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STUDY OF SOME HARMONICITY PROBLEMS CONCERNING THE RESCALED SASAKI METRIC

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Abstract. This paper introduces the concept of harmonicity on the tangent bundle endowed with a rescaled Sasaki metric. Firstly, we study the harmonicity of a vector field on the tangent bundle. Secondly, we investigate the harmonicity of the composition of a vector field and mapping between Riemannian manifolds. Afterwards, we explore the harmonicity of composition between the natural projection of a Riemannian manifold and a map of this manifold to another. Finally, we investigate the harmonicity of the tangent map.

Keywords:Riemannian manifold, tangent bundle, rescaled Sasaki metric, harmonicity.

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1. Introduction

The tangent bundle of a Riemannian manifold can be endowed with Riemannian metrics defined from the Riemannian metric of the base manifold. The most famous of these is the Sasaki metric [19]. Several authors have studied the geometry of the tangent bundle endowed with the Sasaki metric (see [21, 4, 18]). Some authors have constructed other metrics on tangent bundles, which represent deformations of the Sasaki metric on tangent bundles (see [2, 11, 16]). The rescaled metric is between the deformations of the Sasaki metric on the tangent bundle, which have been studied and developed in several recent studies (see [10, 20, 22]).

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The main objective of this research is to investigate the harmonicity concerning the Rescaled Sasaki metric on the tangent bundle. After stating the introduction, we describe the preliminary results of the tangent bundle and basic properties of the Rescaled Sasaki metric. In section 3, We give certain harmonic problems of a vector field concerning this metric. (Theorem 3.2, Theorem 3.3 and Theorem 3.6). In section 4, we investigate the harmonicity of the composition of a vector field and mapping between Riemannian manifolds (Theorem 4.1 and Theorem 4.2). Next, in section 5, we explore the harmonicity of composition between the natural projection of a Riemannian manifold and a map of this manifold to another (Theorem 5.1 and Theorem 5.2). In the last section, we examine the harmonicity of the tangent map (Theorem 6.1, Theorem 6.2 and Theorem 6.3).

2. Preliminary Results

Consider the k-dimensional Riemannian manifold M^k endowed with the Riemannian metric g and the bundle projection (natural projection) $\pi : TM^k \to M^k$. The local coordinates (U, x^i), $i = 1, \ldots, k$ on M^k induces on TM^k a system of local coordinates $(\pi^{-1}(U), x^i, v^i = x^{\bar{i}})$, $i = 1, \ldots, k$ on TM^k. Denote by ∇ the Levi-Civita connection of g and by Γ^s_{ij} the Christoffel symbols of ∇. Let $\Im^1_0(M^k)$ be the module of C^∞ vector fields on M^k over the ring of real-valued C^∞ functions on M^k. There is a direct sum decomposition defined by the Levi Civita connection ∇.

$$T_{(x,v)}TM^k = V_{(x,v)}TM^k \oplus H_{(x,v)}TM^k$$

of the tangent bundle to TM^k at all $(x, v) \in TM^k$ into the vertical distribution

$$V_{(x,v)}TM^k = Ker(d\pi_{(x,v)}) = \{\alpha^i \frac{\partial}{\partial v^i}|_{(x,v)},\ \alpha^i \in \mathbb{R}\}$$

and the horizontal distribution

$$H_{(x,v)}TM^k = \{\alpha^i \frac{\partial}{\partial x^i}|_{(x,v)} - \alpha^i v^j \Gamma^s_{ij} \frac{\partial}{\partial v^s}|_{(x,v)},\ \alpha^i \in \mathbb{R}\}.$$

Given a vector field $Z = Z^i \frac{\partial}{\partial x^i}$ on M^k. The vertical and horizontal lifts of Z are defined by:

$$\begin{aligned} {}^V Z &= Z^i \frac{\partial}{\partial v^i}, \\ {}^H Z &= Z^i (\frac{\partial}{\partial x^i} - v^j \Gamma^s_{ij} \frac{\partial}{\partial v^s}). \end{aligned}$$

We have ${}^H(\frac{\partial}{\partial x^i}) = \frac{\partial}{\partial x^i} - v^j \Gamma^k_{ij} \frac{\partial}{\partial v^k}$ and ${}^V(\frac{\partial}{\partial x^i}) = \frac{\partial}{\partial v^i}$, then $({}^H(\frac{\partial}{\partial x^i}), {}^V(\frac{\partial}{\partial x^i}))_{i=1,\ldots,k}$ is a local adapted frame on TTM^k.

Definition 2.1. [20] *Given a Riemannian manifold* (M^k, g) *and a strictly positive smooth functions* $f : M^k \to]0; +\infty[$*. The Rescaled Sasaki metric on the tangent bundle* TM^k *of* M^k *is defined by:*

$$\begin{aligned} G^f({}^HX, {}^HY)_{(x,v)} &= f(x)g_x(X, Y), \\ G^f({}^VX, {}^HY)_{(x,v)} &= G^f({}^HX, {}^VY)_{(x,v)} = 0, \\ G^f({}^VX, {}^VY)_{(x,v)} &= g_x(X, Y), \end{aligned}$$

for all vector fields X *and* Y *on* M^k *and* $(x, v) \in TM^k$*. Note that, if* $f = 1$*, then* G^f *is the Sasaki metric*[19].

