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STABILITY OF A MICROTEMPERATURES DAMPED POROUS-ELASTIC SYSTEM WITH NONLINEAR DISSIPATION AND NONLINEAR DISTRIBUTED DELAY

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Abstract. In this paper, we consider a one-dimensional porous-elastic system with a nonlinear dissipation and a nonlinear distributed delay subjected to microtemperatures effects. We establish an energy decay rate by using a perturbed energy method and some properties of convex functions, but regardless of the wave speeds of the system. Our result is new and extends some previous results to nonlinearity case.

Keywords: Decay, micro-temperatures, nonlinear distributed delay, nonlinear damping, Porous elasticity.

2010 Mathematics Subject Classification: 35B40, 93D05, 93D15, 74F05.

1. Introduction

In this present work, we aim to study the following nonlinear damped porous elastic system having a nonlinear distributed delay

$$\begin{cases} \rho u_{tt} - \mu u_{xx} - b\varphi_x + \gamma h_1(u_t) + \int_{\tau_1}^{\tau_2} \mu(s) h_2(u_t(x,t-s))\, ds = 0, \text{ in } (0,1)\times\mathbb{R}_+, \\ J\varphi_{tt} - \delta\varphi_{xx} + k_1\omega_x + bu_x + \xi\varphi = 0, \text{ in } (0,1)\times\mathbb{R}_+, \\ \tau w_t + k_2 w + k_1\varphi_{tx} - k_3\omega_{xx} = 0, \text{ in } (0,1)\times\mathbb{R}_+. \end{cases} \tag{1.1}$$

subjected to microtemperature effects and nonlinear damping, with the mixed boundary conditions

$$u(0,t) = \varphi(1,t) = w(0,t) = u_x(1,t) = \varphi_x(0,t) = w_x(1,t) = 0. \tag{1.2}$$

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The evolution equations for one-dimensional theories of porous materials with temperature and microtemperature is given by

$$\begin{cases} \rho u_{tt} = T_x, \quad J\phi_{tt} = H_x + G, \\ \rho\eta_t = q_x, \quad \rho E_t = P_x + q - Q. \end{cases}$$

Here G is the equilibrated body force, T is the stress, H is the equilibrated stress, η is the entropy, q is the heat flux, P is the first heat flux moment, Q is the mean heat flux and E is the first moment of energy. The variables u and ϕ are, respectively, the displacement of the solid elastic material and the volume fraction. The constitutive equations are

$$\begin{cases} T = \mu u_x + b\phi + \gamma u_{tx} - \beta\theta, & H = \delta\phi_x - dw, \\ G = -bu_x - \xi\phi + m\theta - \sigma\phi_t, & \eta = \beta u_x + c\theta + m\phi, \\ q = \kappa\theta_x + \kappa_1 w, & P = -\kappa_2 w_x, \\ Q = \kappa_3 w + \kappa_4\theta_x, & \rho E = -\alpha w - d\phi_x. \end{cases}$$

Where $\rho, J, \mu, b, \gamma, \delta, d, \xi, m, \tau, \beta, c, \kappa, \kappa_1, \kappa_2, \kappa_3, \kappa_4$ and α are the constitutive coefficients whose physical meaning is well known. It is worth noting that θ and w are the temperature and microtemperatures, respectively.

The coefficients of the system, in one-dimensional case, satisfy

$$\xi > 0, \quad \delta > 0, \quad \mu > 0, \quad \rho > 0, \quad J > 0, \quad \text{and} \quad \mu\xi \geq b^2,$$

where b is a real number different from zero. On the other hand, we assume that the thermal conductivity κ and the thermal capacity c are positive, which means that thermal effects are present. While, if microtemperatures are considered, parameters α, κ_2 and κ are positive. γ and σ are nonnegative. If $\sigma > 0$ and $\gamma > 0$, it means that the system is subjected to porous dissipation and viscoelastic dissipation, respectively.

In the absence of thermal effect (i.e $\kappa = 0$), Dridi and Djebabla [17] considered the following system

$$\begin{cases} \rho u_{tt} = \mu u_{xx} + b\varphi_x - \gamma\theta_x, \text{ in } (0,1) \times \mathbb{R}_+, \\ J\varphi_{tt} = \delta\varphi_{xx} - bu_x - \xi\varphi - dw_x + m\theta - \beta\varphi_t, \text{ in } (0,1) \times \mathbb{R}_+, \\ c\theta_t = -\gamma u_{tx} - m\varphi_t - k_1 w_x, \text{ in } (0,1) \times \mathbb{R}_+, \\ \alpha w_t = k_2 w_{xx} - k_3 w - k_1\theta_x - d\varphi_{tx}, \text{ in } (0,1) \times \mathbb{R}_+, \end{cases}$$

with Neumann (on φ, θ)-Dirichlet (on u, w) boundary conditions. In which they proved that the combination of porous-viscosity and microtemperature stabilized the system exponentially regardless of the coefficients of system. In [36], Saci and Djebabla are concerned with the following system

