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ENERGY OF INDU-BALA PRODUCT OF GRAPHS

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ABSTRACT. The energy of a graph Γ is defined as the sum of the absolute values of its eigenvalues. In this article, we compute the energy of the Indu-Bala product of two regular graphs and establish bounds for its energy. Furthermore, we explore the concepts of equienergetic, borderenergetic, orderenergetic, and non-hyperenergetic graphs using the Indu-Bala product of two regular graphs.

Keywords: Graph energy, Indu-Bala product, Equienergetic graphs, Complement of a graph.

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1. Introduction

Let Γ be a simple graph of order n. The degree of a vertex u_i, denoted by d_i, is defined as the number of edges incident to it. A graph Γ is said to be r-regular if and only if each vertex of Γ has degree r. The eigenvalues of the graph Γ of order n are the eigenvalues of its adjacency matrix $A(\Gamma)$, denoted by $\lambda_1, \lambda_2, \ldots, \lambda_n$.

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Let n^0, n^- and n^+ denote the number of zero, negative and positive eigenvalues of the graph Γ, respectively. The energy of a graph Γ is defined as

$$E(\Gamma) = \sum_{j=1}^{n} \mid \lambda_j \mid .$$

The line graph $L(\Gamma)$ of a graph Γ is defined as the graph whose vertex set corresponds to the edge set of Γ, where two vertices in $L(\Gamma)$ are adjacent if and only if their corresponding edges in Γ share a common vertex. The i^{th} iterated line graph of Γ, denoted by $L^i(\Gamma)$ for $i = 1, 2, \ldots$, is defined recursively as $L^i(\Gamma) = L(L^{i-1}(\Gamma))$, with $L^0(\Gamma) = \Gamma$ and $L^1(\Gamma) = L(\Gamma)$.

The concept of graph energy, which originated from Hückel molecular orbital theory, was first introduced by Gutman [6]. If two graphs of the same order have the same energy, they are called equienergetic graphs. If the energy of a graph is equal to the number of vertices n, then the graph is said to be orderenergetic [1]. If $E(\Gamma) \leq 2(n-1)$, then the graph is said to be non-hyperenergetic [17] and if $E(\Gamma) = 2(n-1)$, then Γ is said to be borderenergetic [5]. In the literature, there are various research articles that focus on equienergetic graphs. For recent papers, see [10, 11, 12, 13, 14].

Graph products such as the Cartesian product, tensor product, strong product and their corresponding energies have been well studied in the literature [2, 4, 9, 12, 14, 18]. The distance spectrum, adjacency spectrum, distance Laplacian spectrum and distance signless Laplacian spectrum of another product namely, the Indu-Bala product have been investigated in [7, 8, 16]. However, the energy of the Indu-Bala product has not yet been examined. Therefore, in this paper, we study the energy of the Indu-Bala product, which contributes to the construction of non-regular equienergetic graphs. For undefined terminology and results related to the graph spectra, we follow [3].

Definition 1.1 (Indu–Bala product)**.** [7] *The Indu–Bala product of two graphs* Γ_1 *and* Γ_2*, denoted by* $\Gamma_1 \blacktriangledown \Gamma_2$*, is defined as follows: Let* $\Gamma_1 \vee \Gamma_2$ *denote the join of* Γ_1 *and* Γ_2*, where* $V(\Gamma_1) = \{w_1, w_2, \ldots, w_{n_1}\}$ *and* $V(\Gamma_2) = \{z_1, z_2, \ldots, z_{n_2}\}$*. Take a disjoint copy of* $\Gamma_1 \vee \Gamma_2$*, denoted by* $\Gamma'_1 \vee \Gamma'_2$*, with vertex sets* $V(\Gamma'_1) = \{w'_1, w'_2, \ldots, w'_{n_1}\}$ *and* $V(\Gamma'_2) = \{z'_1, z'_2, \ldots, z'_{n_2}\}$*. Finally, add edges between each vertex* $z_i \in V(\Gamma_2)$ *and its corresponding copy* $z'_i \in V(\Gamma'_2)$*, for all* $i = 1, 2, \ldots, n_2$*.*