Theorem 2.1. [20] *Given a Riemannian manifold* (M^k, g) *and the Levi-Civita connection* $\widetilde{\nabla}$ *of the tangent bundle* (TM^k, G^f) *endowed with the Rescaled Sasaki metric. Then we have the following formulas:*

$$\begin{aligned} 1.\ \widetilde{\nabla}_{{}^HX}{}^HY &= {}^H\big(\nabla_X Y + \frac{1}{2f}(X(f)Y + Y(f)X - g(X, Y)grad f)\big) - \frac{1}{2}{}^V(R(X, Y)v), \\ 2.\ \widetilde{\nabla}_{{}^HX}{}^VY &= \frac{1}{2f}{}^H\big(R(v, Y)X\big) + {}^V(\nabla_X Y), \\ 3.\ \widetilde{\nabla}_{{}^VX}{}^HY &= \frac{1}{2f}{}^H\big(R(v, X)Y\big), \\ 4.\ \widetilde{\nabla}_{{}^VX}{}^VY &= 0, \end{aligned}$$

for all vector fields X *and* Y *on* M^k*, where* R *is the curvature tensor of* ∇ *on* (M^k, g)*.*

Now, we will introduce some basic concepts concerning harmonic maps. Given a smooth map $\psi : (M^k, g) \to (N^n, h)$ between two Riemannian manifolds. If ψ is a critical point of the energy functional,

$$E(\psi) = \int_K e(\psi)\, v_g, \tag{2.1}$$

the map ψ is called harmonic. for all compact domain $K \subseteq M^k$. Here

$$e(\psi) := \frac{1}{2}|d\psi|^2 = \frac{1}{2}Tr_g\, h(d\psi, d\psi) \tag{2.2}$$

is the energy density of ψ, $|d\psi|$ is the Hilbert-Schmitd norm of $d(\psi)$ and v_g is the Riemannian volume form on M^k. The first variation of the energy [13] is expressed by:

$$\frac{d}{dt}E(\psi_t)\Big|_{t=0} = -\int_K h(\tau(\psi), V)v_g, \tag{2.3}$$

for all smooth 1-parameter variation $\{\psi_t\}_{t\in I}$ of ψ and $V = \frac{d}{dt}\psi_t\Big|_{t=0}$.
Then, ψ is to be harmonic if and only if $\tau(\psi) = 0$, where

$$\tau(\psi) := Tr_g \nabla d\psi, \tag{2.4}$$

is called the tension field of ψ, see [6, 7, 8, 12, 14]. Recently, numerous authors have extensively explored this topic, including its application to the tangent bundle [1, 23].

3. Harmonicity of section $X : (M^k, g) \to (TM^k, G^f)$

Lemma 3.1. [14, 15] *Consider a Riemannian manifold (M^k, g). Then the following equation holds:*

$$d_x Y(X_x) = {}^H X_{(x,v)} + {}^V(\nabla_X Y)_{(x,v)} \tag{3.5}$$

for all vector fields X, Y on M^k and $(x, v) \in TM^k$, where $Y_x = v$.

Lemma 3.2. [24] *Consider a Riemannian manifold (M^k, g). Then the following equation holds:*

$$g(\bar{\Delta} Z, Z) = |\nabla Z|^2 - \frac{1}{2}\Delta |Z|^2, \tag{3.6}$$

for all vector field Z on M^k, where $\bar{\Delta} Z := -Tr_g \nabla^2 Z = -Tr_g(\nabla_* \nabla_* - \nabla_{\nabla_* *})Z$ denotes the rough Laplacian of Z.

Lemma 3.3. [24] *Consider a Riemannian manifold (M^k, g). Then the following equation holds:*

$$\bar{\Delta}(\rho Z) = \rho \bar{\Delta} Z - (\Delta \rho) Z - 2\nabla_{grad \rho} Z, \tag{3.7}$$

for all vector field Z on M^k, where ρ is a smooth function of M^k.

Lemma 3.4. *Given a Riemannian manifold (M^k, g) and its tangent bundle (TM^k, G^f) endowed with the Rescaled Sasaki metric. If Z is a vector field on M^k, its corresponding energy density is expressed by:*

$$e(Z) \;=\; \frac{kf}{2} + \frac{1}{2}|\nabla Z|^2. \tag{3.8}$$

Proof. Given a local orthonormal frame field $\{E_i\}_{i=1,\dots,k}$ on M^k and $(x,v)\in TM^k$ such that $Z_x = v$, from (2.2), we have:

$$\begin{aligned} e(Z)_x &= \frac{1}{2}|d_xZ|^2 \\ &= \frac{1}{2}Tr_g\, G^f(dZ,dZ)_{(x,v)} \\ &= \frac{1}{2}\sum_{i=1}^{k} G^f(dZ(E_i),dZ(E_i))_{(x,v)}. \end{aligned}$$

Using (3.5), we obtain:

$$\begin{aligned} e(Z) &= \frac{1}{2}\sum_{i=1}^{k} G^f({}^HE_i + {}^V(\nabla_{E_i}Z), {}^HE_i + {}^V(\nabla_{E_i}Z)) \\ &= \frac{1}{2}\sum_{i=1}^{k}\big(G^f({}^HE_i,{}^HE_i) + G^f({}^V(\nabla_{E_i}Z),{}^V(\nabla_{E_i}Z))\big) \\ &= \frac{1}{2}\sum_{i=1}^{k}\big(fg(E_i,E_i) + g(\nabla_{E_i}Z,\nabla_{E_i}Z)\big) \\ &= \frac{kf}{2} + \frac{1}{2}|\nabla Z|^2. \end{aligned}$$

□

Theorem 3.1. *Given a Riemannian manifold (M^k,g) and its tangent bundle (TM^k,G^f) endowed with the Rescaled Sasaki metric. If Z is a vector field on M^k, its associated tension field is expressed by:*

$$\tau(Z) = {}^H\Big(\frac{2-k}{2f}grad f + \frac{1}{f}Tr_g(R(Z,\nabla_* Z)*)\Big) - {}^V\bar{\Delta}Z. \tag{3.9}$$

Proof. Given a local orthonormal frame field $\{E_i\}_{i=1,\dots,k}$ on M^k and $(x,v)\in TM^k$ such that $(\nabla_{E_i}E_i)_x = 0$ and $Z_x = v$. From (2.4) and (3.5), we have:

$$\begin{aligned} \tau(Z)_x &= \big(Tr_g\nabla dZ\big)_x \\ &= \sum_{i=1}^{k}\big(\nabla^Z_{E_i}dZ(E_i)\big)_x \\ &= \sum_{i=1}^{k}\big(\widetilde{\nabla}_{dZ(E_i)}dZ(E_i)\big)_{(x,v)} \\ &= \sum_{i=1}^{k}\big(\widetilde{\nabla}_{({}^HE_i+{}^V(\nabla_{E_i}Z))}({}^HE_i + {}^V(\nabla_{E_i}Z))\big)_{(x,v)} \\ &= \sum_{i=1}^{k}\big(\widetilde{\nabla}_{{}^HE_i}{}^HE_i + \widetilde{\nabla}_{{}^HE_i}{}^V(\nabla_{E_i}Z) + \widetilde{\nabla}_{{}^V(\nabla_{E_i}Z)}{}^HE_i + \widetilde{\nabla}_{{}^V(\nabla_{E_i}Z)}{}^V(\nabla_{E_i}Z)\big)_{(x,v)}. \end{aligned}$$

