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EXPLORING THE RECIPROCAL FUNCTIONAL EQUATIONS: APPROXIMATIONS IN DIVERSE SPACES

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Abstract. In this study, we explore the generalized Hyers-Ulam-Rassias stability of a specific reciprocal-type functional equation. The equation is given by

$$\Omega(2u+v)+\Omega(2u-v)=\frac{2\Omega(u)\Omega(v)\displaystyle\sum_{\substack{k=0\\ k \text{ is even}}}^{l} 2^{l-k}\binom{l}{k}\Omega(u)^{\frac{k}{l}}\Omega(v)^{\frac{l-k}{l}}}{\left(4\Omega(v)^{\frac{2}{l}}-\Omega(u)^{\frac{2}{l}}\right)^{l}}$$

and we consider its behavior in both non-zero real and non-Archimedean spaces. Additionally, an appropriate counter-example is provided to demonstrate the failure of the stability result in the singular case.

Keywords: Reciprocal functional equation, non-Archimedean space, non-zero real space, approximations, Cauchy sequence, functional inequality, generalized Hyers-Ulam stability, convergence.

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1. Introduction

The exploration of the stability of functional equations began with Ulam's [20] famous question at a Mathematical Colloquium held at the University of Wisconsin in 1940. In the following year, Hyers [9] presented a partial solution to Ulam's question. Subsequently, Th.M. Rassias [11], Aoki [1], J.M. Rassias [12], and Găvruţa [8] expanded and generalized Hyers's findings to include the Cauchy functional equation in various adaptations.

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In 2010, Ravi and Senthil Kumar [13] studied the stability of the reciprocal type functional equation

$$f(x+y)=\frac{f(x)f(y)}{f(x)+f(y)},$$

where $f: X \to \mathbb{R}$ is a mapping with X as the space of non-zero real numbers.

In 2014, Kim and Bodaghi [2] introduced and studied the stability of the quadratic reciprocal functional equation

$$f(2x+y)+f(2x-y)=\frac{2f(x)f(y)[4f(y)+f(x)]}{(4f(y)-f(x))^2}.$$

In 2017, Kim et al. [10] introduced and studied the stability of the reciprocal-cubic functional equation

$$c(2x+y)+c(x+2y)=\frac{9c(x)c(y)[c(x)+c(y)+2c(x)^{\frac{1}{3}}c(y)^{\frac{1}{3}}(c(x)^{\frac{1}{3}}+c(y)^{\frac{1}{3}})]}{[2c(x)^{\frac{2}{3}}+2c(y)^{\frac{2}{3}}+5c(x)^{\frac{1}{3}}c(y)^{\frac{1}{3}}]^3}$$

and the reciprocal-quartic functional equation

$$q(2x+y)+q(2x-y)=\frac{2q(x)q(y)[q(x)+16q(y)+24\sqrt{q(x)q(y)}]}{[4\sqrt{q(y)}-\sqrt{q(x)}]^4}$$

in non-Archimedean fields.

In the same year, Bodaghi and Senthil Kumar [4] introduced and obtained the stability of the following reciprocal-quintic functional equation

$$q(2x+y)+q(2x-y)=\frac{4q(x)q(y)[16q(y)+40q(x)^{\frac{2}{5}}q(y)^{\frac{3}{5}}+5q(x)^{\frac{4}{5}}q(y)^{\frac{1}{5}}]}{[4q(y)^{\frac{2}{5}}-q(x)^{\frac{2}{5}}]^5}$$

and reciprocal-sextic functional equation

$$s(2x+y)+s(2x-y)=\frac{2s(x)s(y)[s(x)+60s(x)^{\frac{2}{3}}s(y)^{\frac{1}{3}}+240s(x)^{\frac{1}{3}}+64s(y)]}{[4s(y)^{\frac{1}{3}}-s(x)^{\frac{1}{3}}]^6}.$$

In 2020, Bodaghi et al [6] considered the following reciprocal-nonic functional equation

$$n(2x+y)+n(2x-y)=\frac{4n(x)n(y)}{(4n(y)^{\frac{2}{9}}-n(x)^{\frac{2}{9}})^9}\Big[256n(y)+2304n(x)^{\frac{2}{9}}n(y)^{\frac{7}{9}}+2016n(x)^{\frac{4}{9}}n(y)^{\frac{5}{9}} +336n(x)^{\frac{6}{9}}n(y)^{\frac{3}{9}}+n(x)^{\frac{8}{9}}n(y)^{\frac{1}{9}}\Big]$$

and the reciprocal-decic functional equation

$$d(2x+y)+d(2x-y)=\frac{2d(x)d(y)}{(4d(y)^{\frac{1}{5}}-d(x)^{\frac{1}{5}})^{10}}\Big[1024d(y)+11520d(x)^{\frac{1}{5}}d(y)^{\frac{4}{5}}+13440d(x)^{\frac{2}{5}}d(y)^{\frac{3}{5}} +3360d(x)^{\frac{3}{5}}d(y)^{\frac{2}{5}}+180d(x)^{\frac{4}{5}}d(y)^{\frac{1}{5}}+d(x)\Big]$$

and obtained various stability results in non-Archimedean fields and some proper examples for their non-stabilities.

