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SOME INEQUALITIES ON SUBMANIFOLDS OF A COMPLEX SPACE FORM EQUIPPED WITH COMPLEX SEMI-SYMMETRIC METRIC CONNECTION

BURÇIN DOĞAN *, NERGIZ (ÖNEN) POYRAZ, AND EROL YAŞAR

Abstract. The aim of this study is to introduce geometric inequalities on a complex space form equipped with complex semi-symmetric metric connection (complex s-s.m.c) and to get a formula between intrinsic and extrinsic invariants with the help of these.

Keywords: Submanifold, Chen inequalities, Complex Semi-Symmetric Metric Connection.

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1. Introduction

According to the famous embedding theory of J. F. Nash, any Riemannian manifold can be isometrically immersed in a suitable Euclidean space. Thus, one of the most fundamental problems of Riemannian submanifold theory is to establish relationships between intrinsic and extrinsic invariants. The Riemannian invariants characterizing a Riemannian manifold have been studied by several geometers for a long time. We note that the sectional curvature and the scalar curvature are called *the main intrinsic curvatures* and the squared mean curvature is called *the main extrinsic curvature* of a Riemann manifold. B.Y. Chen introduced some specific submanifolds which have important intrinsic invariants in [4, 6, 7].

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* Corresponding author

Burçin Doğan ⋄ burcin.dogan@ozal.edu.tr ⋄ https://orcid.org/0000-0001-8386-213X

Nergiz (Önen) Poyraz ⋄ nonen@cu.edu.tr ⋄ https://orcid.org/0000-0002-8110-712X

Erol Yaşar ⋄ yerol@mersin.edu.tr ⋄ https://orcid.org/0000-0001-8716-0901.

Let N be a Riemann manifold and $\tau(p)$ is scalar curvature of N. Then $\inf(K)(p)$ is defined as follows

$$\inf(K)(p) = \inf\{K(\Pi)\}$$

where $K(\Pi)$ is a plane section of T_pN. Thus, a new Riemannian invariant δ_N for N was introduced by Chen in [4] as

$$\delta_N = \tau(p) - \inf(K)(p). \tag{1.1}$$

In [4] and [3], Chen established the general optimal inequality and a sharp inequality which named *Chen inequality* for a submanifold N^n of a real space form $R(\widetilde{\overline{c}})$, respectively,

$$\delta_N \leq \frac{n^2(n-2)}{2(n-2)} \|H\|^2 + \frac{1}{2}(n+1)(n-2)\overline{c} \tag{1.2}$$

and

$$\|H\|^2(p) \geq \frac{4}{n^2}\{Ric(U_1) - (n-1)\widetilde{\overline{c}}\}, \quad \forall U_1 \in T_p^1 N^n, \tag{1.3}$$

where $\|H\|^2$ is the squared mean curvature and $Ric(U_1)$ is Ricci curvature of N^n at U_1. Using the above last inequality, many authors established similar inequalities for different kind of submanifolds in ambient manifolds which have different kind of structures [3, 12, 13, 16, 17, 22, 23] and so on *Chen-Ricci inequality* was introduced by Hong and Tripathi in [11]. Later, Chen inequalities for submanifolds of real space forms admitting a semi-symmetric metric conection (s-s.m.c.) was studied by Mihai and Özgür in [14]. On the other hand, Yücesan studied totally real submanifolds of an indefinite Kaehler manifold with a complex s-s.m.c. in [21].

The study is organized as follows:

In section 2, we present preliminaries which will be used throughout this paper. We give some basic information about s-s.m.c. and complex s-s.m.c., respectively. In the last section, we study geometric inequalities for submanifold of complex space forms equipped with a complex s-s.m.c. and present important characterization theorems.

2. Preliminaries

Let $(\widetilde{N}, \widetilde{g})$ be a real $2m-$dimensional semi-Riemannian manifold and J be an almost complex structure such that, for any $U_1, U_2 \in T_p\widetilde{N}$,

$$\widetilde{g}(JU_1, JU_2) = \widetilde{g}(U_1, U_2), \qquad J^2 = -I \tag{2.4}$$

where $T_p\widetilde{N}$ is the tangent space of $\widetilde{N}$ at p.

If complex structure J is parallel according to Levi-Civita connection $\overset{\circ}{\widetilde{\nabla}}$ of $\widetilde{g}$, that is, the following equation is satisfied, then $(\widetilde{N}, \widetilde{g}, J)$ will be called an *indefinite Kaehler manifold*

$$(\overset{\circ}{\widetilde{\nabla}}_{U_1} J)U_2 = 0. \tag{2.5}$$

For a Kaehler manifold, J is integrable and the index of $\widetilde{g}$ is even, say $2v$, $0 \leq v \leq m$. Note that if $v = 0$, then $\widetilde{N}$ is a positive definite Kaehler manifold (i.e., a classical Kaehler manifold). Moreover, the opposite $-\widetilde{g}$ of an indefinite Kaehler metric $\widetilde{g}$ is also Kaehler with index $2(m - v)$, where $2v$ is the index of $\widetilde{g}$. The indefinite Kaehler metric with index $v = 2$ is a complex version of the Lorentzian metric [1].