Using Theorem 2.1, we obtain:

$$\begin{aligned}\tau(Z) &= \sum_{i=1}^{k}\Big({}^{H}\big(\nabla_{E_i}E_i+\frac{1}{2f}(2E_i(f)E_i-g(E_i,E_i)grad f)\big)-\frac{1}{2}{}^{V}(R(E_i,E_i)Z)\\ &\quad+\frac{1}{2f}{}^{H}(R(Z,\nabla_{E_i}Z)E_i)+{}^{V}(\nabla_{E_i}\nabla_{E_i}Z)+\frac{1}{2f}{}^{H}(R(Z,\nabla_{E_i}Z)E_i)\\ &= \sum_{i=1}^{k}\Big(\frac{1}{2f}{}^{H}\big(2E_i(f)E_i-grad f\big)+\frac{1}{f}{}^{H}(R(Z,\nabla_{E_i}Z)E_i)+{}^{V}(\nabla_{E_i}\nabla_{E_i}Z)\Big).\end{aligned}$$

To simplify the last statement, we use the following equations:

$$\begin{aligned}\sum_{i=1}^{k}\frac{1}{f}R(Z,\nabla_{E_i}Z)E_i &= \frac{1}{f}Tr_g(R(Z,\nabla_{*}Z)*),\\ \sum_{i=1}^{k}\nabla_{E_i}\nabla_{E_i}Z &= -\bar{\Delta}Z.\end{aligned}$$

Then, we find (3.9). □

From Theorem 3.1 we get the following:

Theorem 3.2. *Given a Riemannian manifold (M^k, g) and its tangent bundle (TM^k, G^f) endowed with the Rescaled Sasaki metric. A vector field Z on M^k is a harmonic map if and only if the following conditions hold:*

$$Tr_g(R(Z,\nabla_{*}Z)*)=\frac{k-2}{2}grad f \quad and \quad \bar{\Delta}Z=0.$$

Corollary 3.1. *Given a Riemannian manifold (M^k, g) and its tangent bundle (TM^k, G^f) endowed with the Rescaled Sasaki metric. If $k = 2$ or $f = constant$, then every vector field that is parallel on M^k is harmonic map.*

Given a compact oriented Riemannian manifold (M^k, g) and its tangent bundle (TM^k, G^f) endowed with the Rescaled Sasaki metric. Let Z a vector field on M^k. The energy $E(Z)$ of Z is defined to be the energy of the corresponding map $Z : (M^k, g) \to (TM^k, G^f)$[5]. More precisely, from (3.8), we get

$$E(Z)=\frac{k}{2}\int_{M^k} f\, v_g+\frac{1}{2}\int_{M^k}|\nabla Z|^2 v_g. \tag{3.10}$$

Definition 3.1. [5, 17] *Given a Riemannian manifold (M^k, g) and its tangent bundle (TM^k, G^f) endowed with the Rescaled Sasaki metric. A vector field Z on M^k is called harmonic vector field if the corresponding map $Z : (M^k, g) \to (TM^k, G^f)$ is a critical point for the energy functional E, only considering variations between maps defined by vector fields.*

Theorem 3.3. *Given a Riemannian manifold (M^k, g) and its tangent bundle (TM^k, G^f) endowed with the Rescaled Sasaki metric. Let Z a vector field on M^k and $E : \Im^1_0(M^k) \to [0, +\infty)$ the energy functional restricted to the space of all vector fields. Then the following condition holds:*

$$\frac{d}{dt}E(Z_t)\big|_{t=0} = \int_{M^k} g(\bar{\Delta}Z, V)v_g \tag{3.11}$$

for all smooth 1-parameter variation $\vartheta : M^k \times (-\epsilon, \epsilon) \to TM^k$ of Z i.e. $\vartheta(x, 0) = Z(x)$, $Z_t(x) = \vartheta(x, t) \in T_xM^k$ for all $(x, t) \in M^k \times (-\epsilon, \epsilon)$, $(\epsilon > 0)$ and $V \in \Im^1_0(M^k)$ is the vector field on M^k expressed by:

$$V(x) = \frac{d}{dt}Z_x(t)\Big|_{t=0} = \lim_{t\to 0}\frac{1}{t}(\vartheta(x, t) - \vartheta(x, 0)),$$

where $Z_x(t) = \vartheta(x, t), (x, t) \in M^k \times (-\epsilon, \epsilon)$.

Proof. Given a smooth 1-parameter variation $\vartheta : M^k \times (-\epsilon, \epsilon) \to TM^k$ of Z, such that $Z_t(x) = \vartheta(x, t) \in T_xM^k$ for all $(x, t) \in M^k \times (-\epsilon, \epsilon)$ and $\vartheta(x, 0) = Z(x)$. From (2.1), we have

$$E(Z_t) = \int_{M^k} e(Z_t)v_g.$$

Using (2.3), we get:

$$\frac{d}{dt}E(Z_t)\Big|_{t=0} = -\int_{M^k} G^f(\mathcal{V}, \tau(Z))v_g, \tag{3.12}$$

where $\mathcal{V}$ is the infinitesimal variation induced by ϑ, i.e.,

$$\mathcal{V}(x) = d_{(x,0)}\vartheta(0, \frac{d}{dt})\big|_{t=0} = \frac{d}{dt}Z_t(x)\big|_{t=0}.$$

It is well known that

$$\mathcal{V} = {}^{V}V \circ Z, \tag{3.13}$$

see [5, p.58]. Finally, by (3.9), (3.12) and (3.13), we find:

$$\begin{aligned}\frac{d}{dt}E(Z_t)\Big|_{t=0} &= -\int_{M^k} G^f({}^{V}V, \tau(Z))v_g \\ &= -\int_{M^k} G^f({}^{V}V, -{}^{V}\bar{\Delta}Z)v_g \\ &= \int_{M^k} g(\bar{\Delta}Z, V)v_g.\end{aligned}$$

□

If (M^k, g) is a non-compact oriented Riemannian manifold, then Theorem 3.3 holds. In fact, if M^k is non-compact, we can choose $V \in \Im_0^1(M^k)$ which support is contained in an open subset W in M^k whose closure is compact. Then (3.11) is as follows:

$$\frac{d}{dt}E(Z_t)\big|_{t=0} \;=\; \int_W g(\bar{\Delta}Z, V)v_g.$$

We derive from this a necessary and sufficient condition for a vector field is a harmonic vector field or harmonic map, respectively.

