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LORENTZIAN β-KENMOTSU MANIFOLD ADMITTING GENERALIZED TANAKA-WEBSTER CONNECTION

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Abstract. In this manuscript, we investigate Lorentzian β-Kenmotsu manifold admitting generalized Tanaka-Webster connection (GTWC) $\widetilde{\widetilde{\nabla}}$. We study curvature tensor and its properties with respect to the above connection. Further, we study the connection on extended generalized φ-recurrent Lorentzian β-Kenmotsu manifold. We also investigate the properties of projectively flat, ζ-projectively flat and η-parallel φ-tensor on Lorentzian β-Kenmotsu manifold admitting the connection $\widetilde{\widetilde{\nabla}}$. Moreover, we study Ricci soliton on the above manifold with respect to the connection (GTWC). Finally, we give an example of 3-dimensional Lorentzian β-Kenmotsu manifold verifying our results.

Keywords: Lorentzian β-Kenmotsu manifold, generalized Tanaka-Webster connection, generalized η-Einstein manifold, Ricci soliton, projectively flat.

2010 Mathematics Subject Classification: Primary 53C05, 53C20, 53C25, Secondary 53D15.

1. Introduction

The semi-Riemannian geometry [29] fascinates the researchers because of its abilities to determine the several problems of science, technology, medical and their related areas. A differentiable manifold $\mathfrak{M}$ of dimension $(2n+1)$ equipped with a semi-Riemannian metric g, whose signature is (p, q), $(p+q = 2n+1)$, referred to as $(2n+1)$-dimensional semi-Riemannian manifold. In particular, if we replace p by 1 and q by $2n$, then the semi-Riemannian manifold

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$\mathfrak{M}$ reduces into Lorentzian manifold. The basic characterization of the vectors in a Lorentzian manifold were the starting point to study the geometry of it. As a reason, Lorentzian manifold $\mathfrak{M}$ is the finest choice for the researchers to study the general theory of relativity and cosmological models. The material substance of the cosmos is referred to behave like a perfect fluid space-time in standard cosmological models. In describing the gravity of the space-time, the Riemannian curvature $\mathfrak{R}$, the Ricci tensor $\mathcal{S}$, and the scalar curvature $\mathfrak{r}$ play an essential role.

In the Gray-Hervella classification of almost Hermitian manifolds [7], there appears a class $\mathcal{W}_4$, of Hermitian manifolds which are closely related to locally conformal Kähler manifolds [5]. An almost contact metric structure $(\varphi, \zeta, \eta, g)$ on $\mathfrak{M}$ is referred to as trans-Sasakian structure [15] if $(\mathfrak{M} \times \mathbb{R}, \mathcal{J}, \mathcal{G})$ belongs to the class $\mathcal{W}_4$ [7], where $\mathcal{J}$ is the almost complex structure on $\mathfrak{M} \times \mathbb{R}$ defined by

$$\mathcal{J}\left(\mathfrak{U}_1, \frac{fd}{dt}\right) = \left(\varphi\mathfrak{U}_1 - f\zeta, \eta(\mathfrak{U}_1)\frac{fd}{dt}\right)$$

for all vector fields $\mathfrak{U}_1$ on $\mathfrak{M}$, smooth functions f on $\mathfrak{M} \times \mathbb{R}$ and $\mathcal{G}$ is the product metric on $\mathfrak{M} \times \mathbb{R}$. This can be defined by [4]

$$(\nabla_{\mathfrak{U}_1}\varphi)\mathfrak{U}_2 = \alpha(g(\mathfrak{U}_1, \mathfrak{U}_2)\zeta - \eta(\mathfrak{U}_2)\mathfrak{U}_1) + \beta(g(\varphi\mathfrak{U}_1, \mathfrak{U}_2)\zeta - \eta(\mathfrak{U}_2)\varphi\mathfrak{U}_1) \tag{1.1}$$

for some smooth functions α, β on $\mathfrak{M}$ and we say that the trans-Sasakian structure is of type (α, β).

The concept of α-Sasakian and β-Kenmotsu manifolds was initiated by Janssens and Vanhecke in 1981, where α and β are non-zero real numbers. We know that [11] trans-Sasakian structure of type $(0, 0)$, $(0, \beta)$, and $(\alpha, 0)$ are cosymplectic [3, 4], β-Kenmotsu, and α-Sasakian, respectively. Marrero [13] proved that a trans-Sasakian manifold of dimension $n \geq 5$ is either cosymplectic or α-Sasakian or β-Kenmotsu manifold.

Tanno [25] studied the generalized Tanaka-Webster connection (GTWC) for contact metric manifolds by using the canonical connection. This connection coincides with the Tanaka-Webster connection if the associated CR-structure is integrable. Using this connection, some characterizations of real hypersurfaces in complex space forms [23] have been studied by few geometers. Recently, many authors [6, 12, 16, 18, 20, 22] studied generalized Tanaka-Webster connection (GTWC) in Kenmotsu manifolds.

Hamilton [8] introduced the theory of Ricci flow to establish a canonical metric on a smooth manifold in 1982. The Ricci flow is an evolution equation for metrics on a Riemannian manifold defined by

$$\frac{\partial}{\partial t} g_{ij}(t) = -2\mathfrak{R}_{ij}.$$

A Ricci soliton $(g, \mathcal{V}, \Theta)$ on a Riemannian manifold $(\mathfrak{M}, g)$ is a generalization of an Einstein metric such that it satisfies the following condition [9, 10]:

$$\mathfrak{L}_{\mathcal{V}} g + 2\mathcal{S} + 2\Theta g = 0, \tag{1.2}$$

where $\mathcal{S}$ is the Ricci tensor, $\mathfrak{L}_{\mathcal{V}}$ is the Lie derivative operator along the vector field $\mathcal{V}$ on $(\mathfrak{M}, g)$ and Θ is a real number. The Ricci soliton $(g, \mathcal{V}, \Theta)$ is said to be shrinking, steady, and expanding according to $\Theta < 0, \Theta = 0$, and $\Theta > 0$, respectively.

In this paper, we have taken β as a real constant. Motivated by above studies, the present work is classified as follows: After the introduction, we give a brief account of Lorentzian β-Kenmotsu manifold in section 2. In section 3, we study the expressions for curvature tensor and some results on Lorentzian β-Kenmotsu manifold with respect to GTWC $\widetilde{\widetilde{\nabla}}$. In section 4, we also study extended generalized φ-recurrent Lorentzian β-Kenmotsu manifold admitting the GTWC $\widetilde{\widetilde{\nabla}}$. In section 5, we investigate the properties of projectively flat, ζ-projectively flat and η-parallel φ-tensor on Lorentzian β-Kenmotsu manifold with respect to the GTWC $\widetilde{\widetilde{\nabla}}$. Moreover, in section 6, we study Ricci soliton on Lorentzian β-Kenmotsu manifold admitting the GTWC $\widetilde{\widetilde{\nabla}}$. In the last section, we give an example of 3-dimensional Lorentzian β-Kenmotsu manifold with respect to the GTWC $\widetilde{\widetilde{\nabla}}$ varifying our results.

2. Preliminaries

A differentiable manifold of dimension $(2n + 1)$ is referred to as Lorentzian β-Kenmotsu manifold if it admits a $(1, 1)$-tensor field φ, a contravariant vector field ζ, a covariant vector field η and Lorentzian metric g which satisfy

$$\eta(\zeta) = -1, \quad \varphi\zeta = 0, \quad \eta(\varphi\mathfrak{U}_1) = 0, \tag{2.3}$$

$$\varphi^2(\mathfrak{U}_1) = \mathfrak{U}_1 + \eta(\mathfrak{U}_1)\zeta, \quad g(\mathfrak{U}_1, \zeta) = \eta(\mathfrak{U}_1), \tag{2.4}$$

$$g(\varphi\mathfrak{U}_1, \varphi\mathfrak{U}_2) = g(\mathfrak{U}_1, \mathfrak{U}_2) + \eta(\mathfrak{U}_1)\eta(\mathfrak{U}_2), \quad g(\varphi\mathfrak{U}_1, \mathfrak{U}_2) = g(\mathfrak{U}_1, \varphi\mathfrak{U}_2) \tag{2.5}$$

$\forall\, \mathfrak{U}_1, \mathfrak{U}_2 \in \mathfrak{X}(\mathfrak{M})$, where $\mathfrak{X}(\mathfrak{M})$ is a set of all smooth vector fields on $\mathfrak{M}$. Then such a quartet $(\varphi, \zeta, \eta, g)$ is known as Lorentzian para-contact quartet and the manifold $\mathfrak{M}$ with a Lorentzian para-contact quartet is referred to as a Lorentzian para-contact manifold [14, 19, 21].

On a Lorentzian para-contact manifold, we also have

$$(\nabla_{\mathfrak{U}_1}\varphi)\mathfrak{U}_2 = \beta[g(\varphi\mathfrak{U}_1, \mathfrak{U}_2)\zeta - \eta(\mathfrak{U}_2)\varphi\mathfrak{U}_1] \tag{2.6}$$

$\forall\ \mathfrak{U}_1, \mathfrak{U}_2 \in \mathfrak{X}(\mathfrak{M})$, where ∇ is the Levi-Civita connection with respect to the Lorentzian metric g. Therefore a Lorentzian para-contact manifold satisfying (2.6) is referred to as a Lorentzian β-Kenmotsu manifold [27].

