a variety of algebraic and topological properties.

The concept of norms has a rich history in mathematics, originating in the study of vector spaces and Banach algebras, where norms quantify the size of elements and induce metric spaces [14]. In logical algebras, norms have been adapted to capture algebraic properties, as seen in normed BCK/BCI-algebras [8], where norms relate to implication operations, and in MV-algebras, where norms support quantitative semantics [11]. Unlike these structures, normed GE-algebras, introduced in this paper, define a GE-norm tailored to the non-commutative binary operation of GE-algebras, inducing quasi-metric spaces rather than metric spaces. This generalization extends the applicability of norms to non-linear algebraic systems, offering a novel framework for studying convergence and topological properties in generalized algebraic settings.

In this context, normed GE-algebras represent an important class that combines the algebraic properties of GE-algebras with a GE-norm, enabling the measurement of the magnitude of elements. This paper aims to extend the classical understanding of algebraic norms by introducing the concept of a GE-norm, defined as a real-valued mapping that satisfies specific properties akin to a norm in conventional algebraic systems. We begin by formally defining the notion of a GE-norm and explore its compatibility with the underlying operations of the GE-algebra. Following this, we investigate the properties of the magnitude function derived from the norm and establish a series of results on its behavior. Notably, we prove that normed GE-algebras induce quasi-metric spaces and that these spaces generate a T_0-topology. In subsequent sections, we delve into the properties of convergent sequences in normed GE-algebras, proving the uniqueness of limits and characterizing the boundedness of certain subsequences. We also establish several results concerning the preservation of normed structures under GE-morphisms, culminating in a product theorem for GE-algebras.

This work contributes to the ongoing development of generalized algebraic systems, providing both theoretical insights and practical tools for further exploration of algebraic norms, convergence, and topological spaces in GE-algebras.

2. Preliminaries

Definition 2.1 ([2]). *A GE-algebra is a non-empty set X with a constant 1 and a binary operation "$*$" satisfying the following axioms:*

(GE1) $\mathfrak{a} * \mathfrak{a} = 1$,

(GE2) $1 * \mathfrak{a} = \mathfrak{a}$,

(GE3) $\mathfrak{a} * (\mathfrak{b} * \mathfrak{c}) = \mathfrak{a} * (\mathfrak{b} * (\mathfrak{a} * \mathfrak{c}))$

for all $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in X$.

In a GE-algebra X, a binary relation "$\leq_X$" is defined by

$$(\forall \mathfrak{a}, \mathfrak{b} \in X)\,(\mathfrak{a} \leq_X \mathfrak{b} \;\Leftrightarrow\; \mathfrak{a} * \mathfrak{b} = 1)\,. \tag{2.1}$$

Definition 2.2 ([2, 3]). *A GE-algebra X is said to be*

- *transitive if it satisfies:*

$$(\forall \mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in X)\,(\mathfrak{a} * \mathfrak{b} \leq_X (\mathfrak{c} * \mathfrak{a}) * (\mathfrak{c} * \mathfrak{b}))\,. \tag{2.2}$$

- *commutative if it satisfies:*

$$(\forall \mathfrak{a}, \mathfrak{b} \in X)\,((\mathfrak{a} * \mathfrak{b}) * \mathfrak{b} = (\mathfrak{b} * \mathfrak{a}) * \mathfrak{a})\,. \tag{2.3}$$

Proposition 2.1 ([2]). *Every GE-algebra X satisfies the following items.*

$$\mathfrak{a} * 1 = 1. \tag{2.4}$$

$$\mathfrak{a} * (\mathfrak{a} * \mathfrak{b}) = \mathfrak{a} * \mathfrak{b}. \tag{2.5}$$

$$\mathfrak{a} \leq_X \mathfrak{b} * \mathfrak{a}. \tag{2.6}$$

$$\mathfrak{a} * (\mathfrak{b} * \mathfrak{c}) \leq_X \mathfrak{b} * (\mathfrak{a} * \mathfrak{c}). \tag{2.7}$$

$$1 \leq_X \mathfrak{a} \;\Rightarrow\; \mathfrak{a} = 1. \tag{2.8}$$

$$\mathfrak{a} \leq_X (\mathfrak{b} * \mathfrak{a}) * \mathfrak{a}. \tag{2.9}$$

$$\mathfrak{a} \leq_X (\mathfrak{a} * \mathfrak{b}) * \mathfrak{b}. \tag{2.10}$$

$$\mathfrak{a} \leq_X \mathfrak{b} * \mathfrak{c} \;\Leftrightarrow\; \mathfrak{b} \leq_X \mathfrak{a} * \mathfrak{c}. \tag{2.11}$$

for all $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in X$. *If* X *is transitive, then*

$$\mathfrak{a} \leq_X \mathfrak{b} \;\Rightarrow\; \mathfrak{c} * \mathfrak{a} \leq_X \mathfrak{c} * \mathfrak{b},\; \mathfrak{b} * \mathfrak{c} \leq_X \mathfrak{a} * \mathfrak{c}. \tag{2.12}$$

$$\mathfrak{a} * \mathfrak{b} \leq_X (\mathfrak{b} * \mathfrak{c}) * (\mathfrak{a} * \mathfrak{c}). \tag{2.13}$$

$$\mathfrak{a} \leq_X \mathfrak{b},\; \mathfrak{b} \leq_X \mathfrak{c} \;\Rightarrow\; \mathfrak{a} \leq_X \mathfrak{c}. \tag{2.14}$$

for all $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in X$.

Definition 2.3 ([12])**.** *Let* $(X, *_X, 1_X)$ *and* $(Y, *_Y, 1_Y)$ *be GE-algebras. A mapping* $f : X \to Y$ *is called a GE-morphism if it satisfies:*

$$(\forall \varrho_1, \varrho_2 \in X)(f(\varrho_1 *_X \varrho_2) = f(\varrho_1) *_Y f(\varrho_2)). \tag{2.15}$$

Let $\mathbb{X}_\alpha := \{(X_\alpha, *_\alpha, 1_\alpha) \mid \alpha \in \Lambda\}$ be a family of GE-algebras where Λ is an index set. Let $\prod \mathbb{X}_\alpha$ be the set of all mappings $\eth : \Lambda \to \bigcup\limits_{\alpha \in \Lambda} X_\alpha$ with $\eth(\alpha) \in X_\alpha$, that is,

$$\prod \mathbb{X}_\alpha := \left\{ \eth : \Lambda \to \bigcup_{\alpha \in \Lambda} X_\alpha \mid \eth(\alpha) \in X_\alpha, \alpha \in \Lambda \right\}. \tag{2.16}$$

We define a binary operation $\circledast$ on $\prod \mathbb{X}_\alpha$ and the constant $\mathbf{1}$ by

$$\left(\forall \eth, f \in \prod \mathbb{X}_\alpha\right) ((\eth \circledast f)(\alpha) = \eth(\alpha) *_\alpha f(\alpha)) \tag{2.17}$$

and $\mathbf{1}(\alpha) = 1_\alpha$, respectively, for every $\alpha \in \Lambda$. It is routine to verify that $(\prod \mathbb{X}_\alpha, \circledast, \mathbf{1})$ is a GE-algebra, which is called the *product GE-algebra* (see [3]).

3. Normed GE-algebras

In what follows, let $\mathbb{X} := (X, *, 1_X)$ and $\mathbb{R}$ be a GE-algebra and the set of all real numbers, respectively, unless otherwise specified. In the absence of ambiguity, the GE-algebra $\mathbb{X} := (X, *, 1_X)$ can simply be represented by $\mathbb{X}$.

Definition 3.1. *A GE-norm on* $\mathbb{X} := (X, *, 1_X)$ *is defined to be a mapping* $|| \cdot || : X \to \mathbb{R}$ *that satisfies:*

$$(\forall \varrho \in X)\,(||\varrho|| \geq 0)\,, \tag{3.18}$$

$$(\forall \varrho \in X)\,(||\varrho|| = 0 \;\Leftrightarrow\; \varrho = 1_X)\,, \tag{3.19}$$

$$(\forall \varrho, \varsigma, \varpi \in X)\,(||\varrho * \varpi|| \leq ||\varrho * \varsigma|| + ||\varsigma * \varpi||)\,. \tag{3.20}$$

The GE-norm defined above shares similarities with classical norms, such as those in vector spaces or Banach algebras, where non-negativity and zero norm at the identity (conditions (3.18) and (3.19)) ensure a measure of magnitude [14]. However, it differs significantly due to the non-linear, non-commutative structure of GE-algebras. Unlike classical norms, which induce symmetric metrics, the GE-norm's triangle-like inequality (condition (3.20)) is tailored to the binary operation "$*$", leading to a quasi-metric space (Example 3.3). This formulation is chosen to align with the GE-algebra's axioms (GE1–GE3) and partial order $\leq_X$, ensuring compatibility with algebraic operations and enabling the study of convergence in non-commutative settings.