Theorem 3.4. *Given a Riemannian manifold (M^k, g) and its tangent bundle (TM^k, G^f) endowed with the Rescaled Sasaki metric. A vector field Z on M^k is harmonic vector field if and only if $\bar{\Delta}Z = 0$.*

Using Theorem 3.2 and Theorem 3.4, we obtain the following:

Theorem 3.5. *Given a Riemannian manifold (M^k, g) and its tangent bundle (TM^k, G^f) endowed with the Rescaled Sasaki metric. A vector field Z on M^k is harmonic map if and only if Z is harmonic vector field and*

$$Tr_g(R(Z, \nabla_* Z)*) = \frac{k-2}{2} grad f.$$

It is clear that, all parallel vector field on M^k is harmonic vector field. Conversely, we have:

Theorem 3.6. *Given a compact oriented Riemannian manifold (M^k, g) and its tangent bundle (TM^k, G^f) endowed with the Rescaled Sasaki metric. A vector field Z on M^k is harmonic vector field if and only if Z is parallel.*

Proof. We suppose that Z is a harmonic vector field on M^k, from Theorem 3.4, we find $\bar{\Delta}Z = 0$. By (3.6), we obtain $|\nabla Z|^2 = \frac{1}{2}\Delta|Z|^2$. Applying the divergence Theorem, we get

$$\int_{M^k} |\nabla Z|^2 v^g = 0.$$

Since $|\nabla Z|^2$ is a positive function, hence $\nabla Z = 0$. □

Theorem 3.7. *Given a compact oriented Riemannian manifold (M^k, g) and its tangent bundle (TM^k, G^f) endowed with the Rescaled Sasaki metric. If $k = 2$ or $f =$ constant, a vector field Z on M^k is harmonic map if and only if Z is parallel.*

Example 3.1. *Let $\mathbb{R}^2$ endowed with the Riemannian metric in polar coordinate defined by:*

$$g = dr^2 + r^2 d\theta^2.$$

Relatively to the orthonormal frame

$$E_1 = \partial_r, \quad E_2 = \frac{1}{r}\partial_\theta$$

we have,

$$\nabla_{E_1} E_1 = \nabla_{E_1} E_2 = 0, \quad \nabla_{E_2} E_1 = \frac{1}{r} E_2, \quad \nabla_{E_2} E_2 = -\frac{1}{r} E_1,$$
$$R(E_1, E_2)E_1 = R(E_1, E_2)E_2 = 0.$$

Let $Z = \alpha(r)E_1$ be a vector field, where α is a smooth real function. Using simple calculations, we find:

$$Tr_g(R(Z, \nabla_* Z)*) = 0,$$
$$\bar{\Delta} E_1 = \frac{1}{r^2} E_1,$$

Using (3.7)*, we obtain:*

$$\bar{\Delta} Z = (-\alpha'' - \frac{1}{r}\alpha' + \frac{1}{r^2}\alpha)E_1.$$

i) From Theorem 3.4, we conclude that Z is harmonic vector field equivalently $\bar{\Delta} Z = 0$, then

$$-\alpha'' - \frac{1}{r}\alpha' + \frac{1}{r^2}\alpha = 0. \tag{3.14}$$

The general solution of differential equation (3.14) *is*

$$\alpha(r) = c_1 r + \frac{c_2}{r},$$

where, c_1 and c_2 be real constants.
ii) Since $k = 2$ and $Tr_g(R(Z, \nabla_* Z)*) = 0$, from Theorem 3.5, the vector fields $Z = (c_1 r + \frac{c_2}{r})E_1$ are also harmonic maps.
iii) However the vector field $Y = (r - \frac{1}{r})E_1$ is harmonic but non parallel, because $\nabla_{E_1} Y = (1 + \frac{1}{r^2})E_1 \neq 0$.

Example 3.2. *Consider $\mathbb{R}^2$ with the Riemannian metric*

$$g = e^{2y}dx^2 + dy^2.$$

Relatively to the orthonormal frame $\{E_1 = e^{-y}\partial_x, E_2 = \partial_y\}$ *with respect to* g*, we have:*

$$\nabla_{E_1}E_1 = -E_2,\ \nabla_{E_1}E_2 = E_1,\ \nabla_{E_2}E_1 = \nabla_{E_2}E_2 = 0,$$

$$R(E_1, E_2)E_1 = E_2,\ R(E_1, E_2)E_2 = -E_1.$$

Let $Z = \rho(y)E_2$ *be a vector field, where* ρ *is a smooth non-zero real function. According to simple calculations, we find:*

$$Tr_g(R(Z, \nabla_* Z)*) = -\rho^2 E_2,$$

$$\bar{\Delta}Z = (-\rho'' - \rho' + \rho)E_2.$$

i) From Theorem 3.4, we conclude that $Z = \rho(y)E_2$ *is harmonic vector field if and only if*

$$-\rho'' - \rho' + \rho = 0. \tag{3.15}$$

The general solution of differential equation (3.15) *is*

$$\rho(y) = ae^{\frac{-1-\sqrt{5}}{2}y} + be^{\frac{-1+\sqrt{5}}{2}y},$$

where a *and* b *be non-zero real constants at the same time.*

ii) Since $k = 2$ *and* $Tr_g(R(Z, \nabla_* Z)*) \neq 0$*, from Theorem 3.5, the vector fields* $Z = (ae^{\frac{-1-\sqrt{5}}{2}y} + be^{\frac{-1+\sqrt{5}}{2}y})E_2$ *are never harmonic maps.*