The other results pertaining to the stability of different reciprocal-type functional equations can be found, for instance, in [5, 19, 14, 15, 3, 16, 17, 18].

In this study, we introduce the following $l-$power reciprocal functional equation

$$\Omega(2u+v)+\Omega(2u-v)=\frac{2\Omega(u)\Omega(v)\displaystyle\sum_{\substack{k=0\\ k \text{ is even}}}^{l} 2^{l-k}\binom{l}{k}\Omega(u)^{\frac{k}{l}}\Omega(v)^{\frac{l-k}{l}}}{\left(4\Omega(v)^{\frac{2}{l}}-\Omega(u)^{\frac{2}{l}}\right)^{l}}, \tag{1.1}$$

then, we examine the general solution and its various stability results in non-zero real numbers and in non-Archimedean fields with a proper example for their non-stability.

2. General solution of (1.1)

This section provides the solution for the functional equation (1.1). Assume R^* denotes the set of non-zero real numbers.

We begin with the following lemma.

Lemma 2.1. *Let $a \in \mathbb{N}^*$. Then, we have*

$$\frac{(-1)^l(a-2)^l+(a+2)^l}{2(a)^l}=\sum_{\substack{k=0\\ k \text{ is even}}}^{l}\left(\frac{2}{a}\right)^{l-k}\binom{l}{k}. \tag{2.2}$$

Proof. Let us prove it by mathematical induction. First, for $l=0$, we get

$$\frac{(-1)^0(a-2)^0+(a+2)^0}{2a^0}=\frac{1+1}{2}=\sum_{\substack{k=0\\ k \text{ even}}}^{0}\left(\frac{2}{a}\right)^{0-k}\binom{0}{k}=\binom{0}{0}=1.$$

The statement is true for $l=0$. Next, we assume that for $l=n$, it is true, i.e.

$$\frac{(-1)^n(a-2)^n+(a+2)^n}{2a^n}=\sum_{\substack{k=0\\ k \text{ even}}}^{n}\left(\frac{2}{a}\right)^{n-k}\binom{n}{k}. \tag{2.3}$$

We must now prove that the formula holds for $l=n+1$, i.e.

$$\frac{(-1)^{n+1}(a-2)^{n+1}+(a+2)^{n+1}}{2a^{n+1}}=\sum_{\substack{k=0\\ k \text{ even}}}^{n+1}\left(\frac{2}{a}\right)^{n+1-k}\binom{n+1}{k}. \tag{2.4}$$

To do this, we use the binomial theorem to obtain

$$\begin{aligned}\frac{(-1)^{n+1}(a-2)^{n+1}+(a+2)^{n+1}}{2a^{n+1}}&=\frac{(-1)^{n+1}}{2a^{n+1}}\sum_{k=0}^{n+1}\binom{n+1}{k}a^k(-1)^{n+1-k}2^{n+1-k}\\&\quad+\frac{1}{2a^{n+1}}\sum_{k=0}^{n+1}\binom{n+1}{k}a^k2^{n+1-k}.\end{aligned} \tag{2.5}$$

Next, by simplifications, we get

$$\frac{(-1)^{n+1}(a-2)^{n+1}+(a+2)^{n+1}}{2a^{n+1}}=\sum_{k=0}^{n+1}\binom{n+1}{k}\frac{a^k2^{n+1-k}}{2a^{n+1}}((-1)^{-k}+1). \tag{2.6}$$

Finally, since k is even, we obtain

$$\frac{(-1)^{n+1}(a-2)^{n+1}+(a+2)^{n+1}}{2a^{n+1}}=\sum_{k=0}^{n+1}\binom{n+1}{k}\left(\frac{2}{a}\right)^{n+1-k}. \tag{2.7}$$

The statement is true when $l=n+1$. Hence, by the principle of mathematical induction, the statement is true for all $l\geq 0$. □

Theorem 2.1. *Let $f:\mathbb{R}^*\to\mathbb{R}$ be a continuous function fulfilling the equation (1.1). Assuming $\Omega(u)\neq 0$ and $4\Omega(v)^{\frac{2}{l}}-\Omega(u)^{\frac{2}{l}}\neq 0$ for all $u,v\in\mathbb{R}^*$. Then f takes the form*

$$\Omega(u)=\frac{c}{u^l},\ \textit{for all } u\in\mathbb{R}^*,$$

where $c\neq 0$.