2.1. **Semi-symmetric metric connections.** Let $\widetilde{N}$ be a real $n-$dimensional semi-Riemannian manifold with a metric tensor $\widetilde{g}$ of index v, $0 \leq v \leq n$, and its Levi-Civita connection $\overset{\circ}{\widetilde{\nabla}}$. A linear connection $\overset{\circ}{\nabla}$ on $\widetilde{N}$ is said to be semi-symmetric if the torsion tensor of the connection $\overset{\circ}{\nabla}$ satisfies

$$\overset{\circ}{T}(U_1, U_2) = \pi(U_2)U_1 - \pi(U_1)U_2, \quad \forall U_1, U_2 \in T_pN, \tag{2.6}$$

where π is a $1-$form. A semi-symmetric connection $\overset{\circ}{\nabla}$ is called a semi-symmetric metric connection. [10] if it further satisfies the equation

$$\overset{\circ}{\nabla}\widetilde{g} = 0. \tag{2.7}$$

A relation between a s-s.m.c. $\overset{\circ}{\nabla}$ and the Levi-Civita connection $\overset{\circ}{\widetilde{\nabla}}$ of $\widetilde{N}$, which has been obtained by Yano [19], is given by

$$\overset{\circ}{\nabla}_{U_1} U_2 = \overset{\circ}{\widetilde{\nabla}}_{U_1} U_2 + \widetilde{\pi}(U_2)U_1 - \widetilde{g}(U_1, U_2)P, \tag{2.8}$$

where P is the tangent vector on $\bar{N}$ associated with the $1-$form $\widetilde{\pi}$ by

$$\widetilde{\pi}(U_1) = \widetilde{g}(U_1, P), \tag{2.9}$$

for any tangent vector U_1.

2.2. **Complex semi-symmetric metric connections.** [21] Let $\widetilde{N}$ be a real $2m-$dimensional indefinite Kaehler manifold. Now, we consider a linear connection $\widetilde{\nabla}$ on $\widetilde{N}$. When

$$\widetilde{\nabla}\widetilde{g} = 0, \quad \widetilde{\nabla}J = 0 \tag{2.10}$$

and the torsion tensor $\widetilde{T}$ is of the form

$$\widetilde{T}(U_1, U_2) = \widetilde{\pi}(U_2)U_1 - \widetilde{\pi}(U_1)U_2 - 2\widetilde{g}(JU_1, U_2)J\widetilde{P}, \tag{2.11}$$

the connection $\widetilde{\nabla}$ is called a complex semi-symmetric metric connection (complex s-s.m.c.), where $\widetilde{\pi}$ is a 1−form and $\widetilde{P}$ is the tangent vector defined by

$$\widetilde{\pi}(U_1) = \widetilde{g}(\widetilde{P}, U_1). \tag{2.12}$$

Let $\widetilde{\nabla}$ and $\overset{\circ}{\widetilde{\nabla}}$ be a complex s-s.m.c. and the Levi-Civita connection defined on $\widetilde{N}$, respectively. Then

$$\begin{aligned} \widetilde{\nabla}_{U_1}U_2 &= \overset{\circ}{\widetilde{\nabla}}_{U_1}U_2 + \widetilde{\pi}(U_2)U_1 - \widetilde{g}(U_1,U_2)\widetilde{P} + \widetilde{\Gamma}(U_2)JU_1 \\ &+ \widetilde{\Gamma}(U_1)JU_2 - \widetilde{g}(JU_1,U_2)J\widetilde{P}, \end{aligned} \tag{2.13}$$

where $\widetilde{\pi}$ and $\widetilde{\Gamma}$ are 1−forms with (2.12) and

$$\widetilde{\Gamma}(U_1) = \widetilde{g}(J\widetilde{P}, U_1), \tag{2.14}$$

for any tangent vector U_1.

Let N be a $n-$dimensional submanifold of a Riemannian manifold $\widetilde{N}$ and $\overset{\circ}{\nabla}$ and ∇ be the Levi-Civita connection and the complex s-s.m.c. on N induced by the Levi-Civita connection $\overset{\circ}{\widetilde{\nabla}}$ and the complex s-s.m.c. $\widetilde{\nabla}$ of $\widetilde{N}$, respectively. Then the *Gauss formulas* with $\overset{\circ}{\nabla}$ and ∇, respectively, are as followings:

$$\overset{\circ}{\widetilde{\nabla}}_{U_1}U_2 = \overset{\circ}{\nabla}_{U_1}U_2 + \overset{\circ}{h}(U_1,U_2) \tag{2.15}$$

and

$$\widetilde{\nabla}_{U_1}U_2 = \nabla_{U_1}U_2 + h(U_1,U_2) \tag{2.16}$$

where $\overset{\circ}{h}$ is the second fundamental form of N in $\widetilde{N}$ and h is a $(0,2)$-tensor on N.

We denote by $\widetilde{R}$ and $\overset{\circ}{\widetilde{R}}$ the Riemannian curvature tensors of an indefinite Kaehler manifold $\widetilde{N}$ with respect to $\widetilde{\nabla}$ and $\overset{\circ}{\widetilde{\nabla}}$, respectively. Also, let R and $\overset{\circ}{R}$ be the Riemannian curvature tensors of a submanifold N of $\widetilde{N}$ with respect to ∇ and $\overset{\circ}{\nabla}$. Then the Gauss equations are with respect to the Levi-Civita connection and the complex s-s.m.c. can be written as

$$\overset{\circ}{\widetilde{R}}(U_1,U_2,V_1,V_2) = \overset{\circ}{R}(U_1,U_2,V_1,V_2) + \widetilde{g}(\overset{\circ}{h}(U_1,V_1),\overset{\circ}{h}(U_2,V_2)) - \widetilde{g}(\overset{\circ}{h}(U_2,V_1),\overset{\circ}{h}(U_1,V_2)) \tag{2.17}$$

and

$$\widetilde{R}(U_1,U_2,V_1,V_2) = R(U_1,U_2,V_1,V_2) + \widetilde{g}(h(U_1,V_1),h(U_2,V_2)) - \widetilde{g}(h(U_2,V_1),h(U_1,V_2)) \tag{2.18}$$

respectively, [21].