On a Lorentzian β-Kenmotsu manifold $\mathfrak{M}$, the following relations hold [1, 2]:

$$\nabla_{\mathfrak{U}_1}\zeta = \beta[\mathfrak{U}_1 - \eta(\mathfrak{U}_1)\zeta], \tag{2.7}$$

$$(\nabla_{\mathfrak{U}_1}\eta)\mathfrak{U}_2 = \beta[g(\mathfrak{U}_1, \mathfrak{U}_2) - \eta(\mathfrak{U}_1)\eta(\mathfrak{U}_2)], \tag{2.8}$$

$$\eta(\mathfrak{R}(\mathfrak{U}_1, \mathfrak{U}_2)\mathfrak{U}_3) = \beta^2[g(\mathfrak{U}_1, \mathfrak{U}_3)\eta(\mathfrak{U}_2) - g(\mathfrak{U}_2, \mathfrak{U}_3)\eta(\mathfrak{U}_1)], \tag{2.9}$$

$$\mathfrak{R}(\mathfrak{U}_1, \mathfrak{U}_2)\zeta = \beta^2[\eta(\mathfrak{U}_1)\mathfrak{U}_2 - \eta(\mathfrak{U}_2)\mathfrak{U}_1], \tag{2.10}$$

$$\mathfrak{R}(\zeta, \mathfrak{U}_1)\mathfrak{U}_2 = \beta^2[\eta(\mathfrak{U}_2)\mathfrak{U}_1 - g(\mathfrak{U}_1, \mathfrak{U}_2)\zeta], \tag{2.11}$$

$$\mathcal{S}(\mathfrak{U}_1, \zeta) = -2n\beta^2\eta(\mathfrak{U}_1), \tag{2.12}$$

$$\mathcal{S}(\mathfrak{U}_1, \mathfrak{U}_2) = g(\mathfrak{Q}\mathfrak{U}_1, \mathfrak{U}_2), \tag{2.13}$$

$$\mathfrak{Q}\mathfrak{U}_1 = -2n\beta^2\mathfrak{U}_1, \tag{2.14}$$

$$\mathfrak{Q}\zeta = -2n\beta^2\zeta, \tag{2.15}$$

$$\mathcal{S}(\varphi\mathfrak{U}_1, \varphi\mathfrak{U}_2) = g(\mathfrak{Q}\varphi\mathfrak{U}_1, \varphi\mathfrak{U}_2). \tag{2.16}$$

Using (2.5), (2.13), (2.14) and $\mathfrak{Q}\varphi = \varphi\mathfrak{Q}$, we have

$$\mathcal{S}(\varphi\mathfrak{U}_1, \varphi\mathfrak{U}_2) = \mathcal{S}(\mathfrak{U}_1, \mathfrak{U}_2) - 2n\beta^2\eta(\mathfrak{U}_1)\eta(\mathfrak{U}_2), \tag{2.17}$$

$$\mathcal{S}(\zeta, \zeta) = 2n\beta^2 \tag{2.18}$$

$\forall\ \mathfrak{U}_1, \mathfrak{U}_2, \mathfrak{U}_3 \in \mathfrak{X}(\mathfrak{M})$. Where $\mathfrak{R}$, $\mathcal{S}$, and $\mathfrak{Q}$ denote the curvature tensor of type $(1, 3)$, Ricci tensor of type $(0, 2)$, and Ricci operator, respectively with respect to the connection ∇.

Definition 2.1. *The projective curvature tensor $\mathcal{P}$ in $(2n+1)$-dimensional Lorentzian β-Kenmotsu manifold $\mathfrak{M}$ with respect to the connection ∇ is defined by*

$$\mathcal{P}(\mathfrak{U}_1,\mathfrak{U}_2)\mathfrak{U}_3 = \mathfrak{R}(\mathfrak{U}_1,\mathfrak{U}_2)\mathfrak{U}_3 - \frac{1}{2n}[g(\mathfrak{U}_2,\mathfrak{U}_3)\mathfrak{Q}\mathfrak{U}_1 - g(\mathfrak{U}_1,\mathfrak{U}_3)\mathfrak{Q}\mathfrak{U}_2] \tag{2.19}$$

$\forall\ \mathfrak{U}_1,\mathfrak{U}_2,\mathfrak{U}_3 \in \mathfrak{X}(\mathfrak{M})$. *The manifold is said to be projectively flat if $\mathcal{P}$ vanishes identically on $\mathfrak{M}$.*

Definition 2.2. *A $(2n+1)$-dimensional Lorentzian β-Kenmotsu manifold is said to be ζ-projectively flat with respect to Levi-Civita connection ∇ if*

$$\mathcal{P}(\mathfrak{U}_1,\mathfrak{U}_2)\zeta = 0 \tag{2.20}$$

$\forall\ \mathfrak{U}_1,\mathfrak{U}_2 \in \mathfrak{X}(\mathfrak{M})$.

Definition 2.3. *If the $(1,1)$ tensor φ is η-parallel in a Lorentzian β-Kenmotsu manifold $\mathfrak{M}^{2n+1}$, then we have*

$$g((\nabla_{\mathfrak{U}_1}\varphi)\mathfrak{U}_2,\mathfrak{U}_3) = 0 \tag{2.21}$$

$\forall\ \mathfrak{U}_1,\mathfrak{U}_2,\mathfrak{U}_3 \in \mathfrak{X}(\mathfrak{M})$.

3. The generalized Tanaka-Webster connection (GTWC) $\widetilde{\nabla}$

Tanno defined the generalized Tanaka-Webster connection (GTWC) $\widetilde{\nabla}$ for contact metric manifolds. It is given by[24]

$$\widetilde{\nabla}_{\mathfrak{U}_1}\mathfrak{U}_2 = \nabla_{\mathfrak{U}_1}\mathfrak{U}_2 + (\nabla_{\mathfrak{U}_1}\eta)(\mathfrak{U}_2)\zeta - \eta(\mathfrak{U}_2)\nabla_{\mathfrak{U}_1}\zeta - \eta(\mathfrak{U}_1)\varphi\mathfrak{U}_2 \tag{3.22}$$

$\forall\ \mathfrak{U}_1,\mathfrak{U}_2 \in \mathfrak{X}(\mathfrak{M})$.

By virtue of (2.7) and (2.8), equation (3.22) takes the form

$$\widetilde{\nabla}_{\mathfrak{U}_1}\mathfrak{U}_2 = \nabla_{\mathfrak{U}_1}\mathfrak{U}_2 + \beta g(\mathfrak{U}_1,\mathfrak{U}_2)\zeta - \beta\eta(\mathfrak{U}_2)\mathfrak{U}_1 - \eta(\mathfrak{U}_1)\varphi\mathfrak{U}_2. \tag{3.23}$$

Replacing $\mathfrak{U}_2$ by ζ in (3.23) and using (2.3), (2.4), (2.7), we have

$$\widetilde{\nabla}_{\mathfrak{U}_1}\zeta = 2\beta\mathfrak{U}_1. \tag{3.24}$$

Now

$$(\widetilde{\nabla}_{\mathfrak{U}_1}\varphi)(\mathfrak{U}_2) = \widetilde{\nabla}_{\mathfrak{U}_1}(\varphi\mathfrak{U}_2) - \varphi(\widetilde{\nabla}_{\mathfrak{U}_1}\mathfrak{U}_2). \tag{3.25}$$

Using (2.6) and (3.23) in (3.25), we have

$$(\widetilde{\nabla}_{\mathfrak{U}_1}\varphi)(\mathfrak{U}_2) = \beta g(\varphi\mathfrak{U}_1,\mathfrak{U}_2)\zeta + \eta(\mathfrak{U}_1)\mathfrak{U}_2 + \eta(\mathfrak{U}_1)\eta(\mathfrak{U}_2)\zeta. \tag{3.26}$$

Now

$$(\widetilde{\nabla}_{\mathfrak{U}_1}\eta)(\mathfrak{U}_2) = \widetilde{\nabla}_{\mathfrak{U}_1}\eta(\mathfrak{U}_2) - \eta(\widetilde{\nabla}_{\mathfrak{U}_1}\mathfrak{U}_2). \tag{3.27}$$

Using (3.23) in (3.27), we have

$$(\widetilde{\nabla}_{\mathfrak{U}_1}\eta)(\mathfrak{U}_2) = 2\beta g(\mathfrak{U}_1, \mathfrak{U}_2). \tag{3.28}$$

Now

$$(\widetilde{\nabla}_{\mathfrak{U}_1}g)(\mathfrak{U}_2, \mathfrak{U}_3) = \widetilde{\nabla}_{\mathfrak{U}_1}g(\mathfrak{U}_2, \mathfrak{U}_3) - g(\widetilde{\nabla}_{\mathfrak{U}_1}\mathfrak{U}_2, \mathfrak{U}_3) - g(\mathfrak{U}_2, \widetilde{\nabla}_{\mathfrak{U}_1}\mathfrak{U}_3). \tag{3.29}$$

Using (3.23) in (3.29), we have

$$(\widetilde{\nabla}_{\mathfrak{U}_1}g)(\mathfrak{U}_2, \mathfrak{U}_3) = 2\eta(\mathfrak{U}_1)g(\varphi\mathfrak{U}_2, \mathfrak{U}_3) \neq 0. \tag{3.30}$$