Proof. Assuming $f:\mathbb{R}^*\to\mathbb{R}$ satisfies the functional equation (1.1). Substituting (u,v) by (u,u) in (1.1), yields

$$\Omega(3u)+\Omega(u)=\frac{2\Omega(u)\displaystyle\sum_{\substack{k=0\\ k\text{ is even}}}^{l}2^{l-k}\binom{l}{k}}{3^l}.$$

Setting $a=1$ in (2.2), gives

$$\sum_{\substack{k=0\\ k\text{ is even}}}^{l}2^{l-k}\binom{l}{k}=\frac{3^l+1}{2}. \tag{2.8}$$

Hence

$$\Omega(3u)=\frac{1}{3^l}\Omega(u),\quad\text{for all } u\in\mathbb{R}^*. \tag{2.9}$$

By induction, we prove that for all $k\in\mathbb{N}^*$,

$$\Omega(ku)=\frac{1}{k^l}\Omega(u). \tag{2.10}$$

Assuming this is true for $k\in\{1,2,\ldots,n-1\}$ we prove it for $k=n$. To do this, replacing (u,v) with $(u,(n-2)u)$ in (1.1), we get

$$\Omega(nu)+\Omega(-(n-4)u)=\frac{2\Omega(u)\Omega((n-2)u)\displaystyle\sum_{\substack{k=0\\ k\text{ is even}}}^{l}2^{l-k}\binom{l}{k}\Omega(u)^{\frac{k}{l}}\Omega((n-2)u)^{\frac{l-k}{l}}}{\left(4\Omega((n-2)u)^{\frac{2}{l}}-\Omega(u)^{\frac{2}{l}}\right)^l}. \tag{2.11}$$

Using the recurrence hypothesis:

$$\Omega(nu) + \frac{1}{(n-4)^l}\Omega(-u) = \frac{2\frac{1}{(n-2)^l}\Omega(u) \displaystyle\sum_{\substack{k=0 \\ k \text{ is even}}}^{l} \frac{2^{l-k}}{(n-2)^{l-k}}\binom{l}{k}}{\left(4\frac{1}{(n-2)^2} - 1\right)^l}$$

then,

$$\Omega(nu) + \frac{1}{(n-4)^l}\Omega(-u) = \frac{2(n-2)^l\Omega(u) \displaystyle\sum_{\substack{k=0 \\ k \text{ is even}}}^{l} \frac{2^{l-k}}{(n-2)^{l-k}}\binom{l}{k}}{(-n(n-4))^l}.$$

By taking $a = n-2$ in (2.2), we get

$$(-1)^l n^l (n-4)^l \Omega(nu) + (-1)^l n^l \Omega(-u) = \left((-1)^l (n-4)^l + n^l\right)\Omega(u). \tag{2.12}$$

Now, replacing (u, v) by $(u, (n-3)u)$ in (1.1),

$$\Omega((n-1)u) + \Omega(-(n-5)u) = \frac{2\frac{1}{(n-3)^l}\Omega(u) \displaystyle\sum_{\substack{k=0 \\ k \text{ is even}}}^{l} 2^{l-k}\binom{l}{k}\frac{1}{(n-3)^{l-k}}}{\left(4\frac{1}{(n-3)^2} - 1\right)^l}. \tag{2.13}$$

Using the recurrence hypothesis and by taking $a = n-3$ in (2.2), a simple calculation gives

$$(-1)^l (n-5)^l \Omega(u) + (-1)^l (n-1)^l \Omega(-u) = \left((-1)^l (n-5)^l + (n-1)^l\right)\Omega(u).$$

This implies that

$$(-1)^l \Omega(-u) = \Omega(u). \tag{2.14}$$

By using (2.14) in (2.12), we get

$$(-1)^l n^l (n-4)^l \Omega(nu) + n^l \Omega(u) = \left((-1)^l (n-4)^l + n^l\right)\Omega(u).$$

Then,

$$\Omega(nu) = \frac{1}{n^l}\Omega(u).$$

Thus, the formula (2.10) is true for $k = n$. Therefore Ω is of the form $\frac{c}{u^l}$. □

3. Stability of (1.1) in $\mathbb{R}^*$

For convenience, we introduce the operator $\Lambda : \mathbb{R}^* \to \mathbb{R}$ as

$$\Lambda(u,v) = \Omega(2u+v) + \Omega(2u-v) - \frac{2\Omega(u)\Omega(v) \sum_{\substack{k=0 \\ k \text{ is even}}}^{l} 2^{l-k} \binom{l}{k} \Omega(u)^{\frac{k}{l}} \Omega(v)^{\frac{l-k}{l}}}{\left(4\Omega(v)^{\frac{2}{l}} - \Omega(u)^{\frac{2}{l}}\right)^{l}},$$

for all $u, v \in \mathbb{R}^*$. We are now ready to present our first main result, as follows.

Theorem 3.1. *Let $Q : \mathbb{R}^* \times \mathbb{R}^* \to \mathbb{R}$ be a function fulfilling*

$$\sum_{s=0}^{\infty} \frac{1}{3^{ls}} Q\left(\frac{u}{3^{s+1}}, \frac{v}{3^{s+1}}\right) < \infty \tag{3.15}$$

for all $u, v \in \mathbb{R}^*$. If $\Omega : \mathbb{R}^* \to \mathbb{R}$ fulfilling

$$|\Lambda(u,v)| \leq Q(u,v) \tag{3.16}$$

for all $u, v \in \mathbb{R}^*$, then there is a uniquely defined reciprocal function $G : \mathbb{R}^* \to \mathbb{R}$ that fulfilling (1.1) and the inequality

$$|\Omega(u) - G(u)| \leq \sum_{s=0}^{\infty} \frac{1}{3^{ls}} Q\left(\frac{u}{3^{s+1}}, \frac{u}{3^{s+1}}\right), \text{ for all } u \in \mathbb{R}^*. \tag{3.17}$$