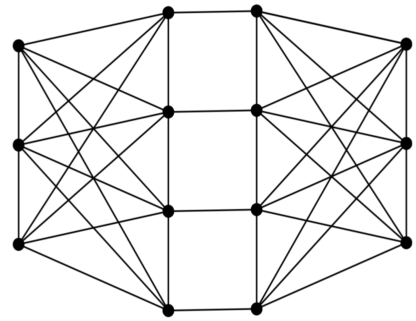

FIGURE 1. The graph $P_3 \blacktriangledown P_4$

Proposition 1.1. [8] *Let Γ_k be an r_k-regular graph of order n_k, for $k = 1, 2$. Then, the spectrum of $\Gamma_1 \blacktriangledown \Gamma_2$ is as follows:*

(a) $\lambda_k(\Gamma_1)$, with multiplicity 2 for $k = 2, 3, \ldots, n_1$;

(b) $\lambda_k(\Gamma_2) + 1$ for $k = 2, 3, \ldots, n_2$;

(c) $\lambda_k(\Gamma_2) - 1$ for $k = 2, 3, \ldots, n_2$;

(d) $\frac{(r_1+r_2+1) \pm \sqrt{(r_1+r_2+1)^2 - 4(r_1(r_2+1) - n_1 n_2)}}{2}$ and $\frac{(r_1+r_2-1) \pm \sqrt{(r_1+r_2-1)^2 - 4(r_1(r_2-1) - n_1 n_2)}}{2}$.

Proposition 1.2. [11] *Let a graph Γ have n vertices with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Then*

$$\sum_{k=1}^{n} \mid \lambda_k + 1 \mid = n + E(\Gamma) - 2n^- + 2 \sum_{\lambda_k \in (-1,0)} (\lambda_k + 1).$$

Proposition 1.3. [15] *Let a graph Γ have n vertices with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Then*

$$\sum_{k=1}^{n} \mid \lambda_k + 2 \mid = 2n + E(\Gamma) - 4n^- + 2 \sum_{\lambda_k \in (-2,0)} (\lambda_k + 2).$$

2. ENERGY OF INDU-BALA PRODUCT OF GRAPHS

Lemma 2.1. *Let a graph Γ have n vertices with eigenvalues $\lambda_n \leqslant \lambda_{n-1} \leqslant \cdots \leqslant \lambda_1$. Then, for $0 \leqslant p < \lambda_1$,*

$$\sum_{k=1}^{n} \mid \lambda_k - p \mid = E(\Gamma) + np - 2pn^+ - 2 \sum_{\lambda_k \in (0,p)} (\lambda_k - p).$$

Proof. Define $n_\lambda(I)$ as the count of eigenvalues of Γ within the interval I.

Let us compute $\sum_{k=1}^{n} \mid \lambda_k - p \mid$,

$$
\begin{aligned}
\sum_{k=1}^{n} \mid \lambda_k - p \mid &= \sum_{\lambda_k \leqslant p} (-\lambda_k + p) + \sum_{\lambda_k > p} (\lambda_k - p) \\
&= \sum_{\lambda_k \leqslant p} -\lambda_k + pn_\lambda[\lambda_n, p] + \sum_{\lambda_k > p} \lambda_k - pn_\lambda(p, \lambda_1] \\
&= pn_\lambda[\lambda_n, p] - pn_\lambda(p, \lambda_1] + \sum_{\lambda_k \leqslant 0} \mid \lambda_k \mid + \sum_{\lambda_k \in (0,p]} -\lambda_k \\
&\quad + \sum_{\lambda_k > p} \lambda_k, \qquad (2.1)
\end{aligned}
$$

The $E(\Gamma)$ can be expressed as,

$$
E(\Gamma) = \sum_{k=1}^{n} \mid \lambda_k \mid = \sum_{\lambda_k \leq 0} \mid \lambda_k \mid + \sum_{\lambda_k \in (0,p]} \lambda_k + \sum_{\lambda_k > p} \lambda_k \qquad (2.2)
$$

The order n can be expressed as,

$$
n = n_\lambda(0, p] + n_\lambda(p, \lambda_1] + n^0 + n^- \qquad (2.3)
$$

or,

$$
n = n_\lambda[\lambda_n, p] + n_\lambda(p, \lambda_1]. \qquad (2.4)
$$

By equalities 2.2 and 2.4, equality 2.1 becomes,

$$
\begin{aligned}
\sum_{k=1}^{n} \mid \lambda_k - p \mid &= p(n - n_\lambda(p, \lambda_1]) - pn_\lambda(p, \lambda_1] + E(\Gamma) - 2 \sum_{\lambda_k \in (0,p]} \lambda_k \\
&= np - 2pn_\lambda(p, \lambda_1] + E(\Gamma) - 2 \sum_{\lambda_k \in (0,p]} \lambda_k \\
&= E(\Gamma) + np - 2pn^+ + 2pn_\lambda(0, p] \\
&\quad -2 \sum_{\lambda_k \in (0,p]} (\lambda_k - p) - 2pn_\lambda(0, p] \quad \text{by the equality 2.3} \\
\sum_{k=1}^{n} \mid \lambda_k - p \mid &= E(\Gamma) + np - 2pn^+ - 2 \sum_{\lambda_k \in (0,p)} (\lambda_k - p).
\end{aligned}
$$

□

Let ξ be the absolute sum of the eigenvalues mentioned in the case (d) of Proposition 1.1.