Thus we can state the following :

Theorem 3.1. *The GTWC $\widetilde{\nabla}$ on a Lorentzian β-Kenmotsu manifold is a non-metric connection.*

Now the torsion tensor $\widetilde{\mathcal{T}}$ of the GTWC $\widetilde{\nabla}$ is given as:

$$\widetilde{\mathcal{T}}(\mathfrak{U}_1, \mathfrak{U}_2) = \widetilde{\nabla}_{\mathfrak{U}_1}\mathfrak{U}_2 - \widetilde{\nabla}_{\mathfrak{U}_2}\mathfrak{U}_1 - [\mathfrak{U}_1, \mathfrak{U}_2]. \tag{3.31}$$

Using (3.23) in (3.31), we have

$$\widetilde{\mathcal{T}}(\mathfrak{U}_1, \mathfrak{U}_2) = \beta\eta(\mathfrak{U}_1)\mathfrak{U}_2 - \beta\eta(\mathfrak{U}_2)\mathfrak{U}_1 - \eta(\mathfrak{U}_1)\varphi\mathfrak{U}_2 + \eta(\mathfrak{U}_2)\varphi\mathfrak{U}_1. \tag{3.32}$$

Now we have the following:

Theorem 3.2. *The GTWC $\widetilde{\nabla}$ on a Lorentzian β-Kenmotsu manifold associated to the connection ∇ of $\mathfrak{M}$ is just the only one affine connection, which is non-metric and its torsion has the form (3.32)*

Let $\mathfrak{R}$ and $\widetilde{\mathfrak{R}}$ denote the curvature tensors of the connections ∇ and $\widetilde{\nabla}$, respectively. Then

$$\widetilde{\mathfrak{R}}(\mathfrak{U}_1, \mathfrak{U}_2)\mathfrak{U}_3 = \widetilde{\nabla}_{\mathfrak{U}_1}\widetilde{\nabla}_{\mathfrak{U}_2}\mathfrak{U}_3 - \widetilde{\nabla}_{\mathfrak{U}_2}\widetilde{\nabla}_{\mathfrak{U}_1}\mathfrak{U}_3 - \widetilde{\nabla}_{[\mathfrak{U}_1, \mathfrak{U}_2]}\mathfrak{U}_3. \tag{3.33}$$

Using (2.3), (2.4), (2.5), (2.6), (2.7) and (3.23) in (3.33), we have

$$\begin{aligned}\widetilde{\mathfrak{R}}(\mathfrak{U}_1, \mathfrak{U}_2)\mathfrak{U}_3 &= \mathfrak{R}(\mathfrak{U}_1, \mathfrak{U}_2)\mathfrak{U}_3 + 3\beta^2[g(\mathfrak{U}_2, \mathfrak{U}_3)\mathfrak{U}_1 - g(\mathfrak{U}_1, \mathfrak{U}_3)\mathfrak{U}_2] \\ &\quad -2\beta[g(\varphi\mathfrak{U}_1, \mathfrak{U}_3)\eta(\mathfrak{U}_2)\zeta - g(\varphi\mathfrak{U}_2, \mathfrak{U}_3)\eta(\mathfrak{U}_1)\zeta]. \end{aligned}\tag{3.34}$$

Contracting (3.34), we have

$$\widetilde{\mathcal{S}}(\mathfrak{U}_2,\mathfrak{U}_3)=\mathcal{S}(\mathfrak{U}_2,\mathfrak{U}_3)+6n\beta^2 g(\mathfrak{U}_2,\mathfrak{U}_3)-2\beta g(\varphi\mathfrak{U}_2,\mathfrak{U}_3). \tag{3.35}$$

Using (2.13) in (3.35), we have

$$\widetilde{\mathfrak{Q}}\mathfrak{U}_2=\mathfrak{Q}\mathfrak{U}_2+6n\beta^2\mathfrak{U}_2-2\beta(\varphi\mathfrak{U}_2). \tag{3.36}$$

Contracting (3.35), we have

$$\widetilde{\mathfrak{r}}=\mathfrak{r}+6n(2n+1)\beta^2-2\beta\Psi, \tag{3.37}$$

where $\Psi=trace(\varphi)$.

Replacing $\mathfrak{U}_3$ by ζ in (3.34) and using (2.3), (2.4), (2.10), we have

$$\widetilde{\mathfrak{R}}(\mathfrak{U}_1,\mathfrak{U}_2)\zeta=-2\beta^2[\eta(\mathfrak{U}_1)\mathfrak{U}_2-\eta(\mathfrak{U}_2)\mathfrak{U}_1]=-2\mathfrak{R}(\mathfrak{U}_1,\mathfrak{U}_2)\zeta. \tag{3.38}$$

Replacing $\mathfrak{U}_1$ by ζ, $\mathfrak{U}_2$ by $\mathfrak{U}_1$ and $\mathfrak{U}_3$ by $\mathfrak{U}_2$ in (3.34) and using (2.3), (2.4), (2.11), we have

$$\widetilde{\mathfrak{R}}(\zeta,\mathfrak{U}_1)\mathfrak{U}_2=-2[\mathfrak{R}(\zeta,\mathfrak{U}_1)\mathfrak{U}_2+\beta g(\varphi\mathfrak{U}_1,\mathfrak{U}_2)\zeta]. \tag{3.39}$$

Replacing $\mathfrak{U}_3$ by ζ in (3.35) and using (2.3), (2.4), (2.12), we have

$$\widetilde{\mathcal{S}}(\mathfrak{U}_2,\zeta)=4n\beta^2\eta(\mathfrak{U}_2). \tag{3.40}$$

Replacing $\mathfrak{U}_2$ by ζ in (3.36) and using (2.3), (2.15), we have

$$\widetilde{\mathfrak{Q}}\zeta=4n\beta^2\zeta. \tag{3.41}$$

Taking the cyclic permutation of $\mathfrak{U}_1,\mathfrak{U}_2$ and $\mathfrak{U}_3$ in (3.34), we have

$$\begin{aligned}\widetilde{\mathfrak{R}}(\mathfrak{U}_2,\mathfrak{U}_3)\mathfrak{U}_1 &= \mathfrak{R}(\mathfrak{U}_2,\mathfrak{U}_3)\mathfrak{U}_1+3\beta^2[g(\mathfrak{U}_3,\mathfrak{U}_1)\mathfrak{U}_2-g(\mathfrak{U}_2,\mathfrak{U}_1)\mathfrak{U}_3]\\ &\quad -2\beta[g(\varphi\mathfrak{U}_2,\mathfrak{U}_1)\eta(\mathfrak{U}_3)\zeta-g(\varphi\mathfrak{U}_3,\mathfrak{U}_1)\eta(\mathfrak{U}_2)\zeta]\end{aligned} \tag{3.42}$$

and

$$\begin{aligned}\widetilde{\mathfrak{R}}(\mathfrak{U}_3,\mathfrak{U}_1)\mathfrak{U}_2 &= \mathfrak{R}(\mathfrak{U}_3,\mathfrak{U}_1)\mathfrak{U}_2+3\beta^2[g(\mathfrak{U}_1,\mathfrak{U}_2)\mathfrak{U}_3-g(\mathfrak{U}_3,\mathfrak{U}_2)\mathfrak{U}_1]\\ &\quad -2\beta[g(\varphi\mathfrak{U}_3,\mathfrak{U}_2)\eta(\mathfrak{U}_1)\zeta-g(\varphi\mathfrak{U}_1,\mathfrak{U}_2)\eta(\mathfrak{U}_3)\zeta].\end{aligned} \tag{3.43}$$

Using Bianchi's first identity in the addition of (3.34), (3.42) and (3.43), we have

$$\widetilde{\mathfrak{R}}(\mathfrak{U}_1,\mathfrak{U}_2)\mathfrak{U}_3+\widetilde{\mathfrak{R}}(\mathfrak{U}_2,\mathfrak{U}_3)\mathfrak{U}_1+\widetilde{\mathfrak{R}}(\mathfrak{U}_3,\mathfrak{U}_1)\mathfrak{U}_2=0. \tag{3.44}$$

Hence we give the following:

Theorem 3.3. *The curvature tensor of a Lorentzian β-Kenmotsu manifold admitting GTWC $\widetilde{\nabla}$ satisfies the equation (3.44).*

4. Extended generalized φ-recurrent Lorentzian β-Kenmotsu manifold admitting the GTWC $\widetilde{\nabla}$

Definition 4.1. *A Lorentzian β-Kenmotsu manifold is said to be an extended generalized φ-recurrent Lorentzian β-Kenmotsu manifold if its curvature tensor $\mathfrak{R}$ satisfies the relation*

$$\begin{aligned} \varphi^2((\nabla_{\mathcal{W}}\mathfrak{R})(\mathfrak{U}_1,\mathfrak{U}_2)\mathfrak{U}_3) &= \mathcal{A}(\mathcal{W})\varphi^2(\mathfrak{R}(\mathfrak{U}_1,\mathfrak{U}_2)\mathfrak{U}_3) \\ &\quad +\mathfrak{B}(\mathcal{W})\varphi^2[g(\mathfrak{U}_2,\mathfrak{U}_3)\mathfrak{U}_1 - g(\mathfrak{U}_1,\mathfrak{U}_3)\mathfrak{U}_2] \end{aligned} \tag{4.45}$$

$\forall\ \mathfrak{U}_1,\mathfrak{U}_2,\mathfrak{U}_3,\mathcal{W}\in\mathfrak{X}(\mathfrak{M})$. *Where $\mathcal{A},\mathfrak{B}$ are two non-vanishing 1-forms such that $g(\mathcal{W},\rho_1) = \mathcal{A}(\mathcal{W})$ and $g(\mathcal{W},\rho_2) = \mathfrak{B}(\mathcal{W})$ for all $\mathcal{W}\in\mathfrak{X}(\mathfrak{M})$ with ρ_1 and ρ_2 being the vector fields associated 1-forms $\mathcal{A}$ and $\mathfrak{B}$, respectively* [17].