Proof. We substitute (u, v) by (u, u) in (3.16) and using (2.8) we get

$$\left|\Omega(3u) - \frac{\Omega(u)}{3^l}\right| \leq Q(u,u) \tag{3.18}$$

for all $u \in \mathbb{R}^*$. Substituting u by $\frac{u}{3}$ in (3.18), we obtain

$$\left|\Omega(u) - \frac{1}{3^l}\Omega\left(\frac{u}{3}\right)\right| \leq Q\left(\frac{u}{3}, \frac{u}{3}\right) \tag{3.19}$$

for all $u \in \mathbb{R}^*$. Now, by setting $u = \frac{u}{3}$ in (3.19), dividing by 3^l, and then adding the resulting inequality to (3.19), we obtain

$$\left|\Omega(u) - \frac{1}{3^{2l}}\Omega\left(\frac{u}{3^2}\right)\right| \leq Q\left(\frac{u}{3}, \frac{u}{3}\right) + \frac{1}{3^l} Q\left(\frac{u}{3^2}, \frac{u}{3^2}\right), \text{ for all } u \in \mathbb{R}^*. \tag{3.20}$$

Similarly, by continuing this process and applying induction on a positive integer m, we obtain

$$\left|\Omega(u) - \frac{1}{3^{ml}}\Omega\left(\frac{u}{3^m}\right)\right| \leq \sum_{s=0}^{m-1} \frac{1}{3^{ls}} Q\left(\frac{u}{3^{s+1}}, \frac{u}{3^{s+1}}\right), \text{ for all } u \in \mathbb{R}^*. \tag{3.21}$$

Thereafter, if we choose any integers m' and m such that $m' > m > 0$, we obtain

$$\begin{aligned}\left|\frac{1}{3^{lm'}}\Omega\left(\frac{u}{3^{m'}}\right) - \frac{1}{3^{lm}}\Omega\left(\frac{u}{3^{m}}\right)\right| &= \left|\frac{1}{3^{lm'}}\Omega\left(\frac{u}{3^{m'}}\right) - \frac{1}{3^{l(m'-1)}}\Omega\left(\frac{u}{3^{l(m'-1)}}\right) + \cdots\right.\\ &\quad \left. + \frac{1}{3^{l(m'-1)}}\Omega\left(\frac{u}{3^{l(m'-1)}}\right) - \cdots + \frac{1}{3^{l(m+1)}}\Omega\left(\frac{u}{3^{m+1}}\right) - \frac{1}{3^{lm}}\Omega\left(\frac{u}{3^{m}}\right)\right|\\ &\le \frac{1}{3^{l(m'-1)}}Q\left(\frac{u}{3^{m'}}, \frac{u}{3^{m'}}\right) + \cdots + \frac{1}{3^{lm}}Q\left(\frac{u}{3^{m+1}}, \frac{u}{3^{m+1}}\right)\\ &\le \sum_{j=m}^{m'-1}\frac{1}{3^{lj}}Q\left(\frac{u}{3^{j+1}}, \frac{u}{3^{j+1}}\right) \end{aligned} \tag{3.22}$$

for all $u \in \mathbb{R}^*$. Letting $m' \to \infty$ in (3.22) and we use (3.15), the sequence $\{\frac{1}{3^{lm}}\Omega(\frac{u}{3^m})\}$ is Cauchy for each $u \in \mathbb{R}^*$. We know that $\mathbb{R}$ is Banach, we can introduce $G : \mathbb{R}^* \to \mathbb{R}$ by $g(u) = \lim_{m\to\infty} \frac{1}{3^{lm}}\Omega\left(\frac{u}{3^m}\right)$. To prove that g fulfilling (1.1), substituting (u, v) by $(3^{-m}u, 3^{-m}v)$ in (3.16) and dividing by 3^{lm}, we arrive

$$|3^{-lm}\Lambda(3^{-m}u,\ 3^{-m}v)| \le 3^{-lm}Q(3^{-m}u,\ 3^{-m}v), \forall u, v \in \mathbb{R}^* \text{ and } m \in \mathbb{N}^*. \tag{3.23}$$

Taking $m \to \infty$ in (3.23) and by (3.15), we find that G fulfilling (1.1) for all $u, v \in \mathbb{R}^*$. One more, setting $m \to \infty$ in (3.21), we arrive at (3.17). Now, we need to demonstrate that G is unique. Suppose $G' : \mathbb{R}^* \to \mathbb{R}$ is another reciprocal mapping that also fulfilling (1.1) and (3.17). Clearly, we have $G'(3^{-m}u) = 3^{lm}G'(u), G(3^{-m}u) = 3^{lm}G(u)$ and utilizing (3.17), we obtain

$$\begin{aligned}|G'(u) - G(u)| &= 3^{-lm}\left|G'(3^{-m}u) - G(3^{-m}u)\right| \\ &\le 3^{-lm}\left(\left|G'(3^{-m}u) - \Omega(3^{-m}u)\right| + \left|\Omega(3^{-m}u) - G(3^{-m}u)\right|\right)\\ &\le 2\sum_{j=0}^{\infty}\frac{1}{3^{l(m+j)}}Q\left(\frac{u}{3^{m+j+1}},\ \frac{u}{3^{m+j+1}}\right)\\ &\le 2\sum_{j=m}^{\infty}\frac{1}{3^{lj}}Q\left(\frac{u}{3^{j+1}}, \frac{u}{3^{j+1}}\right)\end{aligned} \tag{3.24}$$

for all $u \in \mathbb{R}^*$. Letting $m \to \infty$ in (3.24), we obtain the unicity of G. □

The following corollaries are immediate consequences of Theorem 3.1.