On the other hand,we have $\nabla_{E_1}Z = \rho\, E_1 = (ae^{\frac{-1-\sqrt{5}}{2}y} + be^{\frac{-1+\sqrt{5}}{2}y})E_1 \neq 0$*, then the vector fields* Z *is non parallel, i.e. the vector fields* Z *are harmonic vector fields but neither harmonic maps nor parallel vector fields.*

Example 3.3. *The torus* $T^2 = \mathbb{S}^1 \times \mathbb{S}^1$ *(* 2*-dimensional compact oriented Riemannian manifold) endowed with the metric:*

$$g = \frac{4}{(1+x^2)^2}dx^2 + \frac{4}{(1+y^2)^2}dy^2.$$

Let $Z = f_1(x,y)\partial_x + f_2(x,y)\partial_y$ *be a vector field, where* f_1 *and* f_2 *are smooth functions. According to simple calculations, we find:*

$$\nabla_{\partial_x}\partial_x = \frac{-2x}{1+x^2}\partial_x,\ \nabla_{\partial_x}\partial_y = \nabla_{\partial_y}\partial_x = 0,\ \nabla_{\partial_y}\partial_y = \frac{-2y}{1+y^2}\partial_y,$$

$$Tr_g(R(X, \nabla_* X)*) = 0.$$

i) From Theorem 3.6, we conclude that Z is harmonic vector field $\Leftrightarrow \nabla Z = 0$

$$\Leftrightarrow \begin{cases} \nabla_{\partial_x} Z = 0 \\ \nabla_{\partial_y} Z = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} (\frac{\partial f_1}{\partial x} - \frac{2x}{1+x^2} f_1)\partial_x + \frac{\partial f_2}{\partial x}\partial_y = 0 \\ \frac{\partial f_1}{\partial y}\partial_x + (\frac{\partial f_2}{\partial y} - \frac{2y}{1+y^2} f_2)\partial_y = 0 \end{cases}$$

We conclude that the function f_1 is only dependent on x, while the function f_2 is only dependent on y, so we find the following system:

$$\begin{cases} \frac{\partial f_1}{\partial_x} - \frac{2x}{1+x^2} f_1 = 0 \\ \frac{\partial f_2}{\partial_y} - \frac{2y}{1+y^2} f_2 = 0. \end{cases}$$

The general solution for this system is

$$f_1(x) = c_1(1+x^2), \ f_2(y) = c_2(1+y^2)$$

where, c_1 and c_2 be real constants.

ii) Since $k = 2$ and $Tr_g(R(Z, \nabla_* Z)*) = 0$, from Theorem 3.6, we deduce that $Z = c_1(1+x^2)\partial_x + c_2(1+y^2)\partial_y$ are also harmonic maps.

4. Harmonicity of the composition of a vector field and mapping between Riemannian manifolds

Lemma 4.1. [23] *Given a smooth map $\psi : (M^k, g) \to (N^n, h)$ between two Riemannian manifolds, a vector field Y on N^n and the smooth map ξ defined by $\xi := Y \circ \psi$. Then*

$$d\xi(X) = {}^H(d\psi(X)) + {}^V(\nabla^{\psi}_X \xi), \tag{4.16}$$

for all vector field X on M^k, where ∇^{ψ} is the pull-back connection on the pull-back bundle $\psi^{-1}TN^n$.

Theorem 4.1. *Given a smooth map $\psi : (M^k, g) \to (N^n, h)$ between two Riemannian manifolds and the tangent bundle (TN^n, H^f) of N^n endowed with the Rescaled Sasaki metric. Then the tension field of the map $\xi := Y \circ \psi$ is expressed by:*

$$\begin{aligned} \tau(\xi) \quad = \quad & {}^H\big(\tau(\psi) + \frac{1}{f} d\psi(grad^{M^k}(f \circ \psi)) - \frac{1}{2f}|d\psi|^2 grad^{N^n} f + \frac{1}{f} Tr_g(R^{N^n}(\xi, \nabla^{\psi}_* \xi) d\psi(*))\big) \\ & -{}^V\big(\Delta^{\psi}\xi\big), \end{aligned}$$

where $\Delta^{\psi}\xi := -Tr_g(\nabla^{\psi})^2\xi = -Tr_g(\nabla^{\psi}_*\nabla^{\psi}_* - \nabla^{\psi}_{\nabla_* *})\xi$ is the rough Laplacian of ξ on $\psi^{-1}TN^n$.

Proof. Given a local orthonormal frame field $\{E_i\}_{i=1,\dots,k}$ on M^k such that $(\nabla_{E_i}E_i)_x = 0$ and $\xi(x) = (\psi(x), Y_{\psi(x)}), Y_{\psi(x)} = v \in T_{\psi(x)}N^n$.

Using (4.16), we have:

$$\begin{aligned}
\tau(\xi)_x &= (Tr_g \nabla d\xi)_x \\
&= \sum_{i=1}^{k} (\nabla^{\xi}_{E_i} d\xi(E_i))_x \\
&= \sum_{i=1}^{k} \nabla^{TN^n}_{d\xi(E_i)} d\xi(E_i))_{(\psi(x),v)} \\
&= \sum_{i=1}^{k} \big(\nabla^{TN^n}_{({}^H(d\psi(E_i))+{}^V(\nabla^{\psi}_{E_i}\xi))} ({}^H(d\psi(E_i)) + {}^V(\nabla^{\psi}_{E_i}\xi))\big)_{(\psi(x),v)} \\
&= \sum_{i=1}^{k} \big(\nabla^{TN^n}_{{}^H(d\psi(E_i))} {}^H(d\psi(E_i)) + \nabla^{TN^n}_{{}^H(d\psi(E_i))} {}^V(\nabla^{\psi}_{E_i}\xi) + \nabla^{TN^n}_{{}^V(\nabla^{\psi}_{E_i}\xi)} {}^H(d\psi(E_i)) \\
&\quad + \nabla^{TN^n}_{{}^V(\nabla^{\psi}_{E_i}\xi)} {}^V(\nabla^{\psi}_{E_i}\xi)\big)_{(\psi(x),v)}.
\end{aligned}$$