Suppose an extended generalized ϕ-recurrent Lorentzian β-Kenmotsu manifold admitting the GTWC $\widetilde{\nabla}$. Then from definition (4.1), we have

$$\begin{aligned} \varphi^2((\widetilde{\nabla}_{\mathcal{W}}\widetilde{\mathfrak{R}})(\mathfrak{U}_1,\mathfrak{U}_2)\mathfrak{U}_3) &= \mathcal{A}(\mathcal{W})\varphi^2(\widetilde{\mathfrak{R}}(\mathfrak{U}_1,\mathfrak{U}_2)\mathfrak{U}_3) \\ &\quad +\mathfrak{B}(\mathcal{W})\varphi^2[g(\mathfrak{U}_2,\mathfrak{U}_3)\mathfrak{U}_1 - g(\mathfrak{U}_1,\mathfrak{U}_3)\mathfrak{U}_2]. \end{aligned} \tag{4.46}$$

Using (2.4) in (4.46), we have

$$\begin{aligned} (\widetilde{\nabla}_{\mathcal{W}}\widetilde{\mathfrak{R}})(\mathfrak{U}_1,\mathfrak{U}_2)\mathfrak{U}_3 &= -\eta((\widetilde{\nabla}_{\mathcal{W}}\widetilde{\mathfrak{R}})(\mathfrak{U}_1,\mathfrak{U}_2)\mathfrak{U}_3)\zeta + \mathcal{A}(\mathcal{W})[\widetilde{\mathfrak{R}}(\mathfrak{U}_1,\mathfrak{U}_2)\mathfrak{U}_3 \\ &\quad +\eta(\widetilde{\mathfrak{R}}(\mathfrak{U}_1,\mathfrak{U}_2)\mathfrak{U}_3)\zeta] + \mathfrak{B}(\mathcal{W})[g(\mathfrak{U}_2,\mathfrak{U}_3)\mathfrak{U}_1 \\ &\quad -g(\mathfrak{U}_1,\mathfrak{U}_3)\mathfrak{U}_2 + g(\mathfrak{U}_2,\mathfrak{U}_3)\eta(\mathfrak{U}_1)\zeta \\ &\quad -g(\mathfrak{U}_1,\mathfrak{U}_3)\eta(\mathfrak{U}_2)\zeta]. \end{aligned} \tag{4.47}$$

Taking inner product in (4.47) with $\mathcal{V}$ and using (2.4), we have

$$\begin{aligned} g((\widetilde{\nabla}_{\mathcal{W}}\widetilde{\mathfrak{R}})(\mathfrak{U}_1,\mathfrak{U}_2)\mathfrak{U}_3,\mathcal{V}) &= -\eta((\widetilde{\nabla}_{\mathcal{W}}\widetilde{\mathfrak{R}})(\mathfrak{U}_1,\mathfrak{U}_2)\mathfrak{U}_3)\eta(\mathcal{V}) \\ &\quad +\mathcal{A}(\mathcal{W})[g(\widetilde{\mathfrak{R}}(\mathfrak{U}_1,\mathfrak{U}_2)\mathfrak{U}_3,\mathcal{V}) \\ &\quad +\eta(\widetilde{\mathfrak{R}}(\mathfrak{U}_1,\mathfrak{U}_2)\mathfrak{U}_3)\eta(\mathcal{V})] \\ &\quad +\mathfrak{B}(\mathcal{W})[g(\mathfrak{U}_2,\mathfrak{U}_3)g(\mathfrak{U}_1,\mathcal{V}) \\ &\quad -g(\mathfrak{U}_1,\mathfrak{U}_3)g(\mathfrak{U}_2,\mathcal{V}) \\ &\quad +g(\mathfrak{U}_2,\mathfrak{U}_3)\eta(\mathfrak{U}_1)\eta(\mathcal{V}) \\ &\quad -g(\mathfrak{U}_1,\mathfrak{U}_3)\eta(\mathfrak{U}_2)\eta(\mathcal{V})]. \end{aligned} \tag{4.48}$$

Let $\{\varsigma_1,\varsigma_2,\varsigma_3,...,\varsigma_n\}$ be an orthonormal basis for the tangent space of $\mathfrak{M}^{2n+1}$ at a point $p\in\mathfrak{M}^{2n+1}$. Taking $\mathfrak{U}_1=\mathcal{V}=\varsigma_i$ and summation over $i\in[1,n]$ in (4.48), we have

$$\begin{aligned} (\widetilde{\nabla}_{\mathcal{W}}\widetilde{\mathcal{S}})(\mathfrak{U}_2,\mathfrak{U}_3) &= -\sum_{i=1}^{2n+1}\eta((\widetilde{\nabla}_{\mathcal{W}}\widetilde{\mathfrak{R}})(\varsigma_i,\mathfrak{U}_2)\mathfrak{U}_3)\eta(\varsigma_i) \\ &\quad +\mathcal{A}(\mathcal{W})[\widetilde{\mathcal{S}}(\mathfrak{U}_2,\mathfrak{U}_3)+\eta(\widetilde{\mathfrak{R}}(\zeta,\mathfrak{U}_2)\mathfrak{U}_3)] \\ &\quad +\mathfrak{B}(\mathcal{W})[2ng(\mathfrak{U}_2,\mathfrak{U}_3)-g(\mathfrak{U}_2,\mathfrak{U}_3)-\eta(\mathfrak{U}_2)\eta(\mathfrak{U}_3)]. \end{aligned} \tag{4.49}$$

Replacing $\mathfrak{U}_3$ by ζ in (4.49) and using (2.3), (2.4), (3.39), (3.40), we have

$$\begin{aligned} (\widetilde{\nabla}_{\mathcal{W}}\widetilde{\mathcal{S}})(\mathfrak{U}_2,\zeta) &= -\sum_{i=1}^{2n+1}\eta((\widetilde{\nabla}_{\mathcal{W}}\widetilde{\mathfrak{R}})(\varsigma_i,\mathfrak{U}_2)\zeta)\eta(\varsigma_i) \\ &\quad +4n\beta^2\mathcal{A}(\mathcal{W})\eta(\mathfrak{U}_2)+2n\mathfrak{B}(\mathcal{W})\eta(\mathfrak{U}_2). \end{aligned} \tag{4.50}$$

Taking second term of (4.50), we can calculate

$$\begin{aligned} \eta((\widetilde{\nabla}_{\mathcal{W}}\widetilde{\mathfrak{R}})(\varsigma_i,\mathfrak{U}_2)\zeta) &= g(\widetilde{\nabla}_{\mathcal{W}}\widetilde{\mathfrak{R}}(\varsigma_i,\mathfrak{U}_2)\zeta,\zeta)-g(\widetilde{\mathfrak{R}}(\widetilde{\nabla}_{\mathcal{W}}\varsigma_i,\mathfrak{U}_2)\zeta,\zeta) \\ &\quad -g(\widetilde{\mathfrak{R}}(\varsigma_i,\widetilde{\nabla}_{\mathcal{W}}\mathfrak{U}_2)\zeta,\zeta)-g(\widetilde{\mathfrak{R}}(\varsigma_i,\mathfrak{U}_2)\widetilde{\nabla}_{\mathcal{W}}\zeta,\zeta). \end{aligned} \tag{4.51}$$

Let $p\in\mathfrak{M}^{2n+1}$, since ς_i is an orthonormal basis, therefore $\widetilde{\nabla}_{\mathcal{W}}\varsigma_i=0$ at p. Also

$$g(\widetilde{\mathfrak{R}}(\varsigma_i,\mathfrak{U}_2)\zeta,\zeta)=-g(\widetilde{\mathfrak{R}}(\zeta,\zeta)\mathfrak{U}_2,\varsigma_i)=0. \tag{4.52}$$

Since $(\widetilde{\nabla}_{\mathcal{W}}g)=0$, we have

$$g(\widetilde{\nabla}_{\mathcal{W}}\widetilde{\mathfrak{R}}(\varsigma_i,\mathfrak{U}_2)\zeta,\zeta)+g(\widetilde{\mathfrak{R}}(\varsigma_i,\mathfrak{U}_2)\zeta,\widetilde{\nabla}_{\mathcal{W}}\zeta)=0. \tag{4.53}$$