Corollary 3.1. *Let $\Omega : \mathbb{R}^* \to \mathbb{R}$ be a mapping for which there exists $\epsilon > 0$ such that*

$$|\Lambda(u, v)| \le \epsilon$$

holds for all $u, v \in \mathbb{R}^*$. Then,

$$G(u) = \lim_{m\to\infty}\frac{1}{3^{lm}}\Omega\left(\frac{u}{3^m}\right)$$

for all $u \in \mathbb{R}^*, m \in \mathbb{N}$ *and* $G : \mathbb{R}^* \to \mathbb{R}$ *is the unique mapping satisfying the reciprocal functional equation (1.1) such that*

$$|\Omega(u) - G(u)| \leq \frac{3^l}{3^l - 1}\epsilon$$

for every $u \in \mathbb{R}^*$.

Proof. By taking $Q(u,v) = \epsilon$ in Theorem 3.1 we arrive at the desired result. □

Corollary 3.2. *Let* $\epsilon > 0$ *and* $\alpha \neq -l$ *be real numbers, and* $\Omega : \mathbb{R}^* \to \mathbb{R}$ *be a mapping satisfying the functional inequality*

$$|\Lambda(u,v)| \leq \epsilon(|u|^\alpha + |v|^\alpha)$$

for all $u, v \in \mathbb{R}^*$. *Then, there exists a unique reciprocal mapping* $G : \mathbb{R}^* \to \mathbb{R}$ *satisfying the functional equation (1.1) and*

$$|\Omega(u) - G(u)| \leq \frac{2.3^l\epsilon}{3^{\alpha+l} - 1}|u|^\alpha$$

for all $u \in \mathbb{R}^*$.

Proof. By letting $Q(u,v) = \epsilon(|u|^\alpha + |v|^\alpha)$ for all $u, v \in \mathbb{R}^*$ in Theorem 3.1 we get the desired result. □

Corollary 3.3. *Let* $\epsilon > 0$ *and* $\alpha \neq -l$ *be real numbers, and* $\Omega : \mathbb{R}^* \to \mathbb{R}$ *be a mapping satisfying*

$$|\Lambda(u,v)| \leq \epsilon(|u|^{\frac{\alpha}{2}}|v|^{\frac{\alpha}{2}} + |u|^\alpha + |v|^\alpha)$$

for all $u, v \in \mathbb{R}^*$. *Then, there exists a unique reciprocal mapping* $G : \mathbb{R}^* \to \mathbb{R}$ *satisfying the functional equation (1.1) and*

$$|\Omega(u) - G(u)| \leq \frac{\epsilon 3^{l+1}}{3^{\alpha+l} - 1}|u|^\alpha$$

for all $u \in \mathbb{R}^*$.

Proof. By taking $Q(u,v) = \epsilon(|u|^{\frac{\alpha}{2}}|v|^{\frac{\alpha}{2}} + |u|^\alpha + |v|^\alpha)$ for all $u, v \in \mathbb{R}^*$ in Theorem 3.1 we get the desired result. □

Corollary 3.4. *Let* $\Omega : \mathbb{R}^* \to \mathbb{R}$ *be a mapping and there exist* p, q *with* $p + q \neq -l$. *If there exists* $\epsilon \geq 0$ *such that*

$$|\Lambda(u,v)| \leq \epsilon|u|^p|v|^q$$

for all $u, v \in \mathbb{R}^*$, then there exists a unique reciprocal mapping $G : \mathbb{R}^* \to \mathbb{R}$ satisfying the functional equation (1.1) and

$$|\Omega(u) - G(u)| \leq \frac{3^l \epsilon}{3^{p+q+l} - 1} |u|^{p+q}$$

for all $u \in \mathbb{R}^*$.

Proof. Letting $Q(u, v) = \epsilon |u|^p |v|^q$ for all $u, v \in \mathbb{R}^*$ in Theorem 3.1, we obtain the required result. □

4. Stability of (1.1) in non-Archimedean field

In this section, A and B denote a non-Archimedean field and a complete non-Archimedean field, respectively. For any non-Archimedean field A, let $A^* = A \setminus \{0\}$. Familiarity with non-Archimedean fields' properties is assumed.

The second main result can be stated as follows.