Theorem 2.1. *Let the order of an r_k-regular graph Γ_k be n_k, where $k = 1, 2$, then the energy of Indu-Bala product is*

$$\begin{aligned} E(\Gamma_1 \blacktriangledown \Gamma_2) &= 2(E(\Gamma_1) + E(\Gamma_2)) + 2n_2^0 - 2(r_1 + r_2) + 2 \sum_{\lambda_i(\Gamma_2)\in(-1,0)} (\lambda_i(\Gamma_2) + 1) \\ &\quad -2 \sum_{\lambda_i(\Gamma_2)\in(0,1)} (\lambda_i(\Gamma_2) - 1) + \xi. \end{aligned}$$

Proof. Proposition 1.1 provides the eigenvalues of Indu-Bala product of Γ_k; $k = 1, 2$. Therefore,

$$\begin{aligned} E(\Gamma_1 \blacktriangledown \Gamma_2) &= 2\sum_{i=2}^{n_1} \mid \lambda_i(\Gamma_1) \mid + \sum_{i=2}^{n_2} \mid \lambda_i(\Gamma_2) + 1 \mid + \sum_{i=2}^{n_2} \mid \lambda_i(\Gamma_2) - 1 \mid + \xi \\ &= 2\sum_{i=1}^{n_1} \mid \lambda_i(\Gamma_1) \mid -2r_1 + \sum_{i=1}^{n_2} \mid (\lambda_i(\Gamma_2) + 1) \mid -(r_2 + 1) \\ &\quad + \sum_{i=1}^{n_2} \mid (\lambda_i(\Gamma_2) - 1) \mid -(r_2 - 1) + \xi. \end{aligned}$$

By using Lemma 2.1 and Proposition 1.2, we have

$$\begin{aligned} E(\Gamma_1 \blacktriangledown \Gamma_2) &= 2E(\Gamma_1) - 2r_1 + E(\Gamma_2) + n_2 - 2n_2^- + 2 \sum_{\lambda_k(\Gamma_2)\in(-1,0)} (\lambda_k(\Gamma_2) + 1) - r_2 \\ &\quad -1 + E(\Gamma_2) + n_2 - 2n_2^+ - 2 \sum_{\lambda_k(\Gamma_2)\in(0,1)} (\lambda_k(\Gamma_2) - 1) - (r_2 - 1) + \xi \\ &= 2E(\Gamma_1) - 2r_1 + 2E(\Gamma_2) + 2n_2 - 2(n_2^- + n_2^+) - 2r_2 \\ &\quad +2 \sum_{\lambda_k(\Gamma_2)\in(-1,0)} (\lambda_k(\Gamma_2) + 1) - 2 \sum_{\lambda_k(\Gamma_2)\in(0,1)} (\lambda_k(\Gamma_2) - 1) + \xi \qquad (2.5) \\ &= 2(E(\Gamma_1) + E(\Gamma_2)) - 2(r_1 + r_2) + 2n_2^0 + 2 \sum_{\lambda_k(\Gamma_2)\in(-1,0)} (\lambda_k(\Gamma_2) + 1) \\ &\quad -2 \sum_{\lambda_k(\Gamma_2)\in(0,1)} (\lambda_k(\Gamma_2) - 1) + \xi. \end{aligned}$$

□

Corollary 2.1. *Let the order of an r_k-regular graph Γ_k be n_k, where $k = 1, 2$. Then,*

$$\begin{aligned} 2E(\Gamma_1) + 2E(\Gamma_2) - 2(r_1 + r_2) + \xi &\leqslant E(\Gamma_1 \blacktriangledown \Gamma_2) \\ &< 2E(\Gamma_1) + 2E(\Gamma_2) + 2n_2 + \xi. \end{aligned}$$

Equality holds at the left side if and only if there is no eigenvalues in the interval $(-1, 1)$.