A *normed GE-algebra* is a GE-algebra $\mathbb{X} := (X, *, 1_X)$ equipped with a GE-norm $||\cdot|| : X \to \mathbb{R}$ and it is denoted by $(\mathbb{X}, ||\cdot||)$.

Given a GE-algebra $\mathbb{X} := (X, *, 1_X)$, if there exists a function $||\cdot||$ mapping elements of X to non-negative real numbers satisfying the conditions (3.19) and (3.20), then $(\mathbb{X}, ||\cdot||)$ is a normed GE-algebra.

Example 3.1. *For every GE-algebra* $\mathbb{X} := (X, *, 1_X)$*, define a mapping:*

$$||\cdot|| : X \to \mathbb{R},\ \varrho \mapsto \begin{cases} 0 & \text{if } \varrho = 1_X, \\ \varrho_0 & \text{otherwise}, \end{cases}$$

where ϱ_0 *is a positive real number. Then* $||\cdot||$ *is a GE-norm on* $\mathbb{X} := (X, *, 1_X)$*, and so* $(\mathbb{X}, ||\cdot||)$ *is a normed GE-algebra.*

In normed GE-algebras, the "GE-norm" often provides a way to measure the "magnitude" of elements in a way that is compatible with the algebraic operation "$*$".

By the *magnitude* of a normed GE-algebra $(\mathbb{X}, ||\cdot||)$, we mean a real-valued function $\eth$ on $X \times X$ defined as follows:

$$(\forall \varrho, \varsigma \in X)\,(\eth(\varrho, \varsigma) = ||\varrho * \varsigma||)\,. \tag{3.21}$$

We say $\eth(\varrho, \varsigma)$ is the magnitude of (ϱ, ς).

Proposition 3.1. *The magnitude* $\eth : X \times X \to \mathbb{R}$ *of* $(\mathbb{X}, ||\cdot||)$ *has the following assertions:*

$$\eth(\varrho, \varsigma) \geq 0,\ \eth(\varrho, \varrho) = 0 = \eth(\varrho, 1_X), \tag{3.22}$$

$$\eth \text{ satisfies the triangle inequality,} \tag{3.23}$$

$$\eth(1_X, \varrho) = 0 \ \Rightarrow \ \varrho = 1_X, \tag{3.24}$$

$$\varrho \leq_X \varsigma \ \Rightarrow \ \eth(1_X, \varsigma) \leq \eth(1_X, \varrho), \tag{3.25}$$

$$\eth(\varrho, \varsigma) \leq \eth(1_X, \varsigma), \tag{3.26}$$

$$\eth(\varsigma, \varrho * \varpi) \leq \eth(\varrho, \varsigma * \varpi), \tag{3.27}$$

$$\eth(\varsigma * \varrho, \varrho) \leq \eth(1_X, \varrho), \tag{3.28}$$

$$\eth(\varrho * \varsigma, \varsigma) \leq \eth(1_X, \varrho), \tag{3.29}$$

for all $\varrho, \varsigma, \varpi \in X$.

Proof. Let $\varrho, \varsigma, \varpi \in X$. Then (3.22) and (3.23) are clear by (3.18), (3.19) and (3.19). The combination of (GE2) and (3.19) induces (3.24). Let $\varrho, \varsigma \in X$ be such that $\varrho \leq_X \varsigma$. Then $\varrho * \varsigma = 1$, and so

$$\begin{aligned} \eth(1_X, \varsigma) &\overset{(3.21)}{=} ||1_X * \varsigma|| \overset{(3.20)}{\leq} ||1_X * \varrho|| + ||\varrho * \varsigma|| = ||1_X * \varrho|| + ||1|| \\ &\overset{(3.19)}{=} ||1_X * \varrho|| + 0 = ||1_X * \varrho|| \overset{(3.21)}{=} \eth(1_X, \varrho). \end{aligned}$$

Hence (3.25) is valid. By the combination of (GE2), (2.6) and (3.25), we have (3.26). Using (GE2), (2.7) and (3.25), we get (3.27), (3.28) and (3.29). □

Proposition 3.2. *If* $\mathbb{X} := (X, *, 1_X)$ *is transitive, then the magnitude* $\eth : X \times X \to \mathbb{R}$ *of* $(\mathbb{X}, ||\cdot||)$ *satisfies:*

$$(\forall \varrho, \varsigma, \varpi \in X) \left(\eth(\varsigma * \varpi, \varrho * \varpi) \leq \eth(\varrho, \varsigma) \right). \tag{3.30}$$

Proof. Using (GE2), (2.13) and (3.25), we obtain (3.30). □

The following example shows that any magnitude $\eth : X \times X \to \mathbb{R}$ of $(\mathbb{X}, ||\cdot||)$ does not satisfy the following.

$$(\forall \varrho, \varsigma \in X) \left(\eth(\varrho, \varsigma) = 0 = \eth(\varsigma, \varrho) \ \Rightarrow \ \varrho = \varsigma \right). \tag{3.31}$$

Example 3.2. *Consider a non-commutative GE-algebra* $\mathbb{X} := (X, *, 1_X)$*, where* $X = \{1_X, \ell_1, \ell_2, \ell_3, \ell_4\}$ *and a binary operation "$*$" is given in the following table:*

$*$	1_X	ℓ_1	ℓ_2	ℓ_3	ℓ_4
1_X	1_X	ℓ_1	ℓ_2	ℓ_3	ℓ_4
ℓ_1	1_X	1_X	ℓ_2	ℓ_3	1_X
ℓ_2	1_X	ℓ_4	1_X	1_X	ℓ_4
ℓ_3	1_X	ℓ_1	1_X	1_X	ℓ_1
ℓ_4	1_X	1_X	ℓ_2	ℓ_3	1_X

Define a mapping:

$$||\cdot|| : X \to \mathbb{R},\ \varrho \mapsto \begin{cases} 0 & \text{if } \varrho = 1_X, \\ \varrho_0 & \text{otherwise}, \end{cases}$$

where ϱ_0 *is a positive real number. Then* $||\cdot||$ *is a GE-norm on* $\mathbb{X} := (X, *, 1_X)$*, and so* $(\mathbb{X}, ||\cdot||)$ *is a normed GE-algebra. We can observe that* $\eth(\ell_2, \ell_3) = ||\ell_2 * \ell_3|| = ||1_X|| = 0$ *and* $\eth(\ell_3, \ell_2) = ||\ell_3 * \ell_2|| = ||1_X|| = 0$*. Therefore* $\eth(\ell_2, \ell_3) = 0 = \eth(\ell_3, \ell_2)$*. But* $\ell_2 \neq \ell_3$*. Hence* (3.31) *is not valid.*

Theorem 3.1. *If* $\mathbb{X} := (X, *, 1_X)$ *is a commutative GE-algebra, then its magnitude* $\eth : X \times X \to \mathbb{R}$ *satisfies* (3.31)*.*

Proof. Let $\mathbb{X} := (X, *, 1_X)$ be a commutative GE-algebra. Then $(X, \leq_X)$ is antisymmetric. Let $\varrho, \varsigma \in X$ be such that $\eth(\varrho, \varsigma) = 0 = \eth(\varsigma, \varrho)$. Then $||\varrho * \varsigma|| = 0$ and $||\varsigma * \varrho|| = 0$, which imply from (3.19) that $\varrho * \varsigma = 1$ and $\varsigma * \varrho = 1$, i.e., $\varrho \leq_X \varsigma$ and $\varsigma \leq_X \varrho$. Hence $\varrho = \varsigma$, and so (3.31) is valid. □

The following example shows that any magnitude $\eth : X \times X \to \mathbb{R}$ of $(\mathbb{X}, ||\cdot||)$ does not satisfy the following.