Theorem 4.1. *Let $\Upsilon : \mathbb{A}^* \times \mathbb{A}^* \to [0, \infty[$ be a mapping such that*

$$\lim_{m\to\infty} \left|\frac{1}{3^l}\right|^m \Upsilon\left(\frac{u}{3^{m+1}}, \frac{v}{3^{m+1}}\right) = 0, \quad \text{for all } u, v \in \mathbb{A}^*. \tag{4.25}$$

Assuming that $g : \mathbb{A}^* \to \mathbb{B}$ is a mapping fulfilling the following

$$|\Lambda(u, v)| \leq \Upsilon(u, v), \quad \text{for all } u, v \in \mathbb{A}^*. \tag{4.26}$$

Then, there is a uniquely defined reciprocal function $g : \mathbb{A}^* \to \mathbb{B}$ such that

$$|\Omega(u) - g(u)| \leq \max\left\{\left|\frac{1}{3^l}\right|^{k+1} \Upsilon\left(\frac{u}{3^{k+1}}, \frac{u}{3^{k+1}}\right) : k \in \mathbb{N} \cup \{0\}\right\}, \text{ for all } u \in \mathbb{A}^*. \tag{4.27}$$

Proof. Changing (u, v) to (u, u) in (4.26), one finds

$$\left|\Omega(u) - \frac{1}{3^l}\Omega\left(\frac{u}{3}\right)\right| \leq |3^l| \Upsilon\left(\frac{u}{3}, \frac{u}{3}\right) \tag{4.28}$$

for all $u \in \mathbb{A}^*$. Now, considering u as $\frac{u}{3^m}$ in (4.28) and multiplying by $\left|\frac{1}{3^l}\right|^m$, we get

$$\left|\frac{1}{3^{lm}}\Omega\left(\frac{u}{3^m}\right) - \frac{1}{3^{l(m+1)}}\Omega\left(\frac{u}{3^{(m+1)}}\right)\right| \leq \left|\frac{1}{3^l}\right|^m \Upsilon\left(\frac{u}{3^{m+1}}, \frac{u}{3^{m+1}}\right) \tag{4.29}$$

for all $u \in \mathbb{A}^*$. It is easy to obtain from the inequalities (4.25) and (4.29) that the sequence $\left\{\frac{1}{3^{lm}}\Omega\left(\frac{u}{3^{lm}}\right)\right\}$ is Cauchy and converges to a well defined function g since $\mathbb{B}$ is complete. Then, put $g : \mathbb{A}^* \to \mathbb{B}$ as

$$g(u) = \lim_{m\to\infty} \frac{1}{3^{lm}}\Omega\left(\frac{u}{3^m}\right). \tag{4.30}$$

Furthermore, for every element $u \in \mathbb{A}^*$ and each nonnegative integers m, we have the following

$$\begin{aligned}\left|\frac{1}{3^{lm}}\Omega\left(\frac{u}{3^m}\right) - g(u)\right| &= \left|\sum_{k=0}^{m-1}\left[\frac{1}{3^{l(k+1)}}\Omega\left(\frac{u}{3^{(k+1)}}\right) - \frac{1}{3^{lm}}\Omega\left(\frac{u}{3^m}\right)\right]\right| \\ &\leq \max\left\{\left|\frac{1}{3^{l(k+1)}}\right|\Omega\left(\frac{u}{3^{(k+1)}}\right) - \frac{1}{3^{lm}}\Omega\left(\frac{u}{3^m}\right)\right| : 0 \leq k < m\} \qquad (4.31) \\ &\leq \max\left\{\left|\frac{1}{3^l}\right|^m \Upsilon\left(\frac{u}{3^{m+1}}, \frac{u}{3^{m+1}}\right) : 0 \leq k < m\right\}.\end{aligned}$$

As $m \to \infty$ in the inequality (4.31) and by using (4.30), we observe that the inequality (4.27) is valid. By applying inequalities (4.25), (4.26), and (4.30), for all $u, v \in \mathbb{A}^*$, we arrive at the following

$$\begin{aligned}|\Lambda(u,v)| &= \lim_{m\to\infty}\left|\frac{1}{3^l}\right|^m \left|\Lambda\left(\frac{u}{3^m}, \frac{v}{3^m}\right)\right| \\ &\leq \lim_{m\to\infty}\left|\frac{1}{3^l}\right|^m \Upsilon\left(\frac{u}{3^m}, \frac{v}{3^m}\right) \\ &= 0.\end{aligned}$$

Therefore, the mapping g fulfills (4.25), making it a reciprocal mapping. To establish the uniqueness of g, suppose that $g' : \mathbb{A}^* \to \mathbb{B}$ is another reciprocal mapping that also fulfills (4.27). Then

$$\begin{aligned}|g(u) - g'(u)| &= \lim_{n\to\infty}\left|\frac{1}{3^l}\right|^n \left|g\left(\frac{u}{3^n}\right) - g'\left(\frac{u}{3^n}\right)\right| \\ &\leq \lim_{n\to\infty}\left|\frac{1}{3^l}\right|^n \max\left\{\left|g\left(\frac{u}{3^n}\right) - \Omega\left(\frac{u}{3^n}\right)\right|, \left|\Omega\left(\frac{u}{3^n}\right) - g'\left(\frac{u}{3^n}\right)\right|\right\} \\ &\leq \lim_{n\to\infty}\lim_{m\to\infty}\max\left\{\max\left\{\left|\frac{1}{3^l}\right|^{k+n} \Upsilon\left(\frac{u}{3^{k+n+1}}, \frac{u}{3^{k+n+1}}\right) \quad : n \leq k \leq m+n\right\}\right\} \\ &= 0\end{aligned}$$

for all $u \in \mathbb{A}^*$. This shows that g is the only such mapping, thereby concluding the proof. □

As a direct consequence of Theorem 4.1, we have the following corollaries.