Proof. For upper bound, it is observed from the equation 2.5 that

$$n_2^- - \sum_{\lambda_k(\Gamma_2)\in(-1,0)} (\lambda_k(\Gamma_2)+1) > 0 \text{ and } n_2^+ - \sum_{\lambda_k(\Gamma_2)\in(0,1)} (\lambda_k(\Gamma_2)-1) > 0$$

Also, if we can eliminate the values r_1 and r_2 from equation 2.5 as both are positive, we get

$$E(\Gamma_1 \blacktriangledown \Gamma_2) < 2E(\Gamma_1) + 2E(\Gamma_2) + 2n_2 + \xi.$$

For lower bound, it is easy to observe from Theorem 2.1 that

$$\sum_{\lambda_k(\Gamma_2)\in(-1,0)} (\lambda_k(\Gamma_2)+1) > 0 \text{and} - \sum_{\lambda_k(\Gamma_2)\in(0,1)} (\lambda_k(\Gamma_2)-1) > 0,$$

also $n_2^0 \geq 0$, on removing these values from Theorem 2.1, we obtain,

$$2E(\Gamma_1) + 2E(\Gamma_2) - 2(r_1 + r_2) + \xi < E(\Gamma_1 \blacktriangledown \Gamma_2).$$

The equality on the left side is derived from the following fact,

$$\sum_{\lambda_k(\Gamma_2)\in(-1,0)} (\lambda_k(\Gamma_2)+1) = 0, \sum_{\lambda_k(\Gamma_2)\in(0,1)} (\lambda_k(\Gamma_2)-1) = 0 \text{ and } n_2^0 = 0$$

if and only if Γ_2 has no eigenvalues in the interval $(-1, 1)$. □

There are numerous equienergetic graphs with the same regularity and same order, one can find them in the recent articles [11, 12, 13, 14]. With the help of these graphs and Indu-Bala product, one can easily construct non-regular equienergetic graphs.

Corollary 2.2. *Let $H_i; i = 1, 2$ be two r-regular graphs of same order n. Then $H_i \blacktriangledown \Gamma_2; i = 1, 2$ are equienergetic graphs if and only if $H_i; i = 1, 2$ are equienergetic.*

Proof. Proof follows from Theorem 2.1 that $H_i \blacktriangledown \Gamma_2; i = 1, 2$ are equienergetic graphs if and only if

$$\begin{aligned} 2(E(H_1) + E(\Gamma_2)) + 2n_2^0 - 2(r + r_2) + 2 \sum_{\lambda_k(\Gamma_2)\in(-1,0)} (\lambda_k(\Gamma_2)+1) \\ - \sum_{\lambda_k(\Gamma_2)\in(0,1)} (\lambda_k(\Gamma_2)-1) + \xi = 2(E(H_2) + E(\Gamma_2)) + 2n_2^0 - 2(r + r_2) \\ +2 \sum_{\lambda_k(\Gamma_2)\in(-1,0)} (\lambda_k(\Gamma_2)+1) - \sum_{\lambda_k(\Gamma_2)\in(0,1)} (\lambda_k(\Gamma_2)-1) + \xi. \end{aligned}$$

On both sides, the terms of Γ_2 are common. Therefore,

$$E(H_1) = E(H_2).$$

□

Example 2.1. *The regular graphs $K_{n,n}\Box K_{n-1}$ and $K_{n-1,n-1}\Box K_n$ are non-isomorphic having the degree $2n-2$ and order $2n^2-2n$, where $\Box$ denotes the Cartesian product. For all $n \geq 5$ and $k \geq 0$, these graphs $L^k(K_{n,n}\Box K_{n-1})$ and $L^k(K_{n-1,n-1}\Box K_n)$ are equienergetic* [14]*. By Corollary 2.2, $L^k(K_{n,n}\Box K_{n-1})\blacktriangledown\Gamma_2$ and $L^k(K_{n-1,n-1}\Box K_n)\blacktriangledown\Gamma_2$ are equienergetic, non-regular graphs.*

The following finding presents a large collection of non-regular equienergetic graphs.