$$(\forall \varrho, \varsigma \in X)\,(\eth(\varrho, \varsigma) = \eth(\varsigma, \varrho))\,. \tag{3.32}$$

Example 3.3. *Consider a non-commutative GE-algebra* $\mathbb{X} := (X, *, 1_X)$*, where* $X = \{1_X, \ell_1, \ell_2, \ell_3\}$ *and a binary operation "$*$" is given in the following table:*

$*$	1_X	ℓ_1	ℓ_2	ℓ_3
1_X	1_X	ℓ_1	ℓ_2	ℓ_3
ℓ_1	1_X	1_X	1_X	1_X
ℓ_2	1_X	ℓ_1	1_X	1_X
ℓ_3	1_X	ℓ_1	ℓ_2	1_X

Define a mapping:

$$||\cdot|| : X \to \mathbb{R},\ \varrho \mapsto \begin{cases} 0 & \text{if } \varrho = 1_X, \\ \varrho_0 & \text{otherwise,} \end{cases}$$

where ϱ_0 *is a positive real number. Then* $||\cdot||$ *is a GE-norm on* $\mathbb{X} := (X, *, 1_X)$*, and so* $(\mathbb{X}, ||\cdot||)$ *is a normed GE-algebra. We can observe that* $\eth(\ell_2, \ell_3) = ||\ell_2 * \ell_3|| = ||1_X|| = 0$ *and* $\eth(\ell_3, \ell_2) = ||\ell_3 * \ell_2|| = ||\ell_2|| = \varrho_0$*. Therefore* $\eth(\ell_2, \ell_3) \neq \eth(\ell_3, \ell_2)$*. Hence* (3.32) *is not valid.*

Example 3.3 is indicating that the magnitude $\eth : X \times X \to \mathbb{R}$ of $(\mathbb{X}, ||\cdot||)$ cannot be a metric on X, that is, $(X, \eth)$ is not a metric space. But we know that the magnitude $\eth : X \times X \to \mathbb{R}$ of $(\mathbb{X}, ||\cdot||)$ is a quasi metric on X, and thus $(X, \eth)$ is a quasi metric space which generates a T_0-space on X. For the quasi metric $\eth$ on X, we define new real-valued mappings $\eth^{-1}$ and $\eth^{\vee}$ on $X \times X$ as follows:

$$\eth^{-} : X \times X \to \mathbb{R},\ (\varrho, \varsigma) \mapsto \eth(\varsigma, \varrho). \tag{3.33}$$

$$\eth^{\vee} : X \times X \to \mathbb{R},\ (\varrho, \varsigma) \mapsto \max\{\eth(\varrho, \varsigma), \eth^{-}(\varrho, \varsigma)\}. \tag{3.34}$$

It is clear that $\eth^{-}$ and $\eth^{\vee}$ are quasi metrices on X.

The following example illustrates the quasi metrices $\eth^{-}$ and $\eth^{\vee}$ on X.

Example 3.4. *Consider the normed GE-algebra* $(\mathbb{X}, ||\cdot||)$ *in Example 3.3. Then*

$$\begin{aligned} X \times X = \{&(1_X, 1_X), (1_X, \ell_1), (1_X, \ell_2), (1_X, \ell_3), (\ell_1, 1_X), (\ell_1, \ell_1), \\ &(\ell_1, \ell_2), (\ell_1, \ell_3), (\ell_2, 1_X), (\ell_2, \ell_1), (\ell_2, \ell_2), (\ell_2, \ell_3), \\ &(\ell_3, 1_X), (\ell_3, \ell_1), (\ell_3, \ell_2), (\ell_3, \ell_3)\} \end{aligned}$$

and the binary operation "$\circledast$" on $X \times X$ *is given by Table 3.1.*

TABLE 3.1. Tabular representation for the operation "$\circledast$" on $X \times X$

$\circledast$	$(1_X, 1_X)$	$(1_X, \ell_1)$	$(1_X, \ell_2)$	$(1_X, \ell_3)$
$(1_X, 1_X)$	$(1_X, 1_X)$	$(1_X, \ell_1)$	$(1_X, \ell_2)$	$(1_X, \ell_3)$
$(1_X, \ell_1)$	$(1_X, 1_X)$	$(1_X, 1_X)$	$(1_X, 1_X)$	$(1_X, 1_X)$
$(1_X, \ell_2)$	$(1_X, 1_X)$	$(1_X, \ell_1)$	$(1_X, 1_X)$	$(1_X, 1_X)$
$(1_X, \ell_3)$	$(1_X, 1_X)$	$(1_X, \ell_1)$	$(1_X, \ell_2)$	$(1_X, 1_X)$
$(\ell_1, 1_X)$	$(1_X, 1_X)$	$(1_X, \ell_1)$	$(1_X, \ell_2)$	$(1_X, \ell_3)$
(ℓ_1, ℓ_1)	$(1_X, 1_X)$	$(1_X, 1_X)$	$(1_X, 1_X)$	$(1_X, 1_X)$
(ℓ_1, ℓ_2)	$(1_X, 1_X)$	$(1_X, \ell_1)$	$(1_X, 1_X)$	$(1_X, 1_X)$
(ℓ_1, ℓ_3)	$(1_X, 1_X)$	$(1_X, \ell_1)$	$(1_X, \ell_2)$	$(1_X, 1_X)$
$(\ell_2, 1_X)$	$(1_X, 1_X)$	$(1_X, \ell_1)$	$(1_X, \ell_2)$	$(1_X, \ell_3)$
(ℓ_2, ℓ_1)	$(1_X, 1_X)$	$(1_X, 1_X)$	$(1_X, 1_X)$	$(1_X, 1_X)$
(ℓ_2, ℓ_2)	$(1_X, 1_X)$	$(1_X, \ell_1)$	$(1_X, 1_X)$	$(1_X, 1_X)$
(ℓ_2, ℓ_3)	$(1_X, 1_X)$	$(1_X, \ell_1)$	$(1_X, \ell_2)$	$(1_X, 1_X)$
$(\ell_3, 1_X)$	$(1_X, 1_X)$	$(1_X, \ell_1)$	$(1_X, \ell_2)$	$(1_X, \ell_3)$
(ℓ_3, ℓ_1)	$(1_X, 1_X)$	$(1_X, 1_X)$	$(1_X, 1_X)$	$(1_X, 1_X)$
(ℓ_3, ℓ_2)	$(1_X, 1_X)$	$(1_X, \ell_1)$	$(1_X, 1_X)$	$(1_X, 1_X)$
(ℓ_3, ℓ_3)	$(1_X, 1_X)$	$(1_X, \ell_1)$	$(1_X, \ell_2)$	$(1_X, 1_X)$

The quasi metrices $\eth^-$ and $\eth^\vee$ on X appear as follows.

$$\eth^-(\varrho, \varsigma) = \begin{cases} 0 & \text{if } (\varrho, \varsigma) \in (X \times X) \setminus A, \\ \varrho_0 & \text{if } (\varrho, \varsigma) \in A, \end{cases}$$

and

$$\eth^\vee(\varrho, \varsigma) = \begin{cases} 0 & \text{if } (\varrho, \varsigma) \in B, \\ \varrho_0 & \text{if } (\varrho, \varsigma) \in (X \times X) \setminus B, \end{cases}$$

where $A = \{(\ell_1, 1_X), (\ell_1, \ell_2), (\ell_1, \ell_3), (\ell_2, 1_X), (\ell_2, \ell_3), (\ell_3, 1_X)\}$ *and*

$$B = \{(1_X, 1_X), (\ell_1, \ell_1), (\ell_2, \ell_2), (\ell_3, \ell_3)\}.$$

Table 3.1 (continued)