Corollary 4.1. *Let $\mu > 0$ be a constant. If $\Omega : \mathbb{A}^* \to \mathbb{B}$ satisfies*

$$|\Lambda(u,v)| \leq \mu$$

for all $u, v \in \mathbb{A}^*$, then there exists a unique reciprocal mapping $g : \mathbb{A}^* \to \mathbb{B}$ satisfying (1.1) and

$$|\Omega(u) - g(u)| \leq \mu$$

for all $u \in \mathbb{A}^*$.

Proof. Taking $\Upsilon(u,v) = \mu$ in Theorem 4.1, we get the required result. □

Corollary 4.2. *Let $\mu \geq 0$ and $a \neq -l$, be fixed constants. If $\Omega : \mathbb{A}^* \to \mathbb{B}$ satisfies*

$$|\Lambda(u,v)| \leq \mu(|u|^a + |v|^a)$$

for all $u, v \in \mathbb{A}^*$, then there exists a unique reciprocal mapping $g : \mathbb{A}^* \to \mathbb{B}$ satisfying (1.1) and

$$|\Omega(u) - g(u)| \leq \begin{cases} \frac{|2|\mu}{|3|^a}|u|^a, & a > -l, \\ |2|\mu|3|^l|u|^a, & a < -l, \end{cases}$$

for all $u \in \mathbb{A}^*$.

Proof. Considering $\Upsilon(u,v) = \mu(|u|^a + |v|^a)$ in Theorem 4.1, we obtain the desired result. □

Corollary 4.3. *Let $\Omega : \mathbb{A}^* \to \mathbb{B}$ be a mapping and let there exist real numbers $p, q, a = p + q \neq -l$ and $\mu \geq 0$ such that*

$$|\Lambda(u,v)| \leq \mu|u|^p|v|^q$$

for all $u, v \in \mathbb{A}^*$. Then, there exists a unique reciprocal mapping $g : \mathbb{A}^* \to \mathbb{B}$ satisfying (1.1) and

$$|\Omega(u) - g(u)| \leq \begin{cases} \frac{\mu}{|3|^a}|u|^a, & a > -l, \\ \mu|3|^l||u|^a, & a < -l, \end{cases}$$

for all $u \in \mathbb{A}^*$.

Proof. Letting $\Upsilon(u,v) = \mu|u|^p|v|^q$, for all $u, v \in \mathbb{A}^*$ in Theorem 4.1, we acquire the requisite result. □

Corollary 4.4. *Let $\mu \geq 0$ and $a \neq -l$ be real numbers, and $\Omega : \mathbb{A}^* \to \mathbb{B}$ be a mapping satisfying the functional inequality*

$$|\Lambda(u,v)| \leq \mu(|u|^{\frac{a}{2}}|v|^{\frac{a}{2}} + |u|^a + |v|^a)$$

for all $u, v \in \mathbb{A}^*$ Then, there exists a unique reciprocal mapping $g : \mathbb{A}^* \to \mathbb{B}$ satisfying (1.1) and

$$|\Omega(u) - g(u)| \leq \begin{cases} \frac{|3|\mu}{|3|^a}|u|^a, & a > -l, \\ |3|\mu|3|^l|u|^a, & a < -l, \end{cases}$$

for all $u \in \mathbb{A}^*$.

Proof. Letting $\Upsilon(u,v) = \mu(|u|^{\frac{a}{2}}|v|^{\frac{a}{2}} + |u|^a + |v|^a)$ in Theorem 4.1, the result follows directly.

□

5. Counter-Examples

In this section, using the well-known counter-example provided by Gajda [7], we demonstrate that the equation (1.1) is not applicable for $\alpha = -l$ in Corollary 3.2, within the context of non-zero real numbers. Let's define the function

$$\zeta(u) = \begin{cases} \frac{c}{u^l}, & \text{for } u \in (1,\infty) \\ c, & \text{elsewhere} \end{cases} \tag{5.32}$$

where $\zeta : \mathbb{R}^* \to \mathbb{R}$. Let $\Omega : \mathbb{R}^* \to \mathbb{R}$ be a function defined as

$$\Omega(u) = \sum_{m=0}^{\infty} 3^{-lm}\zeta(3^{-m}u) \tag{5.33}$$

for all $u \in \mathbb{R}$. Assume the mapping $\Omega : \mathbb{R}^* \to \mathbb{R}$ defined in (5.33) fulfills the following inequality

$$|\Lambda(u,v)| \leq c\frac{3^{l+1}+1}{3^l-1}\left(|u|^{-l} + |v|^{-l}\right) \tag{5.34}$$

for all $u, v \in \mathbb{R}^*$. We prove that there do not exist a reciprocal mapping $G : \mathbb{R}^* \to \mathbb{R}$ and a constant $\delta > 0$ such that

$$|\Omega(u) - G(u)| \leq \delta\, |u|^{-l} \tag{5.35}$$

for all $u \in \mathbb{R}^*$. Initially, we show that Ω fullfils (5.34). Using (5.32), we have

$$|\Omega(u)| = \left|\sum_{m=0}^{\infty} 3^{-lm}\zeta(3^{-m}u)\right| \leq \sum_{m=0}^{\infty} \frac{c}{3^{lm}} = \frac{3^l}{3^l-1}c.$$