Proposition 2.1. *Let $H_i; i=1,2$ be two $r(\geq 3)$-regular graphs of same order n. Let Γ_2 be any graph. Then $L^k(H_i)\blacktriangledown\Gamma_2; i=1,2$ are equienergetic graphs.*

Proof. If $H_i; i=1,2$ denote $r(\geq 3)$-regular graphs with order n. Then the graphs $L^k(H_i); i=1,2$ and $k\geq 2$ are equienergetic graphs of same degree by Theorem 4.1 of [13]. Therefore, by this observation and Corollary 2.2 completes the proof. $\Box$

Corollary 2.3. *Let the order of an r_k-regular graph Γ_k be n_k, where $k=1,2$. Then $\Gamma_1\blacktriangledown\Gamma_2$ is non-hyperenergetic if $E(\Gamma_1)+E(\Gamma_2)\leqslant 2n_1+n_2-1-\frac{\xi}{2}$.*

Proof. The two graphs Γ_1 and Γ_2 of order n_1 and n_2 then the order of $\Gamma_1\blacktriangledown\Gamma_2$ is $2(n_1+n_2)$. If $E(\Gamma_1)+E(\Gamma_2)\leqslant 2n_1+n_2-1-\frac{\xi}{2}$, then by Corollary 2.1, we have following

$$E(\Gamma_1\blacktriangledown\Gamma_2) < 2(E(\Gamma_1)+E(\Gamma_2))+2n_2+\xi \leqslant 2(2(n_1+n_2)-1).$$

This shows that, the graph $\Gamma_1\blacktriangledown\Gamma_2$ is non-hyperenergetic. $\Box$

Corollary 2.4. *Let the order of an r_k-regular graph Γ_k be n_k, where $k=1,2$. Then $\Gamma_1\blacktriangledown\Gamma_2$ is borderenergetic if and only if*

$$\begin{aligned}E(\Gamma_1)+E(\Gamma_2) = 2(n_1+n_2)+(r_1+r_2)-n_2^0-&\sum_{\lambda_k(\Gamma_2)\in(-1,0)}(\lambda_k(\Gamma_2)+1)\\ &+\sum_{\lambda_k(\Gamma_2)\in(0,1)}(\lambda_k(\Gamma_2)-1)-\frac{\xi}{2}-1.\end{aligned}$$

Specifically, if $\lambda_k(\Gamma_2)\notin(-1,1)$, then $\Gamma_1\blacktriangledown\Gamma_2$ is borderenergetic if and only if $E(\Gamma_1)+E(\Gamma_2)=2(n_1+n_2)+(r_1+r_2)-1-\frac{\xi}{2}$.

Proof. By the definition of borderenergetic graph and Theorem 2.1 together provide the proof. $\Box$

Corollary 2.5. *Let the order of an r_k-regular graph Γ_k be n_k, where $k = 1, 2$. Then $\Gamma_1 \blacktriangledown \Gamma_2$ is orderenergetic if and only if*

$$E(\Gamma_1) + E(\Gamma_2) = (n_1 + n_2) + (r_1 + r_2) - n_2^0 - \sum_{\lambda_k(\Gamma_2)\in(-1,0)} (\lambda_k(\Gamma_2) + 1) + \sum_{\lambda_k(\Gamma_2)\in(0,1)} (\lambda_k(\Gamma_2) - 1) - \frac{\xi}{2}.$$

Specifically if $\lambda_k(\Gamma_2) \notin (-1, 1)$ then, $\Gamma_1 \blacktriangledown \Gamma_2$ is orderenergetic if and only if $E(\Gamma_1) + E(\Gamma_2) = 2(n_1 + n_2) + (r_1 + r_2) - \frac{\xi}{2}$.

Proof. By the definition of orderenergetic graph and Theorem 2.1 together provide the proof. □

Theorem 2.2. *Let the order of an r_k-regular graph Γ_k be n_k, where $k = 1, 2$. Then,*

$$\begin{aligned} E(\overline{\Gamma}_1 \blacktriangledown \overline{\Gamma}_2) &= 2(E(\Gamma_1) + E(\Gamma_2)) + 2(n_1 + n_2) - 2(r_1 + r_2) - 4(n_1^- + n_2^-) - 4 \\ &\quad +4 \sum_{\lambda_k(\Gamma_1)\in(-1,0)} (\lambda_k(\Gamma_1) + 1) + 2 \sum_{\lambda_k(\Gamma_2)\in(-2,0)} (\lambda_k(\Gamma_2) + 2) + \xi_1. \end{aligned}$$

Proof. If $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_n$ are the eigenvalues of any regular graph Γ, then the eigenvalues of complement of Γ are $n - 1 - \lambda_1, -(\lambda_2 + 1), -(\lambda_3 + 1), \cdots, -(\lambda_n + 1)$. From Proposition 1.1, the eigenvalues of Indu-Bala product $\overline{\Gamma_1} \blacktriangledown \overline{\Gamma_2}$ are as follows:

(a) $-(\lambda_k(\Gamma_1) + 1)$, with multiplicity 2 for $k = 2, 3, \cdots, n_1$;

(b) $-\lambda_k(\Gamma_2)$ for $k = 2, 3, \cdots, n_2$;

(c) $-(\lambda_k(\Gamma_2) + 2)$ for $k = 2, 3, \cdots, n_2$;

(d) $\frac{(n_1+n_2)-(r_1+r_2+1)\pm\sqrt{((n_1+n_2)-(r_1+r_2+1))^2-4((n_1-1-r_1)(n_2-r_2)-n_1n_2)}}{2}$ and $\frac{(n_1+n_2)-(r_1+r_2+3)\pm\sqrt{((n_1+n_2)-(r_1+r_2+3))^2-4((n_1-1-r_1)(n_2-r_2-2)-n_1n_2)}}{2}$

Here, we denote the absolute sum of the all eigenvalues in the (d) case as ξ_1

$$\begin{aligned} &E(\overline{\Gamma_1} \blacktriangledown \overline{\Gamma_2}) \\ &= 2\sum_{k=2}^{n_1} | -\lambda_k(\Gamma_1) - 1 | + \sum_{k=2}^{n_2} | -\lambda_k(\Gamma_2) | + \sum_{k=2}^{n_2} | -\lambda_k(\Gamma_2) - 2 | + \xi_1 \\ &= 2\sum_{k=1}^{n_1} | \lambda_k(\Gamma_1) + 1 | - 2(r_1 + 1) + \sum_{k=1}^{n_2} | \lambda_k(\Gamma_2) | - r_2 \\ &\quad + \sum_{k=1}^{n_2} | \lambda_k(\Gamma_2) + 2 | - (r_2 + 2) + \xi_1. \end{aligned}$$

Using Propositions 1.2 and 1.3, we get $E(\overline{\Gamma_1}\blacktriangledown\overline{\Gamma_2})$

$$\begin{aligned}
&= 2E(\Gamma_1) + 2n_1 - 4n_1^- + 4\sum_{\lambda_k(\Gamma_1)\in(-1,0)}(\lambda_k(\Gamma_1)+1) - 2(r_1+1)\\
&\quad +E(\Gamma_2) - r_2 + E(\Gamma_2) + 2n_2 - 4n_2^-\\
&\quad +2\sum_{\lambda_k(\Gamma_2)\in(-2,0)}(\lambda_k(\Gamma_2)+2) - (r_2+2) + \xi_1\\
&= 2E(\Gamma_1) + 2E(\Gamma_2) + 4\sum_{\lambda_k(\Gamma_1)\in(-1,0)}(\lambda_k(\Gamma_1)+1) - 4(n_1^- + n_2^-) - 4\\
&\quad +2(n_1+n_2) - 2(r_1+r_2) + 2\sum_{\lambda_k(\Gamma_2)\in(-2,0)}(\lambda_k(\Gamma_2)+2) + \xi_1\\
&\quad = 2(E(\Gamma_1)+E(\Gamma_2)) + 2(n_1+n_2) - 2(r_1+r_2) - 4(n_1^- + n_2^-) - 4\\
&\quad +4\sum_{\lambda_k(\Gamma_1)\in(-1,0)}(\lambda_k(\Gamma_1)+1) + 2\sum_{\lambda_k(\Gamma_2)\in(-2,0)}(\lambda_k(\Gamma_2)+2) + \xi_1.
\end{aligned}$$

□

Corollary 2.6. *Let the order of an r_k-regular graph Γ_k be n_k, where $k = 1, 2$. Then,*

$$\begin{aligned}
2(E(\Gamma_1)+E(\Gamma_2)) - 4(n_1^- + n_2^-) - 2(r_1+r_2) - 4 + \xi_1 &\le E(\overline{\Gamma_1}\blacktriangledown\overline{\Gamma_2})\\
&< 2(E(\Gamma_1)+E(\Gamma_2)) + 2(n_1+n_2) + \xi_1.
\end{aligned}$$

Equality holds at the left side if and only if Γ_1 has no eigenvalues in the interval $(-1, 0)$ and Γ_2 has no eigenvalues in the interval $(-2, 0)$.