Using (4.53) in (4.51), we have

$$g((\widetilde{\nabla}_{\mathcal{W}}\widetilde{\mathfrak{R}})(\varsigma_i,\mathfrak{U}_2)\zeta,\zeta)$$

$$= -g(\widetilde{\mathfrak{R}}(\varsigma_i,\mathfrak{U}_2)\zeta,\widetilde{\nabla}_{\mathcal{W}}\zeta) - g(\widetilde{\mathfrak{R}}(\widetilde{\nabla}_{\mathcal{W}}\varsigma_i,\mathfrak{U}_2)\zeta,\zeta)$$

$$-g(\widetilde{\mathfrak{R}}(\varsigma_i,\widetilde{\nabla}_{\mathcal{W}}\mathfrak{U}_2)\zeta,\zeta) - g(\widetilde{\mathfrak{R}}(\varsigma_i,\mathfrak{U}_2)\widetilde{\nabla}_{\mathcal{W}}\zeta,\zeta). \quad (4.54)$$

We also know that

$$g(\widetilde{\mathfrak{R}}(\varsigma_i,\widetilde{\nabla}_{\mathcal{W}}\mathfrak{U}_2)\zeta,\zeta) = 0 = g(\widetilde{\mathfrak{R}}(\widetilde{\nabla}_{\mathcal{W}}\varsigma_i,\mathfrak{U}_2)\zeta,\zeta). \quad (4.55)$$

Using (4.55) in (4.54) and using the fact that $\mathfrak{R}$ is skew-symmetric, we obtain

$$\eta((\widetilde{\nabla}_{\mathcal{W}}\widetilde{\mathfrak{R}})(\varsigma_i,\mathfrak{U}_2)\zeta) = 0. \quad (4.56)$$

Therefore second term of (4.50) is zero, i.e.

$$\sum_{i=1}^{2n+1}\eta((\widetilde{\nabla}_{\mathcal{W}}\widetilde{\mathfrak{R}})(\varsigma_i,\mathfrak{U}_2)\zeta)\eta(\varsigma_i) = 0. \quad (4.57)$$

Using (4.57) in (4.50), we have

$$(\widetilde{\nabla}_{\mathcal{W}}\widetilde{\mathcal{S}})(\mathfrak{U}_2,\zeta) = 4n\beta^2\mathcal{A}(\mathcal{W})\eta(\mathfrak{U}_2) + 2n\mathfrak{B}(\mathcal{W})\eta(\mathfrak{U}_2). \quad (4.58)$$

Now we know that

$$(\widetilde{\nabla}_{\mathcal{W}}\widetilde{\mathcal{S}})(\mathfrak{U}_2,\zeta) = \widetilde{\nabla}_{\mathcal{W}}\widetilde{\mathcal{S}}(\mathfrak{U}_2,\zeta) - \widetilde{\mathcal{S}}(\widetilde{\nabla}_{\mathcal{W}}\mathfrak{U}_2,\zeta) - \widetilde{\mathcal{S}}(\mathfrak{U}_2,\widetilde{\nabla}_{\mathcal{W}}\zeta). \quad (4.59)$$

Using (3.24), (3.27) and (3.40) in (4.59), we have

$$(\widetilde{\nabla}_{\mathcal{W}}\widetilde{\mathcal{S}})(\mathfrak{U}_2,\zeta) = 4n\beta^2(\widetilde{\nabla}_{\mathcal{W}}\eta)\mathfrak{U}_2 - 2\beta\mathcal{S}(\mathfrak{U}_2,\mathcal{W})$$

$$-12n\beta^3 g(\mathfrak{U}_2,\mathcal{W}) + 4\beta^2 g(\varphi\mathfrak{U}_2,\mathcal{W}). \quad (4.60)$$

Using (3.28) in (4.60), we have

$$(\widetilde{\nabla}_{\mathcal{W}}\widetilde{\mathcal{S}})(\mathfrak{U}_2,\zeta) = -2\beta\mathcal{S}(\mathfrak{U}_2,\mathcal{W}) - 4n\beta^3 g(\mathfrak{U}_2,\mathcal{W}) + 4\beta^2 g(\varphi\mathfrak{U}_2,\mathcal{W}). \quad (4.61)$$

By virtue of (4.58) and (4.61), we have

$$-\beta\mathcal{S}(\mathfrak{U}_2,\mathcal{W}) - 2n\beta^3 g(\mathfrak{U}_2,\mathcal{W}) + 2\beta^2 g(\varphi\mathfrak{U}_2,\mathcal{W})$$

$$= 2n\beta^2\mathcal{A}(\mathcal{W})\eta(\mathfrak{U}_2) + n\mathfrak{B}(\mathcal{W})\eta(\mathfrak{U}_2). \quad (4.62)$$

Replacing $\mathfrak{U}_2$ by ζ in (4.62) and using (2.3), (2.4), (2.12), we have

$$2n\beta^2\mathcal{A}(\mathcal{W}) + n\mathfrak{B}(\mathcal{W}) = 0. \quad (4.63)$$

By virtue of (4.62) and (4.63), we have

$$\mathcal{S}(\mathfrak{U}_2, \mathcal{W}) = -2n\beta^2 g(\mathfrak{U}_2, \mathcal{W}) + 2\beta g(\varphi\mathfrak{U}_2, \mathcal{W}). \tag{4.64}$$

Thus we can state the following:

Theorem 4.1. *An extended generalized φ-recurrent Lorentzian β-Kenmotsu manifold with respect to the GTWC $\widetilde{\nabla}$ is some class of generalized η-Einstein manifold and the 1-forms $\mathcal{A}$ and $\mathfrak{B}$ are related as $[2\beta^2\mathcal{A}(\mathcal{W}) + \mathfrak{B}(\mathcal{W})] = 0$.*

5. Certain conditions on Lorentzian β-Kenmotsu manifold admitting the GTWC $\widetilde{\nabla}$

The projective curvature tensor [28] $\widetilde{\mathcal{P}}$ on Lorentzian β-Kenmotsu manifold admitting the GTWC $\widetilde{\nabla}$ is defined by

$$\widetilde{\mathcal{P}}(\mathfrak{U}_1, \mathfrak{U}_2)\mathfrak{U}_3 = \widetilde{\mathfrak{R}}(\mathfrak{U}_1, \mathfrak{U}_2)\mathfrak{U}_3 - \frac{1}{2n}[\widetilde{\mathcal{S}}(\mathfrak{U}_2, \mathfrak{U}_3)\mathfrak{U}_1 - \widetilde{\mathcal{S}}(\mathfrak{U}_1, \mathfrak{U}_3)\mathfrak{U}_2]. \tag{5.65}$$

If projective curvature tensor $\widetilde{\mathcal{P}}$ vanishes, then from (5.65), we have

$$\widetilde{\mathfrak{R}}(\mathfrak{U}_1, \mathfrak{U}_2)\mathfrak{U}_3 = \frac{1}{2n}[\widetilde{\mathcal{S}}(\mathfrak{U}_2, \mathfrak{U}_3)\mathfrak{U}_1 - \widetilde{\mathcal{S}}(\mathfrak{U}_1, \mathfrak{U}_3)\mathfrak{U}_2]. \tag{5.66}$$

Using (3.34) and (3.35) in (5.66), we have

$$\begin{aligned}&\mathfrak{R}(\mathfrak{U}_1, \mathfrak{U}_2)\mathfrak{U}_3 - 2\beta[g(\varphi\mathfrak{U}_1, \mathfrak{U}_3)\eta(\mathfrak{U}_2)\zeta - g(\varphi\mathfrak{U}_2, \mathfrak{U}_3)\eta(\mathfrak{U}_1)\zeta]\\ &= \frac{1}{2n}[\mathcal{S}(\mathfrak{U}_2, \mathfrak{U}_3)\mathfrak{U}_1 - \mathcal{S}(\mathfrak{U}_1, \mathfrak{U}_3)\mathfrak{U}_2 + 2\beta g(\varphi\mathfrak{U}_1, \mathfrak{U}_3)\mathfrak{U}_2\\ &-2\beta g(\varphi\mathfrak{U}_2, \mathfrak{U}_3)\mathfrak{U}_1].\end{aligned} \tag{5.67}$$

Taking inner product in (5.67) with $\mathcal{V}$ and using (2.4), we have

$$\begin{aligned}&g(\mathfrak{R}(\mathfrak{U}_1, \mathfrak{U}_2)\mathfrak{U}_3, \mathcal{V}) - 2\beta[g(\varphi\mathfrak{U}_1, \mathfrak{U}_3)\eta(\mathfrak{U}_2)\eta(\mathcal{V}) - g(\varphi\mathfrak{U}_2, \mathfrak{U}_3)\eta(\mathfrak{U}_1)\eta(\mathcal{V})]\\ &= \frac{1}{2n}[\mathcal{S}(\mathfrak{U}_2, \mathfrak{U}_3)g(\mathfrak{U}_1, \mathcal{V}) - \mathcal{S}(\mathfrak{U}_1, \mathfrak{U}_3)g(\mathfrak{U}_2, \mathcal{V}) + 2\beta g(\varphi\mathfrak{U}_1, \mathfrak{U}_3)g(\mathfrak{U}_2, \mathcal{V})\\ &-2\beta g(\varphi\mathfrak{U}_2, \mathfrak{U}_3)g(\mathfrak{U}_1, \mathcal{V})].\end{aligned} \tag{5.68}$$