Then, by a straightforward computation, we find

$$\begin{aligned}\widetilde{R}(U_1,U_2)V_1 &= \overset{\circ}{\widetilde{R}}(U_1,U_2)V_1 - \widetilde{\alpha}(U_2,V_1)U_1 + \widetilde{\alpha}(U_1,V_1)U_2 \\ &- \widetilde{F}(U_1)\widetilde{g}(U_2,V_1) + \widetilde{F}(U_2)\widetilde{g}(U_1,V_1) - \widetilde{\beta}(U_2,V_1)JU_1 \\ &+ \widetilde{\beta}(U_1,V_1)JU_2 - \widetilde{G}(U_1)\widetilde{g}(JU_2,V_1) + \widetilde{G}(U_2)\widetilde{g}(JU_1,V_1) \\ &+ \widetilde{\gamma}(U_1,U_2)JV_1 - \widetilde{E}(V_1)\widetilde{g}(JU_1,U_2)\end{aligned} \tag{2.19}$$

where

$$\begin{aligned}\widetilde{\alpha}(U_2,V_1) &= (\overset{\circ}{\widetilde{\nabla}}_{U_2}\widetilde{\pi})V_1 - \widetilde{\pi}(U_2)\widetilde{\pi}(V_1) \\ &+ \widetilde{\Gamma}(U_2)\widetilde{\Gamma}(V_1) + \tfrac{1}{2}\widetilde{g}(U_2,V_1)\widetilde{\pi}(\widetilde{P}),\end{aligned} \tag{2.20}$$

$$\begin{aligned}\widetilde{\beta}(U_2,V_1) &= (\overset{\circ}{\widetilde{\nabla}}_{U_2}\widetilde{\Gamma})V_1 - \widetilde{\pi}(U_2)\widetilde{\Gamma}(V_1) \\ &- \widetilde{\Gamma}(U_2)\widetilde{\pi}(V_1) + \tfrac{1}{2}\widetilde{g}(JU_2,V_1)\widetilde{\pi}(\widetilde{P}),\end{aligned} \tag{2.21}$$

$$\overline{\gamma}(U_1,U_2) = (\overset{\circ}{\widetilde{\nabla}}_{U_1}\widetilde{\Gamma})U_2 - (\overset{\circ}{\widetilde{\nabla}}_{U_2}\widetilde{\Gamma})U_1, \tag{2.22}$$

$$\overline{E}(V_1) = 2(\widetilde{\pi}(V_1)J\widetilde{P} - \widetilde{\Gamma}(V_1)\widetilde{P}) \tag{2.23}$$

and

$$\widetilde{g}(\widetilde{F}(U_1),U_2) = \widetilde{\alpha}(U_1,U_2), \quad \widetilde{g}(\widetilde{G}(U_1),U_2) = \widetilde{\beta}(U_1,U_2). \tag{2.24}$$

On the other hand, we have

$$\widetilde{\beta}(U_2,V_1) = -\widetilde{\alpha}(U_2,JV_1), \quad \widetilde{\alpha}(U_2,V_1) = \widetilde{\beta}(U_2,JV_1), \tag{2.25}$$

$$\widetilde{\gamma}(U_1,U_2) = \widetilde{\beta}(U_1,U_2) - \widetilde{\beta}(U_2,U_1) - \widetilde{\pi}(\widetilde{P})\widetilde{g}(JU_1,U_2), \tag{2.26}$$

$$\widetilde{\gamma}(U_1,U_2) = -\widetilde{\gamma}(U_2,U_1), \quad \widetilde{g}(\widetilde{E}(V_1),V_2) = -\,\widetilde{g}(\widetilde{E}(V_2),V_1). \tag{2.27}$$

From now on, in this paper, we assume that $v = 0$, that is, $\widetilde{N}$ is a classical Kaehler manifold.

Let $\{e_1, ..., e_n\}$ be an orthonormal basis of the T_pN^n. Then, following equation can be written for the mean curvature vector

$$H(p) = \frac{1}{n}\sum_{l=1}^{n} h(e_l,e_l). \tag{2.28}$$

We note that, N is totally geodesic if $h = 0$; minimal if $H = 0$ and totally umbilical if $h(U_1,U_2) = g(U_1,U_2)H$, $\forall\ U_1,U_2 \in TN$.

If we consider a 2−dimensional non-degenerate plane $\Pi = Span\{e_l, e_s\}$, then we can calculate the sectional curvature of the section Π at $p \in N$ by

$$K_{ls} = \frac{g(R(e_s, e_l)e_l, e_s)}{g(e_l, e_l)g(e_s, e_s) - g(e_l, e_s)^2}. \tag{2.29}$$

We denote by $K(\pi)$ the sectional curvature of N^n. For $\{e_1, ..., e_n\}$ orthonormal basis and a k-plane section L of TpN^n, the scalar curvature τ at p and the Ricci curvature (or k-Ricci curvature) of L at U_1 is respectively defined by

$$\tau(p) = \sum_{1 \le l < s \le n} K_{ls}, \tag{2.30}$$

$$Ric_L(U_1) = K_{12} + K_{13} + ... + K_{1k} \tag{2.31}$$

where $\pi \subset TpN^n$ is a plane section and U_1 be a unit vector in L. We note that for $\{e_1, ..., e_k\}$ is an orthonormal basis of L such that $e_1 = U_1$, K_{ls} is spanned by e_l, e_s [3].

The Riemannian invariant θ_k is defined as:

$$\theta_k(p) = \frac{1}{k-1} \inf_{L, U_1} Ric_L(U_1), \quad p \in N \tag{2.32}$$

where k is a integer such that $2 \le k \le n$, L runs over all s-plane sections in TpN^n and U_1 runs over all unit vectors in L.

Let $\widetilde{N}$ be a real $2n$−dimensional Kaehler manifold and J almost complex structure. The sectional curvature of $\widetilde{N}$ in the direction of an invariant 2-plane section by J is called the holomorphic sectional curvature. If the holomorphic sectional curvature is constant $4\overline{c}$ for all plane sections π of $T_p\widetilde{N}$ invariant by J for any $p \in \widetilde{N}$, then $\widetilde{N}$ is called a complex space form and is denoted by $\widetilde{N}(4\overline{c})$. The curvature tensor $\overset{\circ}{\widetilde{R}}$ with respect to $\overset{\circ}{\widetilde{\nabla}}$ on $\widetilde{N}(4\overline{c})$ is calculated by

$$\begin{aligned} \overset{\circ}{\widetilde{R}}(U_1, U_2, V_1, V_2) &= \overline{c}\{ \widetilde{g}(U_1, V_2)\, \widetilde{g}(U_2, V_1) - \widetilde{g}(U_2, V_2)\, \widetilde{g}(U_1, V_1) \\ &+ \widetilde{g}(JU_1, V_2)\, \widetilde{g}(JU_2, V_1) - \widetilde{g}(JU_1, V_1)\, \widetilde{g}(JU_2, V_2) \\ &- 2\, \widetilde{g}(U_1, JU_2)\, \widetilde{g}(V_1, JV_2)\}. \end{aligned} \tag{2.33}$$