Proof. For upper bound, it can be seen from Theorem 2.2

$$n_1^- - \sum_{\lambda_k(\Gamma_1)\in(-1,0)}(\lambda_k(\Gamma_1)+1) > 0 \text{ and } 2n_2^- - \sum_{\lambda_k(\Gamma_2)\in(-2,0)}(\lambda_k(\Gamma_2)+2) > 0$$

Along these, if we can eliminate the values 4, r_1, r_2 from $E(\overline{\Gamma_1}\blacktriangledown\overline{\Gamma_2})$ in Theorem 2.2 as these are positive, we obtain,

$$E(\overline{\Gamma_1}\blacktriangledown\overline{\Gamma_2}) < 2E(\Gamma_1) + 2E(\Gamma_2) + 2(n_1+n_2) + \xi_1.$$

For lower bound,

$$\sum_{\lambda_k(\Gamma_1)\in(-1,0)}(\lambda_k(\Gamma_1)+1) > 0 \text{ and } \sum_{\lambda_k(\Gamma_2)\in(-2,0)}(\lambda_k(\Gamma_2)+2) > 0$$

and $n_1, n_2 \ge 0$, on removing these values from Theorem 2.2, we obtain, $2(E(\Gamma_1)+E(\Gamma_2)) - 4(n_1^- + n_2^-) - 2(r_1+r_2) - 4 + \xi_1 < E(\overline{\Gamma_1}\blacktriangledown\overline{\Gamma_2})$.

The equality holds at the left side by the following fact

$$\sum_{\lambda_k(\Gamma_1)\in(-1,0)} (\lambda_k(\Gamma_1)+1) = 0 \text{ and } \sum_{\lambda_k(\Gamma_2)\in(-2,0)} (\lambda_k(\Gamma_2)+2) = 0$$

if and only if Γ_1 has no eigenvalues in the interval $(-1,0)$ and Γ_2 has no eigenvalues in the interval $(-2,0)$. □

Corollary 2.7. *Let the order of an r_k-regular graph Γ_k be n_k, where $k=1,2$. Then, $\overline{\Gamma_1}\blacktriangledown\overline{\Gamma_2}$ is non-hyperenergetic if $E(\Gamma_1)+E(\Gamma_2)\leqslant(n_1+n_2)-1-\frac{\xi_1}{2}$.*

Proof. If $\overline{\Gamma_1}$ and $\overline{\Gamma_2}$ are graphs of order n_1 and n_2, then order of $\overline{\Gamma_1}\blacktriangledown\overline{\Gamma_2}$ is $2(n_1+n_2)$. If $E(\Gamma_1)+E(\Gamma_2)\leqslant(n_1+n_2)-1-\frac{\xi_1}{2}$ and by Corollary 2.6, we have the following equation.

$$\text{i.e. } E(\overline{\Gamma_1}\blacktriangledown\overline{\Gamma_2}) < 2(E(\Gamma_1)+E(\Gamma_2))+2(n_1+n_2)+\xi_1 \leqslant 2(2(n_1+n_2)-1).$$

This shows that, the graph $\overline{\Gamma_1}\blacktriangledown\overline{\Gamma_2}$ is non-hyperenergetic. □

Corollary 2.8. *Let the order of an r_k-regular graph Γ_k be n_k, where $k=1,2$. Then, $\overline{G_1}\blacktriangledown\overline{\Gamma_2}$ is borderenergetic if and only if*

$$E(\Gamma_1)+E(\Gamma_2) = (n_1+n_2)+2(n_1^-+n_2^-)+(r_1+r_2)+1$$
$$-2\sum_{\lambda_k(\Gamma_1)\in(-1,0)}(\lambda_k(\Gamma_1)+1) - \sum_{\lambda_k(\Gamma_2)\in(-2,0)}(\lambda_k(\Gamma_2)+2) - \frac{\xi_1}{2}$$

Specifically, if Γ_1 and Γ_2 contains no eigenvalues in the interval $(-1,0)$ and $(-2,0)$ respectively, then $\overline{\Gamma_1}\blacktriangledown\overline{\Gamma_2}$ is borderenergetic if and only if $E(\Gamma_1)+E(\Gamma_2)=2(n_1+n_2)+2(n_1^-+n_2^-)+(r_1+r_2)+1-\frac{\xi_1}{2}$.