$\circledast$	$(\ell_1, 1_X)$	(ℓ_1, ℓ_1)	(ℓ_1, ℓ_2)	(ℓ_1, ℓ_3)
$(1_X, 1_X)$	$(\ell_1, 1_X)$	(ℓ_1, ℓ_1)	(ℓ_1, ℓ_2)	(ℓ_1, ℓ_3)
$(1_X, \ell_1)$	$(\ell_1, 1_X)$	$(\ell_1, 1_X)$	$(\ell_1, 1_X)$	$(\ell_1, 1_X)$
$(1_X, \ell_2)$	$(\ell_1, 1_X)$	(ℓ_1, ℓ_1)	$(\ell_1, 1_X)$	$(\ell_1, 1_X)$
$(1_X, \ell_3)$	$(\ell_1, 1_X)$	(ℓ_1, ℓ_1)	(ℓ_1, ℓ_2)	$(\ell_1, 1_X)$
$(\ell_1, 1_X)$	$(1_X, 1_X)$	$(1_X, \ell_1)$	$(1_X, \ell_2)$	$(1_X, \ell_3)$
(ℓ_1, ℓ_1)	$(1_X, 1_X)$	$(1_X, 1_X)$	$(1_X, 1_X)$	$(1_X, 1_X)$
(ℓ_1, ℓ_2)	$(1_X, 1_X)$	$(1_X, \ell_1)$	$(1_X, 1_X)$	$(1_X, 1_X)$
(ℓ_1, ℓ_3)	$(1_X, 1_X)$	$(1_X, \ell_1)$	$(1_X, \ell_2)$	$(1_X, 1_X)$
$(\ell_2, 1_X)$	$(\ell_1, 1_X)$	(ℓ_1, ℓ_1)	(ℓ_1, ℓ_2)	(ℓ_1, ℓ_3)
(ℓ_2, ℓ_1)	$(\ell_1, 1_X)$	$(\ell_1, 1_X)$	$(\ell_1, 1_X)$	$(\ell_1, 1_X)$
(ℓ_2, ℓ_2)	$(\ell_1, 1_X)$	(ℓ_1, ℓ_1)	$(\ell_1, 1_X)$	$(\ell_1, 1_X)$
(ℓ_2, ℓ_3)	$(\ell_1, 1_X)$	(ℓ_1, ℓ_1)	(ℓ_1, ℓ_2)	$(\ell_1, 1_X)$
$(\ell_3, 1_X)$	$(\ell_1, 1_X)$	(ℓ_1, ℓ_1)	(ℓ_1, ℓ_2)	(ℓ_1, ℓ_3)
(ℓ_3, ℓ_1)	$(\ell_1, 1_X)$	$(\ell_1, 1_X)$	$(\ell_1, 1_X)$	$(\ell_1, 1_X)$
(ℓ_3, ℓ_2)	$(\ell_1, 1_X)$	(ℓ_1, ℓ_1)	$(\ell_1, 1_X)$	$(\ell_1, 1_X)$
(ℓ_3, ℓ_3)	$(\ell_1, 1_X)$	(ℓ_1, ℓ_1)	(ℓ_1, ℓ_2)	$(\ell_1, 1_X)$

Theorem 3.2. *Let $f : X \to Y$ be an onto GE-morphism from a GE-algebra $\mathbb{X} := (X, *, 1_X)$ to a GE-algebra $\mathbb{Y} := (Y, *, 1_Y)$. If $(\mathbb{X}, ||\cdot||)$ is a normed GE-algebra, then so is $(\mathbb{Y}, ||\cdot||)$.*

Proof. Assume that $(\mathbb{X}, ||\cdot||)$ is a normed GE-algebra. Since f is onto, $f^{-1}(\hbar) \neq \emptyset$ for every $\hbar \in Y$. So we can take $||\hbar|| = \inf_{\varrho \in f^{-1}(\hbar)} ||\varrho||$. It is clear that $||\hbar|| \geq 0$. If $||\hbar|| = 0$, then $\inf_{\varrho \in f^{-1}(\hbar)} ||\varrho|| = 0$, and so there exists $\varrho \in X$ such that $||\varrho|| = 0$. Hence $\varrho = 1_X$ which implies that $\hbar = f(\varrho) = f(1_X) = 1_Y$. If $\hbar = 1_Y$, then $||\hbar|| = \inf_{\varrho \in f^{-1}(\hbar)} ||\varrho|| \overset{(3.25)}{=} ||1_X|| = 0$ since $1_X \in f^{-1}(1_Y)$. Let $\hbar, \jmath, \wp \in Y$. Then there exist $\varrho, \varsigma, \varpi \in X$ such that $f(\varrho) = \hbar$, $f(\varsigma) = \jmath$

Table 3.1 (continued)

$\circledast$	$(\ell_2, 1_X)$	(ℓ_2, ℓ_1)	(ℓ_2, ℓ_2)	(ℓ_2, ℓ_3)
$(1_X, 1_X)$	$(\ell_2, 1_X)$	(ℓ_2, ℓ_1)	(ℓ_2, ℓ_2)	(ℓ_2, ℓ_3)
$(1_X, \ell_1)$	$(\ell_2, 1_X)$	$(\ell_2, 1_X)$	$(\ell_2, 1_X)$	$(\ell_2, 1_X)$
$(1_X, \ell_2)$	$(\ell_2, 1_X)$	$(\ell_2, 1_X)$	$(\ell_2, 1_X)$	$(\ell_2, 1_X)$
$(1_X, \ell_3)$	$(\ell_2, 1_X)$	(ℓ_2, ℓ_1)	(ℓ_2, ℓ_2)	$(\ell_2, 1_X)$
$(\ell_1, 1_X)$	$(1_X, 1_X)$	$(1_X, \ell_1)$	$(1_X, \ell_2)$	$(1_X, \ell_3)$
(ℓ_1, ℓ_1)	$(1_X, 1_X)$	$(1_X, 1_X)$	$(1_X, 1_X)$	$(1_X, 1_X)$
(ℓ_1, ℓ_2)	$(1_X, 1_X)$	$(1_X, \ell_1)$	$(1_X, 1_X)$	$(1_X, 1_X)$
(ℓ_1, ℓ_3)	$(1_X, 1_X)$	$(1_X, \ell_1)$	$(1_X, \ell_2)$	$(1_X, 1_X)$
$(\ell_2, 1_X)$	$(1_X, 1_X)$	$(1_X, \ell_1)$	$(1_X, \ell_2)$	$(1_X, \ell_3)$
(ℓ_2, ℓ_1)	$(1_X, 1_X)$	$(1_X, 1_X)$	$(1_X, 1_X)$	$(1_X, 1_X)$
(ℓ_2, ℓ_2)	$(1_X, 1_X)$	$(1_X, \ell_1)$	$(1_X, 1_X)$	$(1_X, 1_X)$
(ℓ_2, ℓ_3)	$(1_X, 1_X)$	$(1_X, \ell_1)$	$(1_X, \ell_2)$	$(1_X, 1_X)$
$(\ell_3, 1_X)$	$(\ell_2, 1_X)$	(ℓ_2, ℓ_1)	(ℓ_2, ℓ_2)	(ℓ_2, ℓ_3)
(ℓ_3, ℓ_1)	$(\ell_2, 1_X)$	$(\ell_2, 1_X)$	$(\ell_2, 1_X)$	$(\ell_2, 1_X)$
(ℓ_3, ℓ_2)	$(\ell_2, 1_X)$	(ℓ_2, ℓ_1)	$(\ell_2, 1_X)$	$(\ell_2, 1_X)$
(ℓ_3, ℓ_3)	$(\ell_2, 1_X)$	(ℓ_2, ℓ_1)	(ℓ_2, ℓ_2)	$(\ell_2, 1_X)$

and $f(\varpi) = \wp$. Hence

$$
\begin{aligned}
||\hbar * \wp|| &= \inf_{u \in f^{-1}(\hbar * \wp)} ||u|| = \inf_{u \in f^{-1}(\hbar) * f^{-1}(\wp)} ||u|| = \inf_{\substack{\varrho \in f^{-1}(\hbar), \\ \varpi \in f^{-1}(\wp)}} ||\varrho * \varpi|| \\
&\overset{(3.20)}{\leq} \inf_{\substack{\varrho \in f^{-1}(\hbar), \\ \varsigma \in f^{-1}(\jmath)}} ||\varrho * \varsigma|| + \inf_{\substack{\varsigma \in f^{-1}(\jmath), \\ \varpi \in f^{-1}(\wp)}} ||\varsigma * \varpi|| \\
&= \inf_{v \in f^{-1}(\hbar) * f^{-1}(\jmath)} ||v|| + \inf_{w \in f^{-1}(\jmath) * f^{-1}(\wp)} ||w|| \\
&= \inf_{v \in f^{-1}(\hbar * \jmath)} ||v|| + \inf_{w \in f^{-1}(\jmath * \wp)} ||w|| \\
&= ||\hbar * \jmath|| + ||\jmath * \wp||.
\end{aligned}
$$

Hence $(\mathbb{Y}, ||\cdot||)$ is a normed GE-algebra. □

Theorem 3.3. *Let $f : X \to Y$ be a one-to-one GE-morphism from a GE-algebra $\mathbb{X} := (X, *, 1_X)$ to a GE-algebra*

$\mathbb{Y} := (Y, *, 1_Y)$. If $(\mathbb{Y}, ||\cdot||)$ is a normed GE-algebra, then so is $(\mathbb{X}, ||\cdot||)$.