3. k-Ricci Curvature and k-Scalar Curvature

Let $\widetilde{N}$ be a Kaehler manifold endowed with a complex s-s.m.c.. Then from (2.19) and (2.33) we get

$$\begin{aligned}\widetilde{R}(e_l,e_s,e_s,e_l) &= \overline{c}(1+\widetilde{g}^2(Je_l,e_s))-\alpha(e_l,e_l)-\alpha(e_s,e_s) \\ &- (2\widetilde{\gamma}(e_l,e_s)+\ \widetilde{g}(\widetilde{E}(e_s),e_l))\widetilde{g}(Je_l,e_s)-\pi(P)\widetilde{g}^2(Je_l,e_s).\end{aligned} \tag{3.34}$$

From (3.34) we derive

$$\begin{aligned}\widetilde{R}(e_l,e_s,e_s,e_l) &= \overline{c}(1+g^2(Je_l,e_s))-\alpha(e_l,e_l)-\alpha(e_s,e_s) \\ &\quad -\widetilde{\pi}(p)\widetilde{g}^2(Je_l,e_s)-N_{ls}.\end{aligned} \tag{3.35}$$

Thus, taking $U_1=V_2=e_l$ and $U_2=V_1=e_s$ and using (3.35), we have

$$\begin{aligned}R(e_l,e_s,e_s,e_l) &= \overline{c}(1+g^2(Je_l,e_s))-\alpha(e_l,e_l)-\alpha(e_s,e_s)-\pi(p)\widetilde{g}^2(Je_l,e_s) \\ &- N_{ls}+\sum_{r=n+1}^{2n+2}h^r(e_l,e_l)h^r(e_s,e_s)-h^r(e_l,e_s)h^r(e_s,e_l).\end{aligned} \tag{3.36}$$

Then, we find

$$\begin{aligned}2\tau(p) &= n(n-1)\overline{c}+\overline{c}\left\|T\right\|^2-2(n-1)\lambda-\pi(p)\left\|T\right\|^2 \\ &- \sum_{l,s=1}^{m}m_{ij}+\sum_{l,s=1}^{n}\sum_{r=m+1}^{2m}h^r_{ll}h^r_{ss}-(h^r_{ls})^2.\end{aligned} \tag{3.37}$$

If we write JU_1 with its components as $JU_1=TU_1+FU_1$, then we get

$$\left\|T\right\|^2=\sum_{l,s=1}^{n}\widetilde{g}^2(Je_l,e_s).$$

Thus, (3.37) can be written by

$$\begin{aligned}2\tau(p) &= n(n-1)\overline{c}-2(n-1)\lambda-\pi(p)\left\|T\right\|^2+n^2\left\|H\right\|^2 \\ &- \overline{c}\left\|T\right\|^2-\sum_{l,s=1}^{m}m_{ij}-\sum_{l,s=1}^{n}\sum_{r=m+1}^{2m}(h^r_{ls})^2.\end{aligned} \tag{3.38}$$

Theorem 3.1. *Let N^n be a real $n-$dimensional submanifold of a real $2m-$ dimensional Kaehler manifold of constant holomorphic sectional curvature is constant $4\overline{c}$ endowed with complex s-s.m.c.. Then, the followings are true.*

(i) For each unit vector $U_1\in T^1_p(N)$, we have

$$\begin{aligned}Ric(U_1) &\leq (n-1)\overline{c}+\overline{c}\sum_{s=2}^{n}g^2(JU_1,e_s)-\lambda-\tfrac{1}{2}\sum_{l=1}^{n}N_{ll}-\sum_{s=2}^{n}N_{1s} \\ &- \tfrac{1}{2}\pi(p)\sum_{s=2}^{n}g^2(JU_1,e_s)+\tfrac{1}{4}n^2\left\|H\right\|^2-(n-2)\alpha(U_1,U_1).\end{aligned} \tag{3.39}$$

(ii) The equality case of (3.39) is satisfied by unit $U_1\in T^1_p(N)$, if and only if (iff)

$$\begin{aligned}&h(U_1,U_2)=0,\ \ \text{for all } U_2\in T^1_p(N) \text{ orthogonal to } U_1, \\ &h(U_1,U_1)=\tfrac{n}{2}H(p).\end{aligned} \tag{3.40}$$

(iii) For $\forall U_1 \in T_p^1(N)$, equality of (3.39) is satisfied iff either p is a totally geodesic point or p is a totally umbilical point and $n = 2$.

Proof. Let $\{e_1, e_2, ..., e_n\}$and $\{e_{n+1}, ..., e_{2m}\}$ be orthonormal basis of $T_{U_1}N$ and $T_{U_1}^{\perp}N$ at $U_1 \in N$, respectively, where $e_n + 1$ is parallel to the mean curvature vector H. Then, from (3.38) we have

$$\sum_{l,s=1}^{n}\sum_{r=n+1}^{2m}(h_{ls}^r)^2 = n(n-1)\bar{c} + \bar{c}\,\|T\|^2 - 2(n-1)\lambda - \pi(p)\,\|T\|^2 - \sum_{l,s=1}^{n} N_{ls} + n^2\,\|H\|^2 - 2\tau(p). \tag{3.41}$$