Proof. The definition of borderenergetic and Theorem 2.2 together provide the proof. □

Corollary 2.9. *Let the order of an r_k-regular graph Γ_k be n_k, where $k=1,2$. Then $\overline{\Gamma_1}\blacktriangledown\overline{\Gamma_2}$ is orderenergetic if and only if*

$$E(\Gamma_1)+E(\Gamma_2) = 2(n_1^-+n_2^-)+(r_1+r_2)+2$$
$$-2\sum_{\lambda_k(\Gamma_1)\in(-1,0)}(\lambda_k(\Gamma_1)+1) - \sum_{\lambda_k(\Gamma_2)\in(-2,0)}(\lambda_k(\Gamma_2)+2) - \frac{\xi_1}{2}.$$

Specifically, if Γ_1 and Γ_2 contains no eigenvalues in the intervals $(-1,0)$ and $(-2,0)$ respectively, then $\overline{\Gamma_1}\blacktriangledown\overline{\Gamma_2}$ is orderenergetic if and only if $E(\Gamma_1)+E(\Gamma_2)=2(n_1+n_2)+2(n_1^-+n_2^-)+(r_1+r_2)-\frac{\xi_1}{2}$.

Proof. The definition of orderenergetic and Theorem 2.2 together provide the proof. □

Corollary 2.10. *Let the order of an r-regular graph H_k; $k = 1, 2$ be n, with no eigenvalues in the interval $(-1, 0)$. Then $\overline{H_k} \blacktriangledown \overline{\Gamma_2}; k = 1, 2$ are equienergetic graphs if and only if $H_k; k = 1, 2$ are equienergetic with same number of negative eigenvalues.*

Proof. Proof follows from Theorem 2.2 that $\overline{H_i} \blacktriangledown \overline{\Gamma_2}; i = 1, 2$ are equienergetic graphs if and only if

$$\begin{aligned}
&2(E(H_1) + E(\Gamma_2)) + 2(n_1 + n_2) - 2(r + r_2) - 4(n_1^- + n_2^-) - 4 \\
&+4 \sum_{\lambda_k(H_1)\in(-1,0)} (\lambda_k(H_1) + 1) + 2 \sum_{\lambda_k(\Gamma_2)\in(-2,0)} (\lambda_k(\Gamma_2) + 2) + \xi_1 = 2(E(H_2) + E(\Gamma_2)) \\
&\qquad +2(n_1 + n_2) - 2(r + r_2) - 4(n_1^{*-} + n_2^-) - 4 \\
&+4 \sum_{\lambda_k(H_2)\in(-1,0)} (\lambda_k(H_2) + 1) + 2 \sum_{\lambda_k(\Gamma_2)\in(-2,0)} (\lambda_k(\Gamma_2) + 2) + \xi_1.
\end{aligned}$$

Here, n_1^{*-} denotes the number negative eigenvalues in H_2.
On both sides, the terms of Γ_2 are common and also, H_1 and H_2 have same regularity. Therefore,

$$E(H_1) - 2n_1^- = E(H_2) - 2n_1^{*-}.$$

□

Example 2.2. *Let us take the graphs in Example 2.1. These are integral graphs, which means no eigenvalues in $(-1, 0)$. These graphs posses same count of negative eigenvalues. Therefore, by Corollary 2.10, $\overline{L^k(K_{n,n} \square K_{n-1})} \blacktriangledown \overline{\Gamma_2}$ and $\overline{L^k(K_{n-1,n-1} \square K_n)} \blacktriangledown \overline{\Gamma_2}$ are equienergetic graphs.*

The following finding presents another large collection of non-regular equienergetic graphs.

Proposition 2.2. *Let the order of an $r(\geq 3)$-regular graph H_k; $k = 1, 2$ be n and Γ_2 be any graph. Then $\overline{L^k(H_i)} \blacktriangledown \overline{\Gamma_2}$; $i = 1, 2$ and $k \geq 2$ are equienergetic graphs.*

Proof. If $H_i; i = 1, 2$ denote $r(\geq 3)$-regular graphs with order n. Then by Theorem 4.1 of [13], the graphs $L^k(H_i); i = 1, 2$ and $k \geq 2$ are equienergetic graphs of same degree. In addition these have all negative eigenvalues equal to -2. Therefore, by this observation and Corollary 2.10 completes the proof. □

3. Conclusion

In this paper, we calculate the energy of the Indu-Bala product two regular graphs. Furthermore, we investigate the properties such as equienergetic, borderenergetic, orderenergetic and non-hyperenergetic characteristics using the Indu-Bala product. Further, one can study the Indu-Bala product of two non-regular graphs.

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