Table 3.1 (continued)

$\circledast$	$(\ell_3, 1_X)$	(ℓ_3, ℓ_1)	(ℓ_3, ℓ_2)	(ℓ_3, ℓ_3)
$(1_X, 1_X)$	$(\ell_3, 1_X)$	(ℓ_3, ℓ_1)	(ℓ_3, ℓ_2)	(ℓ_3, ℓ_3)
$(1_X, \ell_1)$	$(\ell_3, 1_X)$	$(\ell_3, 1_X)$	$(\ell_3, 1_X)$	$(\ell_3, 1_X)$
$(1_X, \ell_2)$	$(\ell_3, 1_X)$	$(\ell_3, 1_X)$	$(\ell_3, 1_X)$	$(\ell_3, 1_X)$
$(1_X, \ell_3)$	$(\ell_3, 1_X)$	(ℓ_3, ℓ_1)	(ℓ_3, ℓ_2)	$(\ell_3, 1_X)$
$(\ell_1, 1_X)$	$(1_X, 1_X)$	$(1_X, \ell_1)$	$(1_X, \ell_2)$	$(1_X, \ell_3)$
(ℓ_1, ℓ_1)	$(1_X, 1_X)$	$(1_X, 1_X)$	$(1_X, 1_X)$	$(1_X, 1_X)$
(ℓ_1, ℓ_2)	$(1_X, 1_X)$	$(1_X, \ell_1)$	$(1_X, 1_X)$	$(1_X, 1_X)$
(ℓ_1, ℓ_3)	$(1_X, 1_X)$	$(1_X, \ell_1)$	$(1_X, \ell_2)$	$(1_X, 1_X)$
$(\ell_2, 1_X)$	$(1_X, 1_X)$	$(1_X, \ell_1)$	$(1_X, \ell_2)$	$(1_X, \ell_3)$
(ℓ_2, ℓ_1)	$(1_X, 1_X)$	$(1_X, 1_X)$	$(1_X, 1_X)$	$(1_X, 1_X)$
(ℓ_2, ℓ_2)	$(1_X, 1_X)$	$(1_X, \ell_1)$	$(1_X, 1_X)$	$(1_X, 1_X)$
(ℓ_2, ℓ_3)	$(1_X, 1_X)$	$(1_X, \ell_1)$	$(1_X, \ell_2)$	$(1_X, 1_X)$
$(\ell_3, 1_X)$	$(1_X, 1_X)$	$(1_X, \ell_1)$	$(1_X, \ell_2)$	$(1_X, \ell_3)$
(ℓ_3, ℓ_1)	$(1_X, 1_X)$	$(1_X, 1_X)$	$(1_X, 1_X)$	$(1_X, 1_X)$
(ℓ_3, ℓ_2)	$(1_X, 1_X)$	$(1_X, \ell_1)$	$(1_X, 1_X)$	$(1_X, 1_X)$
(ℓ_3, ℓ_3)	$(1_X, 1_X)$	$(1_X, \ell_1)$	$(1_X, \ell_2)$	$(1_X, 1_X)$

Proof. Assume that $(\mathbb{Y}, ||\cdot||)$ is a normed GE-algebra. For every $\varrho \in X$, let $||\varrho|| = ||f(\varrho)||$. Then $||\varrho|| = ||f(\varrho)|| \overset{(3.18)}{\geq} 0$ and

$$||\varrho|| = 0 \;\Leftrightarrow\; ||f(\varrho)|| = 0 \;\overset{(3.19)}{\Leftrightarrow}\; f(\varrho) = 1_X = f(1_X) \;\Leftrightarrow\; \varrho = 1_X$$

since f is a one-to-one GE-morphism. For every $\varrho, \varsigma, \varpi \in X$, we get

$$\begin{aligned} ||\varrho * \varpi|| &= ||f(\varrho * \varpi)|| = ||f(\varrho) * f(\varpi)|| \\ &\overset{(3.20)}{\leq} ||f(\varrho) * f(\varsigma)|| + ||f(\varsigma) * f(\varpi)|| \\ &= ||f(\varrho * \varsigma)|| + ||f(\varsigma * \varpi)|| \\ &= ||\varrho * \varsigma|| + ||\varsigma * \varpi||. \end{aligned}$$

Therefore $(\mathbb{X}, ||\cdot||)$ is a normed GE-algebra. □

Theorem 3.4. *Let $\mathbb{X} := (X, *, 1_X)$ and $\mathbb{Y} := (Y, *, 1_Y)$ be GE-algebras and consider the product GE-algebra $\mathbb{X} \times \mathbb{Y} := (X \times Y, \circledast, \mathbf{1})$ of $\mathbb{X} := (X, *, 1_X)$ and $\mathbb{Y} := (Y, *, 1_Y)$. Then*

$\mathbb{X} \times \mathbb{Y}$ *is a normed GE-algebra if and only if* $\mathbb{X} := (X, *, 1_X)$ *and* $\mathbb{Y} := (Y, *, 1_Y)$ *are normed GE-algebras.*

Proof. Assume that $\mathbb{X} \times \mathbb{Y}$ is a normed GE-algebra and consider the projection $f_X : X \times Y \to X$ and $f_Y : X \times Y \to Y$. Then f_X and f_Y are onto GE-morphisms. Hence $\mathbb{X} := (X, *, 1_X)$ and $\mathbb{Y} := (Y, *, 1_Y)$ are normed GE-algebras by Theorem 3.2.

Conversely, suppose that $\mathbb{X} := (X, *, 1_X)$ and $\mathbb{Y} := (Y, *, 1_Y)$ are normed GE-algebras. If $\hbar \in X \times Y$, then $\hbar = (\varrho_\hbar, \varsigma_\hbar)$ for some $\varrho_\hbar \in X$ and $\varsigma_\hbar \in Y$. Define $||\hbar|| = ||\varrho_\hbar|| + ||\varsigma_\hbar||$. Then $||\hbar|| = ||\varrho_\hbar|| + ||\varsigma_\hbar|| \geq 0$ and

$$
\begin{aligned}
||\hbar|| = 0 \ &\Leftrightarrow \ ||\varrho_\hbar|| + ||\varsigma_\hbar|| = 0 \ \Leftrightarrow \ ||\varrho_\hbar|| = 0 = ||\varsigma_\hbar|| \\
&\overset{(3.19)}{\Leftrightarrow} \ \varrho_\hbar = 1_X \text{ and } \varsigma_\hbar = 1_Y \\
&\Leftrightarrow \ \hbar = (\varrho_\hbar, \varsigma_\hbar) = (1_X, 1_Y) = \mathbf{1}.
\end{aligned}
$$

Let $\hbar := (\varrho_\hbar, \varsigma_\hbar)$, $\jmath := (\varrho_\jmath, \varsigma_\jmath)$, $\wp := (\varrho_\wp, \varsigma_\wp) \in X \times Y$. Then

$$
\begin{aligned}
||\hbar \circledast \wp|| &= ||(\varrho_\hbar * \varrho_\wp, \varsigma_\hbar * \varsigma_\wp)|| \\
&= ||\varrho_\hbar * \varrho_\wp|| + ||\varsigma_\hbar * \varsigma_\wp|| \\
&\overset{(3.20)}{\leq} (||\varrho_\hbar * \varrho_\jmath|| + ||\varrho_\jmath * \varrho_\wp||) + (||\varsigma_\hbar * \varsigma_\jmath|| + ||\varsigma_\jmath * \varsigma_\wp||) \\
&= (||\varrho_\hbar * \varrho_\jmath|| + ||\varsigma_\hbar * \varsigma_\jmath||) + (||\varrho_\jmath * \varrho_\wp|| + ||\varsigma_\jmath * \varsigma_\wp||) \\
&= ||(\varrho_\hbar * \varrho_\jmath, \varsigma_\hbar * \varsigma_\jmath)|| + ||(\varrho_\jmath * \varrho_\wp, \varsigma_\jmath * \varsigma_\wp)|| \\
&= ||\hbar \circledast \jmath|| + ||\jmath \circledast \wp||.
\end{aligned}
$$