From Theorem 2.1, we obtain:

$$\begin{aligned}
\tau(\xi) &= \sum_{i=1}^{k} \Big({}^H\big(\nabla^{N^n}_{d\psi(E_i)} d\psi(E_i) + \frac{1}{2f}(2d\psi(E_i)(f)d\psi(E_i) - h(d\psi(E_i), d\psi(E_i))grad^{N^n}f)\big) \\
&\quad + \frac{1}{f}{}^H(R^{N^n}(\xi, \nabla^{\psi}_{E_i}\xi)d\psi(E_i)) + {}^V(\nabla^{N^n}_{d\psi(E_i)} \nabla^{\psi}_{E_i}\xi)\Big) \\
&= \sum_{i=1}^{k} \Big({}^H\big(\nabla^{\psi}_{E_i} d\psi(E_i) + \frac{1}{2f}(2E_i(f\circ\psi)d\psi(E_i) - h(d\psi(E_i), d\psi(E_i))grad^{N^n}f)\big) \\
&\quad + \frac{1}{f}R^{N^n}(\xi, \nabla^{\psi}_{E_i}\xi)d\psi(E_i)\big) + {}^V\big(\nabla^{\psi}_{E_i}\nabla^{\psi}_{E_i}\xi\big)\Big) \\
&= {}^H\big(\tau(\psi) + \frac{1}{f}d\psi(grad^{M^k}(f\circ\psi)) - \frac{1}{2f}|d\psi|^2 grad^{N^n}f + \frac{1}{f}Tr_g(R^{N^n}(\xi, \nabla^{\psi}_{*}\xi)d\psi(*))\big) \\
&\quad - {}^V\big(\Delta^{\psi}\xi\big).
\end{aligned}$$

□

From Theorem 4.1 we obtain:

Theorem 4.2. *Given a smooth map $\psi : (M^k, g) \to (N^n, h)$ between two Riemannian manifolds and (TN^n, H^f) be the tangent bundle of N^n endowed with the Rescaled Sasaki metric. Then the map $\xi := Y \circ \psi$ is harmonic if and only if the following conditions hold:*

$$\begin{cases}
\tau(\psi) = -\frac{1}{f}Tr_g(R^{N^n}(\xi, \nabla^{\psi}_{*}\xi)d\psi(*)) - \frac{1}{f}d\psi(grad^{M^k}(f\circ\psi)) + \frac{1}{2f}|d\psi|^2 grad^{N^n}f, \\
\Delta^{\psi}\xi = 0.
\end{cases}$$

5. HARMONICITY OF COMPOSITION BETWEEN THE NATURAL PROJECTION AND A SMOOTH MAP

Lemma 5.1. *Given a Riemannian manifold (M^k, g) and its tangent bundle (TM^k, G^f) endowed with the Rescaled Sasaki metric. Then The tension field of the canonical projection $\pi : (TM^k, G^f) \to (M^k, g)$ is expressed by:*

$$\tau(\pi) = \frac{k-2}{2f^2}(gradf) \circ \pi. \tag{5.17}$$

Proof. Given a local orthonormal frame field $\{E_i\}_{i=1,\ldots,k}$ on M^k, then $\{\frac{1}{\sqrt{f_1}}{}^HE_i, {}^VE_j\}_{i,j=1,\ldots,k}$ forms a local orthonormal frame field on the tangent bundle TM^k. From Theorem 2.1, we find:

$$\begin{aligned}
\tau(\pi)_{(x,v)} &= (Tr_{G^f}\nabla d\pi)_{(x,v)} \\
&= \sum_{i=1}^{k}\Big(\nabla_{d\pi(\frac{1}{\sqrt{f}}{}^HE_i)}d\pi(\frac{1}{\sqrt{f}}{}^HE_i) - d\pi(\widetilde{\nabla}_{\frac{1}{\sqrt{f}}{}^HE_i}\frac{1}{\sqrt{f}}{}^HE_i)\Big)_{\pi(x,v)} \\
&\quad + \sum_{j=1}^{k}\Big(\nabla_{d\pi({}^VE_j)}d\pi({}^VE_j) - d\pi(\widetilde{\nabla}_{{}^VE_j}{}^VE_j)\Big)_{\pi(x,v)}.
\end{aligned}$$

But because $d\pi({}^VZ) = 0$ and $d\pi({}^HZ) = Z \circ \pi$, for each vector field Z on M^k, we find:

$$\begin{aligned}
\tau(\pi) &= \sum_{i=1}^{k}\Big(\nabla_{(\frac{1}{\sqrt{f}}E_i\circ\pi)}(\frac{1}{\sqrt{f}}E_i\circ\pi) - d\pi\big({}^H(\nabla_{\frac{1}{\sqrt{f}}E_i}\frac{1}{\sqrt{f}}E_i)\big) \\
&\quad -\frac{1}{2f}d\pi\big({}^H\big(2\frac{1}{\sqrt{f}}E_i(f)\frac{1}{\sqrt{f}}E_i - g(\frac{1}{\sqrt{f}}E_i, \frac{1}{\sqrt{f}}E_i)gradf\big)\big)\Big)\circ\pi \\
&= \sum_{i=1}^{k}\Big(\nabla_{\frac{1}{\sqrt{f}}E_i}\frac{1}{\sqrt{f}}E_i - \nabla_{\frac{1}{\sqrt{f}}E_i}\frac{1}{\sqrt{f}}E_i - \frac{1}{2f^2}(2E_i(f)E_i - g(E_i,E_i)gradf)\circ\pi\Big)\circ\pi \\
&= \frac{k-2}{2f^2}(gradf)\circ\pi.
\end{aligned}$$

□

Theorem 5.1. *Given a smooth map between Riemannian manifolds $\psi : (M^k, g) \to (N^n, h)$. The tension field of the map $\zeta := \psi \circ \pi$ is expressed by:*

$$\tau(\zeta) = \frac{1}{f}(\tau(\psi) + \frac{k-2}{2f}d\psi(gradf)) \circ \pi. \tag{5.18}$$