Replacing $\mathcal{V}$ by ζ in (5.68) and using (2.3), (2.4), we have

$$\begin{aligned}&\eta(\mathfrak{R}(\mathfrak{U}_1, \mathfrak{U}_2)\mathfrak{U}_3) - 2\beta[g(\varphi\mathfrak{U}_2, \mathfrak{U}_3)\eta(\mathfrak{U}_1) - g(\varphi\mathfrak{U}_1, \mathfrak{U}_3)\eta(\mathfrak{U}_2)]\\ &= \frac{1}{2n}[\mathcal{S}(\mathfrak{U}_2, \mathfrak{U}_3)\eta(\mathfrak{U}_1) - \mathcal{S}(\mathfrak{U}_1, \mathfrak{U}_3)\eta(\mathfrak{U}_2) + 2\beta g(\varphi\mathfrak{U}_1, \mathfrak{U}_3)\eta(\mathfrak{U}_2)\\ &-2\beta g(\varphi\mathfrak{U}_2, \mathfrak{U}_3)\eta(\mathfrak{U}_1)].\end{aligned} \tag{5.69}$$

Replacing $\mathfrak{U}_1$ by ζ in (5.69) and using (2.3), (2.11), (2.12), we have

$$\mathcal{S}(\mathfrak{U}_2,\mathfrak{U}_3) = -2n\beta^2 g(\mathfrak{U}_2,\mathfrak{U}_3) - 6n\beta^2\eta(\mathfrak{U}_2)\eta(\mathfrak{U}_3) - 2\beta(2n-1)g(\varphi\mathfrak{U}_2,\mathfrak{U}_3). \tag{5.70}$$

Thus we have the following:

Theorem 5.1. *A projectively flat Lorentzian β-Kenmotsu manifold admitting the GTWC $\widetilde{\nabla}$ is a generalized η-Einstein manifold.*

Definition 5.1. *A Lorentzian β-Kenmotsu manifold $\mathfrak{M}^{2n+1}$ with respect to the GTWC $\widetilde{\nabla}$ is said to be ζ-projectively flat* [26] *if*

$$\widetilde{\mathcal{P}}(\mathfrak{U}_1,\mathfrak{U}_2)\zeta = 0$$

$\forall\ \mathfrak{U}_1,\mathfrak{U}_2 \in \mathfrak{X}(\mathfrak{M})$ *orthogonal to ζ, where $\widetilde{\mathcal{P}}$ is the projective curvature tensor of the GTWC $\widetilde{\nabla}$.*

Using (3.34) and (3.35) in (5.66), we have

$$\begin{aligned}\widetilde{\mathcal{P}}(\mathfrak{U}_1,\mathfrak{U}_2)\mathfrak{U}_3 &= \mathcal{P}(\mathfrak{U}_1,\mathfrak{U}_2)\mathfrak{U}_3 - \frac{\beta}{n}[g(\varphi\mathfrak{U}_1,\mathfrak{U}_3)\mathfrak{U}_2 - g(\varphi\mathfrak{U}_2,\mathfrak{U}_3)\mathfrak{U}_1] \\ &\quad -2\beta[g(\varphi\mathfrak{U}_1,\mathfrak{U}_3)\eta(\mathfrak{U}_2)\zeta - g(\varphi\mathfrak{U}_2,\mathfrak{U}_3)\eta(\mathfrak{U}_1)\zeta],\end{aligned} \tag{5.71}$$

where

$$\mathcal{P}(\mathfrak{U}_1,\mathfrak{U}_2)\mathfrak{U}_3 = \mathfrak{R}(\mathfrak{U}_1,\mathfrak{U}_2)\mathfrak{U}_3 - \frac{1}{2n}[\mathcal{S}(\mathfrak{U}_2,\mathfrak{U}_3)\mathfrak{U}_1 - \mathcal{S}(\mathfrak{U}_1,\mathfrak{U}_3)\mathfrak{U}_2] \tag{5.72}$$

is a projective curvature tensor with respect to the connection ∇.
Putting $\mathfrak{U}_3 = \zeta$ in (5.71) and using (2.3), (2.4), we have

$$\widetilde{\mathcal{P}}(\mathfrak{U}_1,\mathfrak{U}_2)\zeta = \mathcal{P}(\mathfrak{U}_1,\mathfrak{U}_2)\zeta. \tag{5.73}$$

Now we give the following:

Theorem 5.2. *A $(2n+1)$-dimensional Lorentzian β-Kenmotsu manifold admitting the GTWC $\widetilde{\nabla}$ is ζ-projectively flat iff the manifold $\mathfrak{M}^{2n+1}$ is ζ-projectively flat with respect to the connection ∇.*

Now using (2.10), (2.12) and (5.72) in (5.73), we have

$$\widetilde{\mathcal{P}}(\mathfrak{U}_1,\mathfrak{U}_2)\zeta = 0. \tag{5.74}$$

Thus we can state the following:

Theorem 5.3. *A $(2n+1)$-dimensional Lorentzian β-Kenmotsu manifold admitting the GTWC $\widetilde{\nabla}$ is ζ-projectively flat.*

Next if the $(1,1)$-tensor φ is η-parallel with respect to the GTWC $\widetilde{\nabla}$, then we have

$$g((\widetilde{\nabla}_{\mathfrak{U}_1}\varphi)\mathfrak{U}_2, \mathfrak{U}_3) = 0 \tag{5.75}$$

$\forall\, \mathfrak{U}_1, \mathfrak{U}_2, \mathfrak{U}_3 \in \mathfrak{X}(\mathfrak{M})$.

By virtue of (3.26) and (5.75), we have

$$\beta g(\varphi\mathfrak{U}_1, \mathfrak{U}_2)\eta(\mathfrak{U}_3) + g(\mathfrak{U}_2, \mathfrak{U}_3)\eta(\mathfrak{U}_1) + \eta(\mathfrak{U}_1)\eta(\mathfrak{U}_2)\eta(\mathfrak{U}_3) = 0. \tag{5.76}$$

Taking $\mathfrak{U}_3 = \zeta$ in (5.76) and using (2.3), (2.4), we have

$$g(\varphi\mathfrak{U}_1, \mathfrak{U}_2) = 0. \tag{5.77}$$

Replacing $\mathfrak{U}_2$ by $\varphi\mathfrak{U}_2$ in (5.77) and using (2.5), we have

$$g(\mathfrak{U}_1, \mathfrak{U}_2) + \eta(\mathfrak{U}_1)\eta(\mathfrak{U}_2) = 0. \tag{5.78}$$

Replacing $\mathfrak{U}_1$ by $\mathfrak{Q}\mathfrak{U}_1$ in (5.78) and using (2.13), (2.14), we have

$$\mathcal{S}(\mathfrak{U}_1, \mathfrak{U}_2) = 2n\beta^2\eta(\mathfrak{U}_1)\eta(\mathfrak{U}_2). \tag{5.79}$$

Hence we have the following:

Theorem 5.4. *If the $(1,1)$-tensor φ is η-parallel on the Lorentzian β-Kenmotsu manifold $\mathfrak{M}^{2n+1}$ admitting the GTWC $\widetilde{\nabla}$, then the manifold $\mathfrak{M}^{2n+1}$ is a special type of η-Einstein manifold.*

6. Ricci soliton on Lorentzian β-Kenmotsu manifold with GTWC $\widetilde{\nabla}$

Let (g, ζ, Θ) be a Ricci soliton on Lorentzian β-Kenmotsu manifold $\mathfrak{M}^{2n+1}$ with respect to the GTWC $\widetilde{\nabla}$. Then we have

$$(\widetilde{\mathfrak{L}}_\zeta g)(\mathfrak{U}_1, \mathfrak{U}_2) + 2\widetilde{\mathcal{S}}(\mathfrak{U}_1, \mathfrak{U}_2) + 2\Theta g(\mathfrak{U}_1, \mathfrak{U}_2) = 0. \tag{6.80}$$

Now

$$(\widetilde{\mathfrak{L}}_\zeta g)(\mathfrak{U}_1, \mathfrak{U}_2) = g(\widetilde{\nabla}_{\mathfrak{U}_1}\zeta, \mathfrak{U}_2) + g(\mathfrak{U}_1, \widetilde{\nabla}_{\mathfrak{U}_2}\zeta). \tag{6.81}$$

Using (3.24) in (6.81), we have

$$(\widetilde{\mathfrak{L}}_\zeta g)(\mathfrak{U}_1, \mathfrak{U}_2) = 4\beta g(\mathfrak{U}_1, \mathfrak{U}_2). \tag{6.82}$$

Using (3.35) and (6.82) in (6.80), we have

$$\mathcal{S}(\mathfrak{U}_1,\mathfrak{U}_2) = -(\Theta + 2\beta + 6n\beta^2)g(\mathfrak{U}_1,\mathfrak{U}_2) + 2\beta g(\varphi\mathfrak{U}_1,\mathfrak{U}_2). \tag{6.83}$$

Now we give the following:

Theorem 6.1. *If (g,ζ,Θ) be a Ricci soliton on a Lorentzian β-Kenmotsu manifold $\mathfrak{M}^{2n+1}$ with the GTWC $\widetilde{\widetilde{\nabla}}$, then the manifold $\mathfrak{M}^{2n+1}$ is some class of generalized η-Einstein manifold.*