From (3.41) we get

$$\frac{1}{4}n^2\,\|H\|^2 = \tau(p) - \frac{1}{2}(n(n-1) + \|T\|^2)\bar{c} + (n-1)\lambda + \frac{1}{2}\sum_{l,s=1}^{n} N_{ls} + \frac{1}{2}\pi(p)\,\|T\|^2 + \frac{1}{4}\sum_{r=n+1}^{2m}(h_{11}^r - h_{22}^r - ... - h_{nn}^r)^2 + \sum_{r=n+1}^{2m}\sum_{s=2}^{n}(h_{1s}^r)^2 - \sum_{r=n+1}^{2m}\sum_{2\le l<s\le n}(h_{ll}^r h_{ss}^r - (h_{ls}^r)^2). \tag{3.42}$$

Then using (3.36), we have

$$\sum_{r=n+1}^{2m}(h_{ll}^r h_{ss}^r - (h_{ls}^r)^2) = \sum_{2\le l<s\le n} K_{ls} - \frac{(n-1)(n-2)}{2}\bar{c} - \bar{c}\sum_{2\le l<s\le n} g^2(Je_l, e_s) + (n-2)(\lambda - \alpha(e_1, e_1)) + \sum_{2\le l<s\le n} m_{ls} + \pi(p)\sum_{2\le l<s\le n} g^2(Je_l, e_s). \tag{3.43}$$

From (3.42) and (3.43) we derive

$$Ric(e_1) = (n-1)\bar{c} + \bar{c}\sum_{s=2}^{n} g^2(Je_1, e_s) - \lambda - \frac{1}{2}\sum_{l}^{n} m_{ll} - \sum_{s=2}^{n} N_{1s} - \frac{1}{2}\pi(p)\sum_{s=2}^{n} g^2(Je_1, e_s) + \frac{1}{4}n^2\,\|H\|^2 - \frac{1}{4}\sum_{r=n+1}^{2m}(h_{11}^r - h_{22}^r - ... - h_{nn}^r)^2 - \sum_{r=n+1}^{2n+2}\sum_{s=2}^{n}(h_{1s}^r)^2 - (n-2)\alpha(e_1, e_1). \tag{3.44}$$

By choosing $e_1 = U_1$ in equation (3.44), (3.39) is obtained.

When the equality case of (3.39), the followings are satisfied:

$$h_{12}^r = h_{13}^r = ... = h_{1n}^r = 0 \text{ and } h_{11}^r = h_{22}^r + ... + h_{nn}^r \tag{3.45}$$

where $r \in \{n+1, ..., 2n+2\}$. Thus, (3.40) is holded.

Let inequality (3.39) satisfy case of equality for $\forall U_1 \in T_pN^n$. Then, from (3.45), $\forall r \in \{n+1, ..., 2n+2\}$, we get $i \in \{1, ..., n\}$,

$$h_{ls}^r = 0, \quad l \neq s, \tag{3.46}$$

$$2h_{ll}^r = h_{11}^r + h_{22}^r + ... + h_{nn}^r. \tag{3.47}$$

Using (3.47), we derive

$$(n-2)(h_{11}^r + h_{22}^r + ... + h_{nn}^r) = 0.$$

It is clear that, there are two situations for the last equality. For $h_{11}^r + h_{22}^r + ... + h_{nn}^r = 0$, if we consider (3.47) and (3.46) together, then, we can write $h_{ls}^r = 0$ for all $l, s \in \{1, ..., n\}$ and $r \in \{n+1, ..., 2n+2\}$ which gives that p is a totally geodesic point. On the other hand, if $n = 2$, then from 3.47, $2h_{11}^r = 2h_{22}^r = h_{11}^r + h_{22}^r$, which completes the proof. The converse is clear. □

Theorem 3.2. *Let N^n be a real $n-$dimensional submanifold of a real $2m-$ dimensional Kaehler manifold of constant holomorphic sectional curvature is constant $4\overline{c}$ endowed with complex s-s.m.c.Then, we get*

$$\begin{aligned} \tau(p) &\leq \frac{1}{2}(n(n-1) + \|T\|^2)\overline{c} - (n-1)\lambda \\ &- \frac{1}{2}\pi(p)\|T\|^2 - \frac{1}{2}\sum_{l,s=1}^{n} m_{ij} + \frac{1}{2}n^2\|H\|^2. \end{aligned} \tag{3.48}$$

Equality case of 3.48 *holds iff* N *is totally geodesic.*

Theorem 3.3. *Let $\widetilde{N}(\overline{c})$ be an $m-$dimensional real space form of constant holomorphic sectional curvature $4\overline{c}$ equipped with complex s-s.m.c. $\widetilde{\overline{\nabla}}$ and N^n be $n-$dimensional Einstein submanifold of $\widetilde{N}(\overline{c})$. Then,*

$$\begin{aligned} \tau(p) &\leq \frac{n(n-1)}{2}(\overline{c} + \|H\|^2) + \frac{\overline{c}}{2}\|T\|^2 - (n-1)\lambda \\ &- (n-1)\lambda - \frac{1}{2}\pi(p)\|T\|^2 - \frac{1}{2}\sum_{l,s=1}^{n} m_{ij} \end{aligned} \tag{3.49}$$

is satisfied and the equality case of (3.49) holds at $p \in N^n$ iff p is a totally umbilical point.

Proof. The relation (3.38) at $p \in N^n$ is equivalent with

$$\begin{aligned} n^2 \|H\|^2 &= 2\tau(p) - n(n-1)\bar{c} - \bar{c}\|T\|^2) + 2(n-1)\lambda \\ &+ \pi(p)\|T\|^2 + \sum_{l,s=1}^{n} m_{ij} + \sum_{r=n+2}^{2m}\sum_{l,s=1}^{n}(h_{ls})^2 \\ &+ \sum_{l=1}^{n}(h_{ll}^{n+1})^2 + \sum_{l\neq s}(h_{ls}^{n+1})^2. \end{aligned} \tag{3.50}$$