Proof. Given a local orthonormal frame field $\{E_i\}_{i=1,\ldots,k}$ on M^k, then $\{\frac{1}{\sqrt{f_1}}{}^HE_i, {}^VE_j\}_{i,j=1,\ldots,k}$ forms a local orthonormal frame field on the tangent bundle TM^k. Then the tension field of the map $\zeta := \psi \circ \pi$ is given by [6, 8]

$$\tau(\zeta) = \tau(\psi\circ\pi) = d\psi(\tau(\pi)) + Tr_{G^f}\nabla d\psi(d\pi, d\pi),$$

then, we have:

$$
\begin{aligned}
Tr_{G^f}\nabla d\psi(d\pi,d\pi) &= \sum_{i=1}^{k}\Big(\nabla^{N^n}_{d\psi(d\pi(\frac{1}{\sqrt{f}}{}^HE_i))}d\psi(d\pi(\frac{1}{\sqrt{f}}{}^HE_i)) - d\psi(\nabla_{d\pi(\frac{1}{\sqrt{f}}{}^HE_i)}d\pi(\frac{1}{\sqrt{f}}{}^HE_i))\Big)\circ\pi \\
&+ \sum_{j=1}^{k}\Big(\nabla^{N^n}_{d\psi(d\pi({}^VE_j))}d\psi(d\pi({}^VE_j)) - d\psi(\nabla_{d\pi({}^VE_j)}d\pi({}^VE_j))\Big)\circ\pi \\
&= \sum_{i=1}^{k}\Big(\nabla^{\psi}_{(\frac{1}{\sqrt{f}}E_i)\circ\pi}d\psi((\frac{1}{\sqrt{f}}E_i)\circ\pi) - d\psi\big(\nabla_{(\frac{1}{\sqrt{f}}E_i)\circ\pi}(\frac{1}{\sqrt{f}}E_i)\circ\pi\big)\Big)\circ\pi \\
&= \sum_{i=1}^{k}\Big(\nabla^{\psi}_{\frac{1}{\sqrt{f}}E_i}\frac{1}{\sqrt{f}}d\psi(E_i) - d\psi\big(\nabla_{\frac{1}{\sqrt{f}}E_i}\frac{1}{\sqrt{f}}E_i\big)\Big)\circ\pi \\
&= \sum_{i=1}^{k}\Big(\frac{1}{\sqrt{f}}E_i(\frac{1}{\sqrt{f}})d\psi(E_i) + \frac{1}{f}\nabla^{\psi}_{E_i}d\psi(E_i) \\
&- d\psi(\frac{1}{\sqrt{f}}E_i(\frac{1}{\sqrt{f}})E_i + \frac{1}{f}\nabla_{E_i}E_i)\Big)\circ\pi \\
&= \sum_{i=1}^{k}\Big(\frac{1}{f}\nabla^{\psi}_{E_i}d\psi(E_i) - \frac{1}{f}d\psi\big(\nabla_{E_i}E_i\big)\Big)\circ\pi \\
&= \frac{1}{f}\tau(\psi)\circ\pi.
\end{aligned}
$$

Using (5.17), we obtain (5.18). □

Theorem 5.2. *Given a smooth map $\psi:(M^k,g)\to(N^n,h)$ between two Riemannian manifolds. Then the map $\zeta:=\psi\circ\pi$ is harmonic if and only if*

$$\tau(\psi)=\frac{2-n}{2f}d\psi(gradf).$$

6. Harmonicity of the tangent map

Given Riemannian manifolds (M^k,g), (N^n,h) and their tangent bundles (TM^k,G^{f_1}), (TN^n,H^{f_2}) respectively, equipped with the Rescaled Sasaki metrics, such that f_1, f_2 are strictly positive smooth functions on M^k, N^n respectively.

Lemma 6.1. [9] *Given a smooth map $\psi:(M^k,g)\to(N^n,h)$ between two Riemannian manifolds and its tangent map*

$$
\begin{aligned}
\Psi = d\psi : TM^k &\longrightarrow TN^n \\
(x,v) &\longmapsto (\psi(x), d\psi(v))
\end{aligned}
$$

we have:

$$\begin{cases} d\Psi\big({}^{V}X\big) = {}^{V}\big(d\psi(X)\big), \\ d\Psi\big({}^{H}X\big) = {}^{H}\big(d\psi(X)\big) + {}^{V}\big(\nabla d\psi(x,v)\big), \end{cases}$$

for all vector field X on M^k.

Theorem 6.1. *Given a smooth map between Riemannian manifolds $\psi : (M^k, g) \to (N^n, h)$, then the tension field associated to the tangent map $\Psi : (TM^k, G^{f_1}) \longrightarrow (TN^n, H^{f_2})$ is given by:*

$$\begin{aligned} \tau(\Psi) &= {}^{H}\Big(\frac{1}{f_1}\tau(\psi) + \frac{k-2}{2f_1^2} d\psi(grad^{M^k} f_1) + \frac{1}{f_1 f_2} d\psi(grad^{M^k}(f_2 \circ \psi)) \\ &\quad - \frac{1}{2f_1 f_2}|d\psi|^2 grad^{N^n} f_2 + \frac{1}{f_1 f_2} Tr_g(R^{N^n}(d\psi(u), \nabla d\psi(*,u))d\psi(*))\Big) \\ &\quad + {}^{V}\Big(\frac{k-2}{2f_1^2}\nabla d\psi(grad^{M^k} f_1, u) + \frac{1}{f_1} Tr_g(\nabla^{\psi}_{*}(\nabla d\psi(*,u)))\Big). \end{aligned}$$