Using (6.82) in (6.80), we have

$$\widetilde{\widetilde{\mathcal{S}}}(\mathfrak{U}_1,\mathfrak{U}_2) = -(2\beta + \Theta)g(\mathfrak{U}_1,\mathfrak{U}_2). \tag{6.84}$$

Contracting (6.84), we have

$$\widetilde{\widetilde{\mathfrak{r}}} = -(2n+1)(2\beta + \Theta). \tag{6.85}$$

Replacing $\mathfrak{U}_2$ by ζ in (6.83) and using (2.3), (2.4), (2.12), we have

$$\Theta = -2\beta(1 + 2n\beta). \tag{6.86}$$

Thus we have the following:

Theorem 6.2. *A Ricci soliton (g,ζ,Θ) in a Lorentzian β-Kenmotsu manifold $\mathfrak{M}^{2n+1}$ admitting the GTWC $\widetilde{\widetilde{\nabla}}$ is either steady or shrinking.*

Let $(g,\mathcal{V},\Theta)$ be the Ricci soliton in a Lorentzian β-Kenmotsu manifold $\mathfrak{M}^{2n+1}$ admitting the GTWC $\widetilde{\widetilde{\nabla}}$ such that $\mathcal{V}$ is pointwise collinear with ζ, i.e., $\mathcal{V} = \mathfrak{b}\zeta$, where $\mathfrak{b}$ is a function. Then (1.2) holds and follows that

$$\begin{aligned}&\mathfrak{b}g(\widetilde{\widetilde{\nabla}}_{\mathfrak{U}_1}\zeta,\mathfrak{U}_2) + (\mathfrak{U}_1\mathfrak{b})\eta(\mathfrak{U}_2) + \mathfrak{b}g(\mathfrak{U}_1,\widetilde{\widetilde{\nabla}}_{\mathfrak{U}_2}\zeta)\\ &+(\mathfrak{U}_2\mathfrak{b})\eta(\mathfrak{U}_1) + 2\widetilde{\widetilde{\mathcal{S}}}(\mathfrak{U}_1,\mathfrak{U}_2) + 2\Theta g(\mathfrak{U}_1,\mathfrak{U}_2) = 0.\end{aligned} \tag{6.87}$$

Replacing $\mathfrak{U}_2$ by ζ in (6.87) and using (2.3), (2.4), (3.24), (3.40), we have

$$(\mathfrak{U}_1\mathfrak{b}) = (2\Theta + \zeta\mathfrak{b} + 4\mathfrak{b}\beta + 4\mathfrak{b}\beta + 8n\beta^2)\eta(\mathfrak{U}_1). \tag{6.88}$$

Replacing $\mathfrak{U}_1$ by ζ in (6.88) and using (2.3), we have

$$(\zeta\mathfrak{b}) = -(\Theta + 2\mathfrak{b}\beta + 4n\beta^2). \tag{6.89}$$

Equations (6.88) and (6.89), yield

$$(d\mathfrak{b}) = (\Theta + 2\mathfrak{b}\beta + 4n\beta^2)\eta. \tag{6.90}$$

Applying d on (6.90), we have

$$(\Theta + 2\mathfrak{b}\beta + 4n\beta^2)d\eta = 0. \tag{6.91}$$

Since $d\eta \neq 0$, from (6.91), we have

$$\Theta = -2\beta(\mathfrak{b} + 2n\beta). \tag{6.92}$$

Putting (6.92) in (6.90), we obtain $d\mathfrak{b} = 0$, i.e., $\mathfrak{b}$ is a constant. Hence we have the following:

Theorem 6.3. *If $(g, \mathcal{V}, \Theta)$ be the Ricci soliton in a Lorentzian β-Kenmotsu manifold $\mathfrak{M}^{2n+1}$ admitting the GTWC $\widetilde{\widetilde{\nabla}}$ such that $\mathcal{V} = \mathfrak{b}\zeta$, then $\mathcal{V}$ is a constant multiple of ζ and the Ricci soliton is either steady or shrinking.*

7. Example of Lorentzian β-Kenmotsu manifold

Example 7.1. *Let $\mathfrak{M} = \{(\mathfrak{t}_1, \mathfrak{t}_2, \mathfrak{t}_3) \in \mathbb{R}^3 : \mathfrak{t}_3 > 0\}$ be a 3-dimensional manifold, where $(\mathfrak{t}_1, \mathfrak{t}_2, \mathfrak{t}_3)$ are the standard coordinates of $\mathbb{R}^3$. The vector fields* [27]

$$\varsigma_1 = e^{\mathfrak{t}_3}\frac{\partial}{\partial \mathfrak{t}_2}, \quad \varsigma_2 = e^{\mathfrak{t}_3}\left(\frac{\partial}{\partial \mathfrak{t}_1} + \frac{\partial}{\partial \mathfrak{t}_2}\right), \quad \varsigma_3 = \beta\frac{\partial}{\partial \mathfrak{t}_3}$$

are linearly independent at each point of $\mathfrak{M}$, where β is a real constant. Let g be the Lorentzian metric defined by

$$g(\varsigma_1, \varsigma_2) = g(\varsigma_1, \varsigma_3) = g(\varsigma_2, \varsigma_3) = 0,$$
$$g(\varsigma_1, \varsigma_1) = g(\varsigma_2, \varsigma_2) = -g(\varsigma_3, \varsigma_3) = 1. \tag{7.93}$$

Let η be the 1-form defined by $\eta(\mathfrak{U}_1) = g(\mathfrak{U}_1, \varsigma_3)$ for any $\mathfrak{U}_1 \in \mathfrak{X}(\mathfrak{M})$ and φ be the $(1,1)$-tensor field defined by

$$\varphi(\varsigma_1) = -\varsigma_2, \quad \varphi(\varsigma_2) = -\varsigma_1, \quad \varphi(\varsigma_3) = 0. \tag{7.94}$$

Now using the linearity of φ and g, we have

$$\eta(\varsigma_3) = -1, \quad \varphi^2(\mathfrak{U}_1) = \mathfrak{U}_1 + \eta(\mathfrak{U}_1)\varsigma_3 \tag{7.95}$$

and

$$g(\varphi\mathfrak{U}_1, \varphi\mathfrak{U}_2) = g(\mathfrak{U}_1, \mathfrak{U}_2) + \eta(\mathfrak{U}_1)\eta(\mathfrak{U}_2) \tag{7.96}$$

$\forall\ \mathfrak{U}_1, \mathfrak{U}_2 \in \mathfrak{X}(\mathfrak{M})$. *Therefore for* $\varsigma_3 = \zeta$, *the structure* $(\varphi, \zeta, \eta, g)$ *defines a Lorentzian para-contact structure on* $\mathfrak{M}$. *Let* ∇ *be the Levi-Civita connection with respect to the Lorentzian metric* g. *Then we have*

$$[\varsigma_1, \varsigma_2] = 0, \quad [\varsigma_1, \varsigma_3] = -\beta\varsigma_1, \quad [\varsigma_2, \varsigma_3] = -\beta\varsigma_2. \tag{7.97}$$

We recall Koszul's formula as

$$\begin{aligned} 2g(\nabla_{\mathfrak{U}_1}\mathfrak{U}_2, \mathfrak{U}_3) &= \mathfrak{U}_1 g(\mathfrak{U}_2, \mathfrak{U}_3) + \mathfrak{U}_2 g(\mathfrak{U}_3, \mathfrak{U}_1) - \mathfrak{U}_3 g(\mathfrak{U}_1, \mathfrak{U}_2) \\ &\quad - g(\mathfrak{U}_1, [\mathfrak{U}_2, \mathfrak{U}_3]) - g(\mathfrak{U}_2, [\mathfrak{U}_1, \mathfrak{U}_3]) \\ &\quad + g(\mathfrak{U}_3, [\mathfrak{U}_1, \mathfrak{U}_2]). \end{aligned} \tag{7.98}$$

By virtue of (7.98), we have

$$\begin{aligned} &\nabla_{\varsigma_1}\varsigma_1 = -\beta\varsigma_3, \quad \nabla_{\varsigma_1}\varsigma_2 = 0, \quad \nabla_{\varsigma_1}\varsigma_3 = -\beta\varsigma_1, \\ &\nabla_{\varsigma_2}\varsigma_1 = 0, \quad \nabla_{\varsigma_2}\varsigma_2 = -\beta\varsigma_3, \quad \nabla_{\varsigma_2}\varsigma_3 = -\beta\varsigma_2, \\ &\nabla_{\varsigma_3}\varsigma_1 = 0, \quad \nabla_{\varsigma_3}\varsigma_2 = 0, \quad \nabla_{\varsigma_3}\varsigma_3 = 0. \end{aligned} \tag{7.99}$$