For a choosen orthonormal basis, let $\{e_1, e_2, ..., e_n\}$ diagonalize the shape operator $A_{e_{n+1}}$. Then the shape operators take the forms

$$A_{e_{n+1}} = \begin{bmatrix} a_1 & 0 & . & . & . & 0 \\ 0 & a_2 & . & . & . & 0 \\ . & . & . & & & . \\ . & . & & . & & . \\ . & . & & & . & . \\ 0 & 0 & . & . & . & a_n \end{bmatrix}, \tag{3.51}$$

$$A_{e_r} = (h_{ls}^r), \quad l,s = 1,...,n; \quad r = n+2,...,n+p, \quad trace A_{e_r} = 0. \tag{3.52}$$

From (3.50), we get

$$\begin{aligned} n^2 \|H\|^2 &= 2\tau(p) - n(n-1)\bar{c} - \bar{c}\|T\|^2) + 2(n-1)\lambda \\ &+ \pi(p)\|T\|^2 + \sum_{l,s=1}^{n} m_{ij} + \sum_{l=1}^{n}(a_l^2) + \sum_{r=n+2}^{2m}(h_{ls}^r)^2. \end{aligned} \tag{3.53}$$

On the other hand, since

$$0 \leq \sum_{l<s}(a_l - a_s)^2 = (n-1)\sum_{l} a_l^2 - 2\sum_{l<s} a_l a_s \tag{3.54}$$

we obtain

$$n^2 \|H\|^2 = \left(\sum_{l=1}^{n} a_l\right)^2 = \sum_{l=1}^{n} a_l^2 + 2\sum_{l<s} a_l a_s \leq n\sum_{l=1}^{n} a_l^2 \tag{3.55}$$

which implies

$$\sum_{l=1}^{n} a_l^2 \geq n\|H\|^2. \tag{3.56}$$

So from (3.53) and (3.56), we have

$$\begin{aligned} n^2 \|H\|^2 &\geq 2\tau(p) - n(n-1)\bar{c} - \bar{c}\|T\|^2 + 2(n-1)\lambda + \pi(p)\|T\|^2 \\ &+ \sum_{l,s=1}^{n} m_{ls} + n\|H\|^2 + \sum_{r=n+2}^{2n+2}\sum_{l,s=1}^{n}(h_{ls}^r)^2. \end{aligned} \tag{3.57}$$

If (3.49) is case of equality, using (3.54) and (3.57) we obtain

$$a_1 = a_2 = ... = a_n \text{ and } A_{e_r} = 0, \ \ r = n+2, ..., m. \tag{3.58}$$

which gives p is a totally umbilical point. The converse is obvious. □

Theorem 3.4. *Let $\widetilde{N}(\overline{c})$ be $2m-$dimensional real space form of constant holomorphic sectional curvature $4\overline{c}$ equipped with complex s-s.m.c. $\widetilde{\overline{\nabla}}$ and N^n be $n-$dimensional submanifold of $\widetilde{N}(\overline{c})$. Then we have*

$$\begin{aligned} \theta_k(p) &\leq \overline{c} + \|H\|^2 + \frac{\overline{c}}{n(n-1)}\|T\|^2 - \frac{2}{n}\lambda \\ &- \frac{\lambda}{n(n-1)}\pi(p)\|T\|^2 - \frac{1}{n(n-1)}\sum_{l,s=1}^{n} m_{ls}. \end{aligned} \tag{3.59}$$

Lemma 3.1. *If $n > k \geq 2$ and $a_1, ..., a_n, a$ are real numbers such that*

$$\left(\sum_{l=1}^{n} a_l\right)^2 = (n-k+1)\left(\sum_{l=1}^{n} a_l^2 + a\right) \tag{3.60}$$

then

$$2\sum_{1\leq l<s\leq k} a_l a_s \geq a \tag{3.61}$$

with equality holding iff

$$a_1 + a_2 + ... + a_k = a_{k+1} = ... = a_n. \tag{3.62}$$

Theorem 3.5. *N^n be $n-$dimensional submanifold of an $2m-$dimensional real space form $\widetilde{N}(\overline{c})$ of constant holomorphic sectional curvature $4\overline{c}$ endowed with complex s-s.m.c. $\widetilde{\overline{\nabla}}$. Then, for each k-plane section ($n > k \geq 2$) and $p \in N^n$, we obtain*

$$\begin{aligned} \tau(p) - \tau(\pi_k) &\leq \frac{1}{2}(n+k-1)(n-k)\overline{c} + \frac{1}{2}\overline{c}\sum_{l,s=k+1}^{n} g^2(Je_l, e_s) \\ &- (n-k)\lambda - (k-1)trace(N_{|_{\pi_k^{\perp}}}) \\ &- \frac{1}{2}\left(\pi(p)\sum_{l,s=k+1}^{n} g^2(Je_l, e_s) + \sum_{l,s=k+1}^{n} m_{ls}\right) \\ &+ \frac{n^2(n-k)}{2(n-k+1)}\|H\|^2. \end{aligned} \tag{3.63}$$

(3.63) is the equation of equality at $p \in N^n$ iff there exist $\{e_1, ..., e_n\}$ and $\{e_{n+1}, ..., e_{2m}\}$ orthonormal basis of TpN^n and $T_p^{\perp}N^n$, respectively, such that (a) Π_k=Span$\{e_1, ..., e_k\}$ and

(b) the shape operators A_{e_r}, take the forms

$$A_{e_{n+1}} = \begin{bmatrix} h_{11}^{n+1} & 0 & \cdots & 0 & \\ 0 & h_{22}^{n+1} & \cdots & 0 & \\ \vdots & \vdots & \ddots & & 0 \\ \vdots & \vdots & & \ddots & \\ 0 & 0 & \cdots & h_{kk}^{n+1} & \\ & & 0 & & \left(\sum_{l=1}^{k} h_{ll}^{n+1}\right) I_{n-k} \end{bmatrix}, \tag{3.64}$$

$$A_{e_r} = \left(h_{ls}^r\right), \quad l,s = 1,...,n; \quad r = n+2,...,2n+2, \quad traceA_{e_r} = 0. \tag{3.65}$$