Proof. Given a local orthonormal frame field $\{E_i\}_{i=1,\dots,k}$ on M^k such that $(\nabla^{M^k}_{E_i} E_i)_x = 0$, then $\big\{\frac{1}{\sqrt{f_1}}{}^{H}E_i, {}^{V}E_j\big\}_{i,j=1,\dots,k}$ forms a local orthonormal frame field on TM^k, we have:

$$\begin{aligned} \tau(\Psi)_{(x,v)} &= \sum_{i=1}^{k} \big(\nabla^{\Psi}_{\frac{1}{\sqrt{f_1}}{}^{H}E_i} d\Psi(\frac{1}{\sqrt{f_1}}{}^{H}E_i) - d\Psi(\nabla^{TM^k}_{\frac{1}{\sqrt{f_1}}{}^{H}E_i}\frac{1}{\sqrt{f_1}}{}^{H}E_i)\big)_{(\psi(x),d\psi(u))} \\ &\quad + \sum_{j=1}^{n} \big(\nabla^{\Psi}_{{}^{V}E_j} d\Psi({}^{V}E_j) - d\Psi(\nabla^{TM^k}_{{}^{V}E_j}{}^{V}E_j)\big)_{(\psi(x),d\psi(u))} \\ &= \Big(\sum_{i=1}^{k} (\frac{1}{\sqrt{f_1}}{}^{H}E_i(\frac{1}{\sqrt{f_1}})d\Psi({}^{H}E_i) + \frac{1}{f_1}\nabla^{\Psi}_{{}^{H}E_i} d\Psi({}^{H}E_i) + \sum_{j=1}^{n} \big(\nabla^{TN^n}_{d\Psi({}^{V}E_j)} d\Psi({}^{V}E_j)\big) \\ &\quad - d\Psi\big(\frac{1}{\sqrt{f_1}}{}^{H}E_i(\frac{1}{\sqrt{f_1}}){}^{H}E_i + \frac{1}{f_1}\nabla^{TM^k}_{{}^{H}E_i}{}^{H}E_i)\big)\Big)_{(\psi(x),d\psi(u))}. \end{aligned}$$

Using Theorem 2.1 and Lemma 6.1, we obtain:

$$\begin{aligned} \tau(\Psi) &= \frac{1}{f_1}\sum_{i=1}^{k} \Big(\nabla^{TN^n}_{d\Psi({}^{H}E_i)} d\Psi({}^{H}E_i) - d\Psi\big(\nabla^{TM^k}_{{}^{H}E_i}{}^{H}E_i\big)\Big) + \sum_{j=1}^{n} \big(\nabla^{TN^n}_{d\Psi({}^{V}E_j)} d\Psi({}^{V}E_j)\big) \\ &= \frac{1}{f_1}\sum_{i=1}^{k} \Big(\nabla^{TN^n}_{{}^{H}(d\psi(E_i))}{}^{H}(d\psi(E_i)) + \nabla^{TN^n}_{{}^{H}(d\psi(E_i))}{}^{V}(\nabla d\psi(E_i,u)) + \nabla^{TN^n}_{{}^{V}(\nabla d\psi(E_i,u))}{}^{H}(d\psi(E_i)) \\ &\quad + \nabla^{TN^n}_{{}^{V}(\nabla d\psi(E_i,u))}{}^{V}(\nabla d\psi(E_i,u)) - d\Psi\big({}^{H}\big(\frac{1}{2f_1}(2E_i(f_1)E_i - g(E_i,E_i)grad^{M^k} f_1)\big)\big)\Big) \\ &\quad + \sum_{j=1}^{n} \big(\nabla^{TN^n}_{{}^{V}(d\psi(E_j))}{}^{V}(d\psi(E_j))\big) \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{f_1}\sum_{i=1}^{k}\Big({}^{H}\big(\nabla^{N^n}_{d\psi(E_i)}d\psi(E_i)+\frac{1}{2f_2}(2d\psi(E_i)(f_2)d\psi(E_i)-h(d\psi(E_i),d\psi(E_i))grad^{N^n}f_2)\big)\\
&\quad +\frac{1}{f_2}{}^{H}(R^{N^n}(d\psi(u),\nabla d\psi(E_i,u))d\psi(E_i))+{}^{V}(\nabla^{N^n}_{d\psi(E_i)}\nabla d\psi(E_i,u))\Big)\\
&\quad +\frac{k-2}{2f_1^2}d\Psi\big({}^{H}\big(grad^{M^k}f_1)\big)\\
&= {}^{H}\big(\frac{1}{f_1}\tau(\psi)+\frac{k-2}{2f_1^2}d\psi(grad^{M^k}f_1)+\frac{1}{f_1f_2}d\psi(grad^{M^k}(f_2\circ\psi))\\
&\quad -\frac{1}{2f_1f_2}|d\psi|^2grad^{N^n}f_2+\frac{1}{f_1f_2}Tr_g(R^{N^n}(d\psi(u),\nabla d\psi(*,u))d\psi(*))\big)\\
&\quad +{}^{V}\big(\frac{k-2}{2f_1^2}\nabla d\psi(grad^{M^k}f_1,u)+\frac{1}{f_1}Tr_g(\nabla^{\psi}_{*}(\nabla d\psi(*,u)))\big).
\end{aligned}
$$

□

Theorem 6.2. *Given a smooth map between Riemannian manifolds $\psi:(M^k,g)\to(N^n,h)$, then the tangent map $\Psi:(TM^k,G^{f_1})\longrightarrow(TN^n,H^{f_2})$ is harmonic if and only if*

$$
\begin{aligned}
\tau(\psi) &= \frac{2-k}{2f_1}d\psi(grad^{M^k}f_1)-\frac{1}{f_2}d\psi(grad^{M^k}(f_2\circ\psi))+\frac{1}{2f_2}|d\psi|^2grad^{N^n}f_2\\
&\quad -\frac{1}{f_2}Tr_g(R^{N^n}(d\psi(u),\nabla d\psi(*,u))d\psi(*))\big),
\end{aligned}
$$

and

$$
Tr_g\nabla^{\psi}_{*}(\nabla d\psi(u,*))=\frac{2-n}{2f_1}\nabla d\psi(grad^{M^k}f_1,u).
$$

Theorem 6.3. *If $\psi:(M^k,g)\to(N^n,h)$ is totally geodesic, then the tangent map $\Psi:(TM^k,G^{f_1})\longrightarrow(TN^n,H^{f_2})$ is harmonic if and only if*

$$
\tau(\psi)=\frac{2-k}{2f_1}d\psi(grad^{M^k}f_1)-\frac{1}{f_2}d\psi(grad^{M^k}(f_2\circ\psi))+\frac{1}{2f_2}|d\psi|^2grad^{N^n}f_2.
$$

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