Now for $\mathfrak{U}_1 = \mathfrak{U}_1^1\varsigma_1 + \mathfrak{U}_1^2\varsigma_2 + \mathfrak{U}_1^3\varsigma_3$ *and* $\zeta = \varsigma_3$, *we have*

$$\nabla_{\mathfrak{U}_1}\zeta = \nabla_{\mathfrak{U}_1^1\varsigma_1 + \mathfrak{U}_1^2\varsigma_2 + \mathfrak{U}_1^3\varsigma_3}\varsigma_3 = -\beta(\mathfrak{U}_1^1\varsigma_1 + \mathfrak{U}_1^2\varsigma_2) \tag{7.100}$$

and

$$\beta[\mathfrak{U}_1 - \eta(\mathfrak{U}_1)\zeta] = \beta[\mathfrak{U}_1^1\varsigma_1 + \mathfrak{U}_1^2\varsigma_2 + 2\mathfrak{U}_1^3\varsigma_3], \tag{7.101}$$

where $\mathfrak{U}_1^1, \mathfrak{U}_1^2$ *and* $\mathfrak{U}_1^3$ *are scalars.*
Now using (7.100) and (7.101), we have

$$2\beta(\mathfrak{U}_1^1\varsigma_1 + \mathfrak{U}_1^2\varsigma_2 + \mathfrak{U}_1^3\varsigma_3) = 0.$$

Since $(\mathfrak{U}_1^1\varsigma_1 + \mathfrak{U}_1^2\varsigma_2 + \mathfrak{U}_1^3\varsigma_3) \neq 0$, *therefore we have*

$$\beta = 0. \tag{7.102}$$

Hence it can be easily see that the structure $(\mathfrak{M}^3, \varphi, \zeta, \eta, g)$ *is a Lorentzian* β*-Kenmotsu manifold.*

By using (7.97) and (7.99), we can obtain the components of the curvature tensor $\mathfrak{R}$ *with respect to the connection* ∇ *as follows:*

$$\begin{aligned}
&\mathfrak{R}(\varsigma_1,\varsigma_2)\varsigma_3 = 0, \quad \mathfrak{R}(\varsigma_1,\varsigma_3)\varsigma_2 = 0, \quad \mathfrak{R}(\varsigma_2,\varsigma_3)\varsigma_1 = 0,\\
&\mathfrak{R}(\varsigma_2,\varsigma_3)\varsigma_3 = -\beta^2\varsigma_2, \quad \mathfrak{R}(\varsigma_1,\varsigma_3)\varsigma_3 = -\beta^2\varsigma_1, \quad \mathfrak{R}(\varsigma_1,\varsigma_2)\varsigma_2 = \beta^2\varsigma_1,\\
&\mathfrak{R}(\varsigma_3,\varsigma_1)\varsigma_1 = \beta^2\varsigma_3, \quad \mathfrak{R}(\varsigma_2,\varsigma_1)\varsigma_1 = \beta^2\varsigma_2, \quad \mathfrak{R}(\varsigma_3,\varsigma_2)\varsigma_2 = \beta^2\varsigma_3.
\end{aligned} \tag{7.103}$$

Along with $\mathfrak{R}(\varsigma_i,\varsigma_i)\varsigma_i = 0$, $\forall\ i = 1,2,3$. *By using (7.103), we can verify equations (2.9), (2.10) and (2.11).*

Now using (3.23), (7.93), (7.94) and (7.99), we obtain

$$\begin{aligned}
&\widetilde{\nabla}_{\varsigma_1}\varsigma_1 = 0, \quad \widetilde{\nabla}_{\varsigma_1}\varsigma_2 = 0, \quad \widetilde{\nabla}_{\varsigma_1}\varsigma_3 = 0,\\
&\widetilde{\nabla}_{\varsigma_2}\varsigma_1 = 0, \quad \widetilde{\nabla}_{\varsigma_2}\varsigma_2 = 0, \quad \widetilde{\nabla}_{\varsigma_2}\varsigma_3 = 0,\\
&\widetilde{\nabla}_{\varsigma_3}\varsigma_1 = -\varsigma_2, \quad \widetilde{\nabla}_{\varsigma_3}\varsigma_2 = -\varsigma_1, \quad \widetilde{\nabla}_{\varsigma_3}\varsigma_3 = 0.
\end{aligned} \tag{7.104}$$

By using (3.30) and (3.32), we have

$$(\widetilde{\nabla}_{\varsigma_1}g)(\varsigma_2,\varsigma_3) = 0, \quad (\widetilde{\nabla}_{\varsigma_2}g)(\varsigma_3,\varsigma_1) = 0, \quad (\widetilde{\nabla}_{\varsigma_3}g)(\varsigma_1,\varsigma_2) = 2 \neq 0$$

and also, we have

$$\widetilde{\mathcal{T}}(\varsigma_1,\varsigma_2) = 0, \quad \widetilde{\mathcal{T}}(\varsigma_1,\varsigma_3) = \beta\varsigma_1 - \varsigma_2, \quad \widetilde{\mathcal{T}}(\varsigma_2,\varsigma_3) = \beta\varsigma_2 + \varsigma_1.$$

Along with $\widetilde{\mathcal{T}}(\varsigma_i,\varsigma_i) = 0$; $\forall\ i = 1,2,3$. *Hence* $\mathfrak{M}^3$ *is a* 3*-dimensional Lorentzian* β*-Kenmotsu manifold admitting the GTWC* $\widetilde{\nabla}$ *which is a non-metric connection.*

Now using (3.33), (7.97) and (7.104), we can easily obtain the components of curvature tensor $\widetilde{\mathfrak{R}}$ *with respect to the GTWC* $\widetilde{\nabla}$ *as follows:*

$$\widetilde{\mathfrak{R}}(\varsigma_i,\varsigma_j)\varsigma_k = 0 \tag{7.105}$$

$\forall\ i,j,k = 1,2,3$. *In view of (7.105), we can verify equations (3.34), (3.38), (3.39), (3.42), (3.43) and (3.44). Therefore it is clear that the Theorem (3.3) is well satisfied.*

The Ricci tensor $\mathcal{S}(\varsigma_j,\varsigma_k)$; $j,k = 1,2,3$ *of the connection* ∇ *can be calculated as under:*

$$\mathcal{S}(\varsigma_j,\varsigma_k) = \sum_{i=1}^{3} g(\mathfrak{R}(\varsigma_i,\varsigma_j)\varsigma_k,\varsigma_i).$$

It follows that

$$\mathcal{S}(\varsigma_1,\varsigma_1) = 0, \quad \mathcal{S}(\varsigma_2,\varsigma_2) = 0, \quad \mathcal{S}(\varsigma_3,\varsigma_3) = -2\beta^2. \tag{7.106}$$

Along with $\mathcal{S}(\varsigma_j, \varsigma_k) = 0$; $\forall$ $(j \neq k) = 1, 2, 3$. *By virtue of (7.106), we can verify equations (4.64), (5.70) and (5.79).*

The Ricci tensor $\widetilde{\mathcal{S}}(\varsigma_j, \varsigma_k)$; $j, k = 1, 2, 3$ *of the connection* $\widetilde{\widetilde{\nabla}}$ *can be calculated as under:*

$$\widetilde{\mathcal{S}}(\varsigma_j, \varsigma_k) = \sum_{i=1}^{3} g(\widetilde{\mathfrak{R}}(\varsigma_i, \varsigma_j)\varsigma_k, \varsigma_i).$$

It follows that

$$\widetilde{\mathcal{S}}(\varsigma_j, \varsigma_k) = 0 \tag{7.107}$$

$\forall$ $j, k = 1, 2, 3$.

By virtue of (7.107), we can verify equations (3.35) and (3.40).

The scalar curvature $\mathfrak{r}$ *is given by*

$$\begin{aligned}\mathfrak{r} &= \sum_{i=1}^{3} g(\varsigma_i, \varsigma_i)\mathcal{S}(\varsigma_i, \varsigma_i) \\ &= g(\varsigma_1, \varsigma_1)\mathcal{S}(\varsigma_1, \varsigma_1) + g(\varsigma_2, \varsigma_2)\mathcal{S}(\varsigma_2, \varsigma_2) + g(\varsigma_3, \varsigma_3)\mathcal{S}(\varsigma_3, \varsigma_3) \\ &= 2\beta^2. \end{aligned} \tag{7.108}$$

Also, the scalar curvature $\widetilde{\mathfrak{r}}$ *is given by*

$$\begin{aligned}\widetilde{\mathfrak{r}} &= \sum_{i=1}^{3} g(\varsigma_i, \varsigma_i)\widetilde{\mathcal{S}}(\varsigma_i, \varsigma_i) \\ &= g(\varsigma_1, \varsigma_1)\widetilde{\mathcal{S}}(\varsigma_1, \varsigma_1) + g(\varsigma_2, \varsigma_2)\widetilde{\mathcal{S}}(\varsigma_2, \varsigma_2) + g(\varsigma_3, \varsigma_3)\widetilde{\mathcal{S}}(\varsigma_3, \varsigma_3) \\ &= 0. \end{aligned} \tag{7.109}$$

If (g, ζ, Θ) *be the Ricci soliton on* $\mathfrak{M}^3$ *with respect to the GTWC* $\widetilde{\widetilde{\nabla}}$, *then from (7.109) and (6.85), we have*

$$-(2n + 1)(2\beta + \Theta) = 0,$$

i.e.

$$\Theta = -2\beta. \tag{7.110}$$

Thus the Ricci soliton (g, ζ, Θ) *on a Lorentzian* β*-Kenmotsu manifold* $\mathfrak{M}^3$ *admitting the GTWC* $\widetilde{\widetilde{\nabla}}$ *is steady, expanding, and shrinking according to* $\beta = 0$, $\beta < 0$, *and* $\beta > 0$, *respectively. Hence Theorem (6.2) is verified.*

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