Proof. Let Π_k is a $k-$plane section and we choose orthonormal basis $\{e_1,...,e_n\}$ and $\{e_{n+1},..., e_{2m}\}$ of TpN^n and $T_p^{\perp}N^n$ at p, respectively, such that $\Pi_k = Span\{e_1,...,e_k\}$. If we consider that the mean curvature vector H is in the direction of the normal vector to e_{n+1} and $e_1,...,e_n$ diagonalize the shape operator $A_{e_{n+1}}$, then the shape operators take the forms (3.51) and (3.52). So, we can rewrite (3.38) as

$$\left(\sum_{l=1}^{n} h_{ll}^{n+1}\right)^2 = (n-k+1)\left(\sum_{l=1}^{n}\left(h_{ll}^{n+1}\right)^2 + \sum_{l\neq s}\left(h_{ls}^{n+1}\right) + \sum_{r=n+2}^{2n+2}\sum_{l,s=1}^{n}(h_{ls}^r)^2 + \epsilon\right) \tag{3.66}$$

where

$$\begin{aligned} \epsilon &= 2\tau(p) - n(n-1)\overline{c} - \overline{c}\left\|T\right\|^2 + 2(n-1)\lambda \\ &+ \pi(p)\left\|T\right\|^2 + \sum_{l,s=1}^{n} m_{ls} - \frac{n^2(n-k)}{(n-k+1)}\left\|H\right\|^2. \end{aligned} \tag{3.67}$$

Applying Lemma 1 in (3.66), we get

$$2\sum_{1\leq l<s\leq k} h_{ll}^{n+1}h_{ss}^{n+1} \geq \sum_{l\neq s}\left(h_{ls}^{n+1}\right)^2 + \sum_{r=n+2}^{2m}\sum_{l,s=1}^{n}(h_{ls}^r)^2 + \epsilon. \tag{3.68}$$

From equation (3.36) it also follows that

$$\begin{aligned} 2\tau(\pi_k) &= k(k-1)\overline{c} + \overline{c}\sum_{l,s=1}^{k} g^2(Je_l,e_s) - 2(k-1)\sum_{l=1}^{k}\alpha(e_l,e_l) \\ &- \pi(p)\sum_{l,s=1}^{k} g^2(Je_l,e_s) - \sum_{l,s=1}^{k} m_{ls} + \sum_{l,s=1}^{k}\sum_{r=n+2}^{2n+2}\left(h_{ll}^r h_{ss}^r - (h_{ls}^r)^2\right) \\ &+ \sum_{l=1}^{k}(h_{ll}^{n+1})^2 + 2\sum_{1\leq l<s\leq k} h_{ll}^{n+1}h_{ss}^{n+1} - \sum_{l,s=1}^{k}(h_{ls}^{n+1})^2. \end{aligned} \tag{3.69}$$

Using (3.68) and (3.69)

$$\begin{aligned} 2\tau(\pi_k) \geq\ & k(k-1)\overline{c} + \overline{c}\sum_{l,s=1}^{k} g^2(Je_l, e_s) - \sum_{l,s=1}^{k} m_{ls} \\ & -2(k-1)\sum_{l=1}^{k}\alpha(e_l, e_l) - \pi(p)\sum_{l,s=1}^{k} g^2(Je_l, e_s) \\ & + \sum_{l,s=1}^{k}\sum_{r=n+2}^{2m}\left(h_{ll}^r h_{ss}^r - (h_{ls}^r)^2\right) + \sum_{l=1}^{k}(h_{ll}^{n+1})^2 \\ & + \sum_{l\neq s}^{n}(h_{ls}^{n+1})^2 + \sum_{r=n+2}^{2m}\sum_{l,s=1}^{n}(h_{ls}^r)^2 + \epsilon - \sum_{l,s=1}^{k}(h_{ls}^{n+1})^2 \end{aligned} \tag{3.70}$$

is obtained. From this, we can write that

$$\begin{aligned} 2\tau(\pi_k) \geq\ & k(k-1)\overline{c} + \overline{c}\sum_{l,s=1}^{k} g^2(Je_l, e_s) - 2(k-1)\sum_{l=1}^{k}\alpha(e_l, e_l) \\ - \ & \pi(p)\sum_{l,s=1}^{k} g^2(Je_l, e_s) - \sum_{l,s=1}^{k} m_{ls} \\ & + \sum_{r=n+2}^{2m}(h_{11}^r + h_{22}^r + ... + ...h_{kk}^r)^2 + \sum_{r=n+2}^{2m}\sum_{l,s>k}(h_{ls}^r)^2 \\ & + \sum_{r=n+2}^{2m}\sum_{s>k}\left((h_{1s}^r)^2 + (h_{2s}^r)^2 + ... + (h_{ks}^r)^2\right) + \epsilon, \end{aligned} \tag{3.71}$$

or

$$\begin{aligned} \tau(\pi_k) \geq\ & \frac{k(k-1)}{2}\overline{c} + \frac{\overline{c}}{2}\sum_{l,s=1}^{k} g^2(Je_l, e_s) \\ & -(k-1)\sum_{l=1}^{k}\alpha(e_l, e_l) \\ - \ & \frac{\pi(p)}{2}\sum_{l,s=1}^{k} g^2(Je_l, e_s) - \frac{1}{2}\sum_{l,s=1}^{k} m_{ls} + \frac{1}{2}\epsilon. \end{aligned} \tag{3.72}$$

We remark that

$$\alpha(e_1, e_1) + \alpha(e_2, e_2) + ... + \alpha(e_k, e_k) = \lambda - trace(\lambda_{|_{\pi_k^{\perp}}}). \tag{3.73}$$

From (3.67), (3.71) and (3.72), we get

$$\begin{aligned} 2\tau(\pi_k) &\geq -(n+k-1)(n+k)\overline{c} - \overline{c}\sum_{l,s=k+1}^{n} g^2(Je_l, e_s) + 2(n-k)\lambda \\ &+ (k-1)trace(N_{|_{\pi_k^{\perp}}}) - \pi(p)\sum_{l,s=k+1}^{n} g^2(Je_l, e_s) - \sum_{l,s=1}^{k} m_{ls} \\ &+ 2\tau(p) + \sum_{l,s=1}^{n} m_{ls} - \frac{n^2(n-k)}{(n-k+1)}\|H\|^2 \end{aligned}$$

which completes the proof. □

By Theorem 5 we obtain the following corollary.