Therefore $\mathbb{X} \times \mathbb{Y}$ is a normed GE-algebra. □

Definition 3.2. *Let* $(\mathbb{X}, ||\cdot||)$ *be a normed GE-algebra and consider a sequence* $\{\hbar_n\}$ *in* X*. Then* $\{\hbar_n\}$ *is said to be convergent in* X *if there exists a number* $\hbar_0$ *in* X *such that for every* $\varepsilon > 0$ *(no matter how small), there exists a natural number* k_0 *such that the magnitude for* $(\hbar_n, \hbar_0)$ *and* $(\hbar_0, \hbar_n)$ *is less than* ε *for all* $n \geq k_0$*, that is, it can be written as:*

$$
\lim_{n\to\infty} \hbar_n = \hbar_0 \text{ if and only if for every } \varepsilon > 0 \text{ there exists } k_0 \in \mathbb{N} \text{ such that}
$$
$$
n \geq k_0 \ \Rightarrow \ \eth(\hbar_n, \hbar_0) < \varepsilon \text{ and } \eth(\hbar_0, \hbar_n) < \varepsilon.
$$

In this case, we say that $\hbar_0$ *is the limit of* $\{\hbar_n\}$*.*

Theorem 3.5. *Let $\mathbb{X} := (X, *, 1_X)$ be a commutative GE-algebra. In a normed GE-algebra $(\mathbb{X}, ||\cdot||)$, a convergent sequence cannot have two different limits, that is, If a sequence $\{\hbar_n\}$ converges to a limit $\hbar_0$, then that limit is unique.*

Proof. Let $\{\hbar_n\}$ be a convergent sequence in X, and let $\hbar_0$ and $\jmath_0$ be two limits of $\{\hbar_n\}$. Then for every $\varepsilon > 0$, there exists a natural number k_0 such that $\eth(\hbar_n, \hbar_0) < \frac{\varepsilon}{2}$, $\eth(\hbar_0, \hbar_n) < \frac{\varepsilon}{2}$, $\eth(\hbar_n, \jmath_0) < \frac{\varepsilon}{2}$ and $\eth(\hbar_0, \jmath_n) < \frac{\varepsilon}{2}$ for all $n \geq k_0$. Hence

$$\begin{aligned}\eth(\hbar_0, \jmath_0) &\overset{(3.21)}{=} ||\hbar_0 * \jmath_0|| \overset{(3.20)}{\leq} ||\hbar_0 * \hbar_n|| + ||\hbar_n * \jmath_0|| \\ &\overset{(3.21)}{=} \eth(\hbar_0, \hbar_n) + \eth(\hbar_n, \jmath_0) < \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

By the similarly way, we have $\eth(\jmath_0, \hbar_0) \leq \varepsilon$. Since ε is arbitrary, it follows that $\eth(\hbar_0, \jmath_0) = 0 = \eth(\jmath_0, \hbar_0)$. Using Theorem 3.1, we conclude that $\hbar_0 = \jmath_0$. Therefore $\{\hbar_n\}$ has a unique limit. □

Theorem 3.6. *In a normed GE-algebra $(\mathbb{X}, ||\cdot||)$, every convergent sequence $\{\hbar_n\}$ in X satisfies:*

$$(\forall \varepsilon > 0)(\exists k_0 \in \mathbb{N})\,(n, m \geq k_0 \;\Rightarrow\; \eth(\hbar_n, \hbar_m) < \varepsilon \text{ and } \eth(\hbar_m, \hbar_n) < \varepsilon)\,. \tag{3.35}$$

Proof. Let $\mathbb{X} := \langle X, *, 1_X\rangle$ be a normed GE-algebra with GE-norm $\|\cdot\|$, and let $\eth(\varrho, \varsigma) = \|\varrho * \varsigma\|$ be the magnitude function. Suppose $\{\hbar_n\}$ is a sequence in X that converges to $\hbar_0$ in X. By definition 3.2, for every $\varepsilon > 0$, there exists $k_0 \in \mathbb{N}$ such that for all $n \geq k_0$,

$$\eth(\hbar_n, \hbar_0) = \|\hbar_n * \hbar_0\| < \varepsilon \quad \text{and} \quad \eth(\hbar_0, \hbar_n) = \|\hbar_0 * \hbar_n\| < \varepsilon.$$

To prove that $\{\hbar_n\}$ satisfies condition (3.35), fix $\varepsilon > 0$. Since $\{\hbar_n\}$ converges to $\hbar_0$, there exists $k_0 \in \mathbb{N}$ such that for all $n \geq k_0$,

$$\eth(\hbar_n, \hbar_0) < \frac{\varepsilon}{2} \quad \text{and} \quad \eth(\hbar_0, \hbar_n) < \frac{\varepsilon}{2}.$$

We need to show that for all $n, m \geq k_0$, $\eth(\hbar_n, \hbar_m) < \varepsilon$ and $\eth(\hbar_m, \hbar_n) < \varepsilon$. Consider $\eth(\hbar_n, \hbar_m) = \|\hbar_n * \hbar_m\|$. By the triangle-like inequality of the GE-norm (Definition 3.1, condition (3.20)), for any $\varrho, \varsigma, \varpi \in X$,

$$\|\varrho * \varpi\| \leq \|\varrho * \varsigma\| + \|\varsigma * \varpi\|.$$

Set $\varrho = \hbar_n$, $\varpi = \hbar_m$, and $\varsigma = \hbar_0$. Then,

$$\|\hbar_n * \hbar_m\| \leq \|\hbar_n * \hbar_0\| + \|\hbar_0 * \hbar_m\|,$$

i.e.,

$$\eth(\hbar_n, \hbar_m) \leq \eth(\hbar_n, \hbar_0) + \eth(\hbar_0, \hbar_m).$$

Since $n, m \geq k_0$, we have:

$$\eth(\hbar_n, \hbar_0) = \|\hbar_n * \hbar_0\| < \frac{\varepsilon}{2}, \quad \eth(\hbar_0, \hbar_m) = \|\hbar_0 * \hbar_m\| < \frac{\varepsilon}{2}.$$

Thus,

$$\eth(\hbar_n, \hbar_m) \leq \eth(\hbar_n, \hbar_0) + \eth(\hbar_0, \hbar_m) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Similarly, we can show that $\eth(\hbar_m, \hbar_n) < \varepsilon$. □

The converse of Theorem 3.6 is not valid as seen in the following example.

Example 3.5. *(i) For the normed GE-algebra* $(\mathbb{X}, ||\cdot||)$ *in Example 3.2, we can observe that if*

$$\hbar_n = \begin{cases} \ell_1 & \text{if } n \text{ is odd,} \\ \ell_4 & \text{if } n \text{ is even,} \end{cases}$$

then the sequence $\{\hbar_n\}$ *in* X *satisfies* (3.35)*. If we take* $\varepsilon > 0$ *such that* $\varrho_0 \geq \varepsilon$*, then*

$$\eth(\hbar_7, \ell_2) = ||\ell_1 * \ell_2|| = ||\ell_2|| = \varrho_0 \not< \varepsilon$$

and/or $\eth(\ell_2, \hbar_7) = ||\ell_2 * \ell_1|| = ||\ell_4|| = \varrho_0 \not< \varepsilon$*. Hence* $\{\hbar_n\}$ *is not convergent.*

(ii) Let $(0, 1] \subseteq \mathbb{R}$ *and define a binary operation "$*$" on* $(0, 1]$ *as follows:*

$$\varrho * \varsigma = \begin{cases} \varsigma & \text{if } \varrho = 1, \\ 1 & \text{otherwise.} \end{cases}$$

Then $((0, 1], *, 1)$ *is a GE-algebra. If we take a sequence* $\{\frac{1}{n+1}\}_{n\in\mathbb{N}}$*, then it satisfies* (3.35) *but does not converge in* $(0, 1]$*.*