We can see that Ω is bounded by $\frac{c3^l}{3^l-1}$ on $\mathbb{R}$. If $|u|^{-l} + |v|^{-l} \geq 1$, then the left hand side of (5.34) is less than $\frac{c(3^{l+1}+1)}{3^l-1}$. Now, assume that $0 < |u|^{-l} + |v|^{-l} < 1$. Therefore, there exists a positive integer m such that

$$\frac{1}{3^{l(m+1)}} \leq |u|^{-l} + |v|^{-l} < \frac{1}{3^{lm}}. \tag{5.36}$$

Thus, the inequality (5.36) yields $3^{lm}\left(|u|^{-l} + |v|^{-l}\right) < 1$, or equivalently: $3^{lm}u^{-l} < 1$, $3^{lm}v^{-l} < 1$. So,

$$\frac{u^l}{3^{lm}} > 1, \quad \frac{v^l}{3^{lm}} > 1.$$

Hence, the last inequalities imply $\frac{u^l}{3^{l(m-1)}} > 3^l > 1$, $\frac{v^l}{3^{l(m-1)}} > 3^l > 1$ and thus we find

$$\frac{1}{3^{m-1}}(u) > 1, \quad \frac{1}{3^{m-1}}(v) > 1, \quad \frac{1}{3^{m-1}}(2u+v) > 1, \quad \frac{1}{3^{m-1}}(2u-v) > 1.$$

Hence, for every value of $m = 0, 1, 2, \ldots, n-1$, we get

$$\frac{1}{3^n}(u) > 1, \quad \frac{1}{3^n}(v) > 1, \quad \frac{1}{3^n}(2u+v) > 1, \quad \frac{1}{3^n}(2u-v) > 1,$$

and $\Delta(3^{-n}u, 3^{-n}v) = 0$ for $m = 0, 1, 2, \ldots, n-1$. Applying (5.32) and the definition of Ω, we get

$$|\Delta(u,v)| \le \sum_{m=n}^{\infty} \frac{c}{3^{lm}} + \sum_{m=n}^{\infty} \frac{c}{3^{lm}} + \frac{3^l+1}{3^l} \sum_{m=n}^{\infty} \frac{c}{3^{lm}}$$
$$\le c\frac{3^{l+1}+1}{3^l} \cdot \frac{1}{3^{lm}} \left(1 - \frac{1}{3^l}\right)^{-1}$$
$$\le c\left(\frac{3^{l+1}+1}{3^l-1}\right) \cdot \frac{1}{3^{l(m+1)}}$$
$$\le c\left(\frac{3^{l+1}+1}{3^l-1}\right)\left(|u|^{-l} + |v|^{-l}\right)$$

for all $u, v \in \mathbb{R}^*$. This means that the inequality (5.34) holds. We claim that the $l-$power reciprocal functional equation (1.1) is not stable for $\alpha = -l$ in Corollary 3.2. Suppose that there exists a reciprocal mapping $\Omega : \mathbb{R}^* \longrightarrow \mathbb{R}$ satisfying (5.35). So, we have

$$|\Omega(u)| \le (\delta + 1)|u|^{-l}. \tag{5.37}$$

Furthermore, a positive integer m can be choosen with the condition $m\,c > \delta + 1$. If $u \in (1, 3^{m-1})$, then $3^{-n}u \in (1, \infty)$ for all $m = 0, 1, 2, \ldots, n-1$ and therefore

$$|\Omega(u)| = \sum_{m=0}^{\infty} \frac{\zeta(3^{-m}u)}{3^{lm}} \ge \sum_{m=0}^{n-1} \frac{3^{lm}c}{u^l \cdot 3^{lm}} = \frac{m\,c}{u^l} > (\delta+1)u^{-l}$$

which contradicts (5.37). Thus, the $l-$power functional equation (1.1) is not stable for $\alpha = -l$ in Corollary 3.2.

6. Conclusion

In this paper, we have successfully explored the generalized Hyers-Ulam-Rassias stability of a reciprocal-type functional equation, focusing on its behavior in non-zero real and non-Archimedean spaces with suitable counter-examples.

Through detailed analysis, we derived a general solution for the functional equation in the real number space and established the conditions for stability using various inequality techniques. Furthermore, our study extends these findings to non-Archimedean fields, highlighting the unique characteristics and behaviors of solutions in such spaces.

Further research could explore additional types of functional equations and their stability across various mathematical fields, enhancing the framework established in this study.

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