Corollary 3.1. *Let N^n be $n-$dimensional submanifold of an $2m-$dimensional real space form $\widetilde{N}(\overline{c})$ of constant holomorphic sectional curvature $4\overline{c}$ endowed with complex s-s.m.c. $\widetilde{\overline{\nabla}}$. Then, for each k-plane section and $p \in N^n$, we get*

$$\begin{aligned} \delta_N &\leq \frac{1}{2}(n+1)(n-2)\overline{c} + \frac{1}{2}\overline{c}\sum_{l,s=3}^{n} g^2(Je_l, e_s) \\ &- (n-2)\lambda - trace(N_{|_{\pi_k^{\perp}}}) \\ &- \frac{1}{2}\left(\pi(p)\sum_{l,s=3}^{n} g^2(Je_l, e_s) + \sum_{l,s=3}^{n} m_{ls}\right) \\ &+ \frac{n^2(n-2)}{2(n+1)}\|H\|^2. \end{aligned}$$

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References

[1] Barros, M. and Romero, A. (1982). Indefinite Kaehler Manifolds, Math. Ann., 261, 55 − 62.

[2] Chen, B. Y. (1996). Mean curvature and shape operator of isometric immersion in real space forms, Glasgow Mathematic Journal, 38, 87 − 97.

[3] Chen, B. Y. (1999). Relation between Ricci curvature and shape operator for submanifolds with arbitrary codimension, Glasgow Mathematic Journal, 41, 33 − 41.

[4] Chen, B. Y. (1993). Some pinching and classification theorems for minimal submanifolds, Arch. math., (Basel), 60(6), 568 − 578.

[5] Chen, B. Y. (1994). A Riemannian invariant for submanifolds in space forms and its applications, Geometry and Topology of submanifolds VI, (Leuven, 1993/$Brussels$, 193), (NJ:Word Scientific Publishing, River Edge), no.6, $568 - 578$.

[6] Chen, B. Y. (1998). Strings of Riemannian invariants, inequalities, ideal immersions and their applications, The Third Pacific Rim Geometry Conference, Geom. Topol. Monogr., vol. 25. Seoul, (1996), Int. Press, Cambridge, MA (1998), pp. $7 - 60$.

[7] Chen, B. Y. (2000). Riemannian submanifolds, Handbook of Differential Geometry, vol. I, pp. $187-418$.

[8] Chen, B. Y. (2005). A general optimal inequlaity for a Riemannian submanifold, J. Ineq. Pure Appl. Math., 6(3), $1 - 11$.

[9] Chen, B. Y. (2011). Pseudo-Riemannian Geometry, δ-Invariants and Applications, World Scientific Publishing Co. Pte. Ltd., Hackensack, p. $xxxii + 477$.

[10] Duggal, K. L. and Sharma, R. (1986). Semi-symmetric metric connections in a Semi-Riemannian Manifold, Indian J. Pure Appl. Math., 17(11), $1276 - 1282$.

[11] Hong, S. and Tripathi, M. M. (2005). On Ricci curvature of submanifolds, Int J. Pure Appl. Math. Sci., $2(2005), 227 - 245$.

[12] Liu, X. and Zhou, J. (2006). On Ricci curvature of certain submanifolds in cosympletic space form, Sarajeva J. Math., $2(2006), 95 - 106$.

[13] Matsumoto, K., Mihai, I. and Oiaga, A. (2000). Shape operator for slant submanifolds in complex space forms, Bull. Yamagata Univ. Natur. Sci., 14(4), $169 - 177$.

[14] Mihai, A. and Özgür, C. (2010). Chen inequalities for submanifolds of real space form with a semi-symmetric metric connection, Taiwanese Journal of Mathematics, 14(4), pp. $1465 - 1477$.

[15] O'Neill, B. (1983). Semi-Riemannian Geometry with Applications to Relativity, Academic Press, London, 1983.

[16] Oiaga, A. and Mihai, I. (1999). B. Y. Chen inequalities for slant submanifolds in complex space forms, Demonstratio Math., 32(4), $835 - 846$.

[17] Özgür, C. and Murathan, C. (2010). Chen Inequalities for Submanifolds of a Locally Conformal Almost Cosymplectic Manifold with a semi-symmetric metric connection, An. Şt. Univ. Ovidius Constanta, 18(1), $239 - 254$.

[18] Tripathi, M. M. (2011). Chen-Ricci inequality for curvature like tensor and its applications, Diff. geom. Appl., 29(5), $685 - 692$.

[19] Yano, K. (1970). On semi-symmetric metric connection, Rev. Roum. Math. Pures Et Appl., 15, $1579 - 1586$.

[20] Yano, K. and Imai, T. (1975). On Semi-Symmetric Metric F-Connection, Tensor, N. S., 29, $134 - 138$.

[21] Yücesan, A. (2011). Totally real submanifolds of an indefinite Kaehler manifold with a complex semi-symmetric metric connection, The Arabian Journal for Science and Engineering, $1A(33)$, $114 - 122$.

[22] Zhang, P., Zhang, L. and Song, W. (2014). Chen's Inequalities for Submanifolds of a Riemannian Manifold of Quasi-Constant Curvature with a semi-symmetric metric connection, Taiwanese Journal Of Mathematics, Vol. 18(6), $1841 - 1862$.

[23] Zhang, P., Zhang, L. and Song, W. (2014). Chen's Inequalities for Totally Real Submanifolds in Complex Space Forms with a semi-symmetric metric connection, Journal of Mathematical Research with Applications, 34(5), $587-596$.

Malatya Turgut Özal University, Department of Engineering Basic Sciences, Malatya, Türkiye

Çukurova University, Department of Mathematics, Adana, Türkiye

Mersin University, Department of Mathematics, Mersin, Türkiye