Theorem 3.7. *Let* $\{\hbar_n\}$ *be a sequence in a normed GE-algebra* $(\mathbb{X}, ||\cdot||)$ *where* $\mathbb{X} := (X, *, 1_X)$ *is a commutative GE-algebra. Then it is convergent if and only if all of its non-trivial subsequences converge.*

Proof. Assume that $\{\hbar_n\}$ is a convergent sequence in $(\mathbb{X}, ||\cdot||)$ and let $\hbar_0$ be its limit. For every $\varepsilon > 0$ there exists $k_0 \in \mathbb{N}$ such that

$$n \geq k_0 \;\Rightarrow\; \eth(\hbar_n, \hbar_0) < \varepsilon \text{ and } \eth(\hbar_0, \hbar_n) < \varepsilon.$$

Let $\{\hbar_{\phi(n)}\}$ be a non-trivial subsequence of $\{\hbar_n\}$. If $n \geq k_0$, then $\phi(n) \geq n \geq k_0$, and so $\eth(\hbar_{\phi(n)}, \hbar_0) < \varepsilon$ and $\eth(\hbar_0, \hbar_{\phi(n)}) < \varepsilon$. This shows that $\{\hbar_{\phi(n)}\}$ is convergent.

Conversely, suppose that all of non-trivial subsequences of $\{\hbar_n\}$ converge. If $\{\hbar_n\}$ is not convergent, then there are at least two non-trivial subsequences, say $\{\hbar_{\phi(n)}\}$ and $\{\hbar_{\phi(m)}\}$, with different limits $\hbar_0$ and $\jmath_0$, respectively. This is a contradiction by Theorem 3.5, and thus $\{\hbar_n\}$ is a convergent sequence in $(\mathbb{X}, ||\cdot||)$. $\square$

Theorem 3.8. *Let $\{\hbar_n\}$ be a sequence in $(\mathbb{X}, ||\cdot||)$. If $\hbar_0$ is a limit of $\{\hbar_n\}$, then 1_X is a limit of the sequences $\{\hbar_n * \hbar_0\}$ and $\{\hbar_0 * \hbar_n\}$.*

Proof. If $\hbar_0$ is a limit of $\{\hbar_n\}$, then for every $\varepsilon > 0$ there exists $k_0 \in \mathbb{N}$ such that

$$n \geq k_0 \;\Rightarrow\; \eth(\hbar_n, \hbar_0) < \varepsilon \text{ and } \eth(\hbar_0, \hbar_n) < \varepsilon.$$

Hence $\eth(\hbar_n * \hbar_0, 1_X) \overset{(3.22)}{=} 0 < \varepsilon$ and

$$\begin{aligned}\eth(1_X, \hbar_n * \hbar_0) &\overset{(3.21)}{=} ||1_X * (\hbar_n * \hbar_0)|| \overset{(GE2)}{=} ||\hbar_n * \hbar_0|| \\ &\overset{(3.21)}{=} \eth(\hbar_n, \hbar_0) < \varepsilon.\end{aligned}$$

Therefore 1_X is a limit of $\{\hbar_n * \hbar_0\}$. Similarly, $\{\hbar_0 * \hbar_n\}$ has a limit 1_X. $\square$

Theorem 3.9. *Let $\{\hbar_n\}$ be a sequence in a normed GE-algebra $(\mathbb{X}, ||\cdot||)$. If $\hbar_0$ is a limit of $\{\hbar_n\}$, then $\{\eth(\hbar_n, \jmath_0)\}$ and $\{\eth(\jmath_0, \hbar_n)\}$ are bounded above for all $\jmath_0 \in X$.*

Proof. Assume that $\{\hbar_n\}$ converges to $\hbar_0$. By the definition of convergence, for every $\varepsilon > 0$ there exists a natural number k_0 such that $\eth(\hbar_n, \hbar_0) < \varepsilon$ and $\eth(\hbar_0, \hbar_n) < \varepsilon$ for all $n \geq k_0$. It follows from (3.20) that

$$\eth(\hbar_n, \jmath_0) \leq \eth(\hbar_n, \hbar_0) + \eth(\hbar_0, \jmath_0) < \varepsilon + \eth(\hbar_0, \jmath_0)$$

and $\eth(\jmath_0, \hbar_n) \leq \eth(\jmath_0, \hbar_0) + \eth(\hbar_0, \hbar_n) < \eth(\jmath_0, \hbar_0) + \varepsilon$. If $n < k_0$, then $\eth(\hbar_n, \jmath_0) = ||\hbar_n * \jmath_0|| \leq M$ and $\eth(\jmath_0, \hbar_n) = ||\jmath_0 * \hbar_n|| \leq M$ where

$$M := \max\{||\hbar_n * \jmath_0||, ||\jmath_0 * \hbar_n||\}.$$

This completes the proof. $\square$

Let $\eth$ be the magnitude of a normed GE-algebra $(\mathbb{X}, ||\cdot||)$. Consider the following:

$$(\forall \varrho, \varsigma, \varpi \in X)\left(\varrho \leq_X \varsigma \;\Rightarrow \left\{ \begin{array}{l} \eth(\varrho, \varpi) \leq \eth(\varsigma, \varpi) \\ \eth(\varpi, \varsigma) \leq \eth(\varpi, \varrho) \end{array} \right. \right). \tag{3.36}$$

The following example shows that (3.36) is not valid in general.

Example 3.6. *Consider a GE-algebra* $\mathbb{X} := (X, *, 1_X)$*, where* $X = \{1_X, \ell_1, \ell_2, \ell_3, \ell_4, \ell_5\}$ *and a binary operation "$*$" is given in the following table:*

$*$	1_X	ℓ_1	ℓ_2	ℓ_3	ℓ_4	ℓ_5
1_X	1_X	ℓ_1	ℓ_2	ℓ_3	ℓ_4	ℓ_5
ℓ_1	1_X	1_X	1_X	ℓ_3	ℓ_3	ℓ_5
ℓ_2	1_X	ℓ_1	1_X	ℓ_4	ℓ_4	ℓ_5
ℓ_3	1_X	1_X	ℓ_2	1_X	1_X	ℓ_5
ℓ_4	1_X	1_X	1_X	1_X	1_X	ℓ_5
ℓ_5	1_X	ℓ_1	ℓ_2	ℓ_3	ℓ_4	1_X

Define a norm $||\cdot||$ *on* $\mathbb{X} := (X, *, 1_X)$ *as follows:*

$$||\cdot|| : X \to \mathbb{R},\ \varrho \mapsto \begin{cases} 0 & \text{if } \varrho = 1_X, \\ \varrho_0 & \text{otherwise}, \end{cases}$$

where ϱ_0 *is a positive real number. Then* $(\mathbb{X}, ||\cdot||)$ *is a normed GE-algebra. Note that* $\ell_3 * \ell_1 = 1_X$ *and* $\ell_4 * \ell_2 = 1_X$*, i.e.,* $\ell_3 \leq_X \ell_1$ *and* $\ell_4 \leq_X \ell_2$*. We can observe that*

$$\eth(\ell_3, \ell_2) = ||\ell_3 * \ell_2|| = ||\ell_2|| = \varrho_0 \nleq 0 = ||1_X|| = ||\ell_1 * \ell_2|| = \eth(\ell_1, \ell_2)$$

and

$$\eth(\ell_3, \ell_2) = ||\ell_3 * \ell_2|| = ||\ell_2|| = \varrho_0 \nleq 0 = ||1_X|| = ||\ell_3 * \ell_4|| = \eth(\ell_3, \ell_4).$$

We now discuss the squeeze theorem for convergence sequences.

Theorem 3.10. *Assume that every magnitude* $\eth$ *of a normed GE-algebra* $(\mathbb{X}, ||\cdot||)$ *satisfies* (3.36)*. Let* $\{\hbar_n\}$*,* $\{\jmath_n\}$ *and* $\{\wp_n\}$ *be sequences in* $(\mathbb{X}, ||\cdot||)$ *such that* $\{\jmath_n\}$ *is trapped between* $\{\hbar_n\}$ *and* $\{\wp_n\}$ *for a sufficiently large* n*, that is, there exists a natural number* k_0 *such that* $\hbar_n \leq_X \jmath_n \leq_X \wp_n$ *for all* $n > k_0$*. If* $\{\hbar_n\}$ *and* $\{\wp_n\}$ *converge to* $\hbar_0$*, then* $\{\jmath_n\}$ *also converges to* $\hbar_0$*.*

Proof. If $\{\hbar_n\}$ and $\{\wp_n\}$ converge to $\hbar_0$, then for every $\varepsilon > 0$ there exist natural numbers $k_\hbar$ and $k_\jmath$ such that

$$n \geq k_\hbar \ \Rightarrow\ \eth(\hbar_n, \hbar_0) < \varepsilon \text{ and } \eth(\hbar_0, \hbar_n) < \varepsilon$$

and

$$n \geq k_\jmath \ \Rightarrow\ \eth(\wp_n, \hbar_0) < \varepsilon \text{ and } \eth(\hbar_0, \wp_n) < \varepsilon.$$

Using (3.36), we have

$$\eth(\hbar_n, \hbar_0) \leq \eth(\jmath_n, \hbar_0) \leq \eth(\wp_n, \hbar_0)$$

and

$$\eth(\wp_0, \hbar_n) \leq \eth(\jmath_0, \hbar_n) \leq \eth(\hbar_0, \hbar_n)$$

for all $k_0 := \max\{k_\hbar, k_\jmath\}$. It follows that if $n \geq k_0$, then $\eth(\jmath_n, \hbar_0) < \varepsilon$ and $\eth(\hbar_0, \jmath_n) < \varepsilon$. Thus $\{\jmath_n\}$ converges to $\hbar_0$. □

Theorem 3.11. *Let f be a GE-morphism from a GE-algebra $\mathbb{X} := (X, *_X, 1_X)$ to a GE-algebra $\mathbb{Y} := (Y, *_Y, 1_Y)$. Assume that $||\varrho|| = ||f(\varrho)||$ for all $\varrho \in X$. Then a sequence $\{\hbar_n\}$ in $(\mathbb{X}, ||\cdot||)$ converges to $\hbar_0$ if and only if the sequence $\{f(\hbar_n)\}$ in $(\mathbb{Y}, ||\cdot||)$ converges to $f(\hbar_0)$.*

Proof. Assume that a sequence $\{\hbar_n\}$ in $(\mathbb{X}, ||\cdot||)$ converges to $\hbar_0$. Then for every $\varepsilon > 0$, there exists a natural number k_0 such that $\eth(\hbar_n, \hbar_0) < \varepsilon$ and $\eth(\hbar_0, \hbar_n) < \varepsilon$ for all $n \geq k_0$. Using (2.15) and (3.21), we have

$$\begin{aligned}\eth(f(\hbar_n), f(\hbar_0)) &= ||f(\hbar_n) *_Y f(\hbar_0)|| = ||f(\hbar_n *_X \hbar_0)|| \\ &= ||\hbar_n *_X \hbar_0|| = \eth(\hbar_n, \hbar_0) < \varepsilon\end{aligned}$$

and

$$\begin{aligned}\eth(f(\hbar_0), f(\hbar_n)) &= ||f(\hbar_0) *_Y f(\hbar_n)|| = ||f(\hbar_0 *_X \hbar_n)|| \\ &= ||\hbar_0 *_X \hbar_n|| = \eth(\hbar_0, \hbar_n) < \varepsilon\end{aligned}$$

Therefore the sequence $\{f(\hbar_n)\}$ converges to $f(\hbar_0)$.

Conversely, suppose that the sequence $\{f(\hbar_n)\}$ in $(\mathbb{Y}, ||\cdot||)$ converges to $f(\hbar_0)$. For every $\varepsilon > 0$ there exists a natural number k_0 such that $\eth(f(\hbar_n), f(\hbar_0)) < \varepsilon$ and $\eth(f(\hbar_0), f(\hbar_n)) < \varepsilon$ for all $n \geq k_0$. It follows that

$$\begin{aligned}\eth(\hbar_n, \hbar_0) &= ||\hbar_n *_X \hbar_0|| = ||f(\hbar_n *_X \hbar_0)|| \\ &= ||f(\hbar_n) *_Y f(\hbar_0)|| = \eth(f(\hbar_n), f(\hbar_0)) < \varepsilon\end{aligned}$$

and

$$\begin{aligned}\eth(\hbar_0, \hbar_n) &= ||\hbar_0 *_X \hbar_n|| = ||f(\hbar_0 *_X \hbar_n)|| \\ &= ||f(\hbar_0) *_Y f(\hbar_n)|| = \eth(f(\hbar_0), f(\hbar_n)) < \varepsilon\end{aligned}$$

for all $n \geq k_0$. Consequently, $\{\hbar_n\}$ converges to $\hbar_0$. □

4. Conclusion

This paper introduces normed GE-algebras, equipping GE-algebras with a GE-norm to measure element magnitudes. We defined a magnitude function $\eth(\varrho, \varsigma) = \|\varrho * \varsigma\|$ that induces a quasi-metric space, generating a T_0-topology (Theorem 3.1, Example 3.3). Key results include the Cauchy-like property of convergent sequences (Theorem 3.6), preservation of normed structures under GE-morphisms (Theorem 3.2), and properties of product spaces (Theorem 3.4). These findings establish normed GE-algebras as a robust framework for studying convergence and topological properties in generalized algebraic systems. The significance of this work lies in bridging algebraic and geometric concepts, enabling the analysis of non-commutative structures in a topological context. The quasi-metric and T_0-topology support applications in functional analysis, modeling asymmetric distances, and in mathematical logic, quantifying logical distances in non-classical logics [13]. The GE-morphism and product theorems facilitate the study of complex algebraic systems. Future work includes exploring additional topological properties, such as compactness or connectedness, in the T_0-topology. Extending GE-norms to BCK/BCI-algebras or residuated lattices could broaden their scope [8]. Applications in functional analysis (e.g., asymmetric function spaces) and topology (e.g., non-Hausdorff spaces) are promising. Open problems, such as characterizing complete normed GE-algebras, encourage further interdisciplinary research.

Normed GE-algebras offer promising applications across several mathematical disciplines. In *functional analysis*, the quasi-metric spaces induced by GE-norms (Example 3.3) provide a framework for studying function spaces with asymmetric distances, which are relevant in asymmetric functional analysis [10]. These spaces can model non-reversible processes or directed convergence, extending traditional Banach space techniques. In *topology*, the T_0-topology generated by normed GE-algebras facilitates the study of non-Hausdorff topological spaces, which are prevalent in computational topology and data analysis. This topology supports the analysis of convergence properties in generalized settings. In *mathematical logic*, normed GE-algebras, as extensions of Hilbert algebras linked to intuitionistic logic, enable quantitative semantics where the GE-norm measures the "distance" between logical propositions [13]. This can enhance reasoning frameworks in non-classical logics, such as those used in artificial intelligence and formal verification. These applications underscore the versatility of normed GE-algebras and pave the way for future interdisciplinary research.

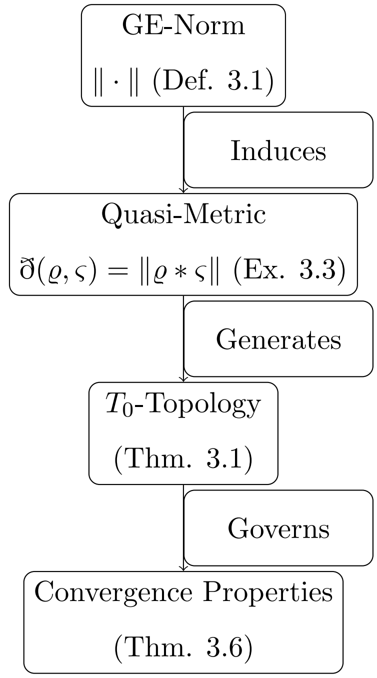

FIGURE 1. Flowchart illustrating the relationships between GE-norms, quasi-metrics, T_0-topology, and convergence properties in normed GE-algebras. The GE-norm induces a quasi-metric, which generates a T_0-topology, governing sequence convergence (e.g., Cauchy-like property in Theorem 3.6).

Acknowledgments. The authors would like to thank the referee for some useful comments and their helpful suggestions that have improved the quality of this paper.

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