$$\begin{cases} \rho u_{tt} = \mu u_{xx} + b\varphi_x - \gamma\theta_x, \text{ in } (0,1)\times(0,\infty), \\ J\varphi_{tt} = \delta\varphi_{xx} - bu_x - \xi\varphi - dw_x + m\theta, \text{ in } (0,1)\times(0,\infty), \\ c\theta_t = -\gamma u_{tx} - m\varphi_t - k_1 w_x, \text{ in } (0,1)\times(0,\infty), \\ \alpha w_t = k_2 w_{xx} - k_3 w - k_1\theta_x - d\varphi_{tx}, \text{ in } (0,1)\times(0,\infty), \end{cases} \tag{1.3}$$

they improved the result obtained in Dridi and Djebabla [17] by proving that a unique dissipation given by microtemperatures is sufficiently strong enough to produce exponential stability in absence of both thermal conductivity and the porous dissipation (i.e $\kappa = \beta = 0$), under a new stability number given by

$$\chi = \frac{\mu}{\rho} - \frac{\delta}{J} - \frac{\gamma^2}{c\rho}.$$

By neglecting the nonlinear damping (i.e. $\gamma = 0$) and the nonlinear distributed delay, we obtain the system

$$\begin{cases} \rho u_{tt} - \mu u_{xx} - b\varphi_x = 0, \text{ in } (0,L)\times(0,\infty), \\ J\varphi_{tt} - \delta\varphi_{xx} + k_1\omega_x + bu_x + \xi\varphi = 0, \text{ in } (0,L)\times(0,\infty), \\ \tau w_t + k_2 w + k_1\varphi_{tx} - k_3\omega_{xx} = 0, \text{ in } (0,L)\times(0,\infty). \end{cases} \tag{1.4}$$

This system has been studied by Santos et al. [37] with fractional dissipation damping ($\sigma\varphi_t$) in the second equaton, they concluded that the case (i.e $\sigma = 0$) is an interesting open problem. Apalara [4] provided a solution to this last by considered the system (1.4) under the following Neumann (on φ)-Dirichlet (on u, w) boundary conditions

$$u(0,t) = u(1,t) = \varphi_x(0,t) = \varphi_x(1,t) = w(0,t) = w(1,t) = 0, t > 0,$$

and established the same results of Santos et al. [37] in the absence of porous dissipation (i.e $\sigma = 0$), he showed that the unique dissipation given by microtemperature damping is strong enough to exponentially stabilize the system if and only if the wave speeds of the system are equal

$$\left(\frac{\rho}{\mu} = \frac{J}{\delta}\right).$$

Now, let us recall some results about the effect of nonlinear damping mechanisms on similar problems, Apalara [3] considered the following porous system:

$$\begin{cases} \rho u_{tt} - \mu u_{xx} - b\phi_x = 0, & x \in (0,1), t > 0, \\ J\phi_{tt} - \delta\phi_{xx} + bu_x + \xi\phi + \alpha(t)g(\phi_t) = 0, & x \in (0,1), t > 0, \end{cases} \tag{1.5}$$

the term $\alpha(t)g(\phi_t)$ is the nonlinear damping, which subjected on the second equation. He established a general and an explicit decay rate result for the energy of system (1.5) with the condition of same speed of propagation, that is

$$\frac{\mu}{\rho} = \frac{\delta}{J}. \tag{1.6}$$

It is worth mentioning that in the case of $\mu = b = \xi$, the system (1.5) becomes

$$\begin{cases} \rho u_{tt} - \mu (u_x + \phi)_x = 0, & x \in (0,1), t > 0, \\ J\phi_{tt} - \delta\phi_{xx} + \mu (u_x + \phi) + \alpha(t)g(\phi_t) = 0, & x \in (0,1), t > 0, \end{cases} \tag{1.7}$$

which is a Timoshenko system with nonlinear damping. Alabau-Boussouira [2] studied (1.7) with $\alpha(t) = 1$ and proved a general semi-explicit formula for the decay rate of the energy at infinity with the condition (1.6). Mustafa and Messaoudi [28] considered (1.7) with all the coefficients $\rho = \mu = J = \delta = 1$ and obtained a general and an explicit decay result, depending on α and g.

Among the most important property of a physical system is the time delay by which the response to a subjected force is delayed in its effect (see [38]). The original study of this effect on a system was first introduced by Datko et al. [15] in 1986 when they showed that the presence of the delay may not only destabilize a system which is asymptotically stable in the absence of the delay but may also lead to ill-posedness (see also [30] and [32]). On the other hand, it has been established that voluntary introduction of delay can benefit the control (see [1]). Choucha et al. [13] considered a porous thermoelastic system with microtemperature effect, temperatures and distributed delay terms. they proved the well posedness of the system, and established an exponential stability of its solution. Moumen et al. [29] are concerned with one-dimensional porous–elastic systems with nonlinear damping, infinite memory and distributed delay terms, they proved that the solution energy has an explicit and optimal decay for the cases of equal and nonequal speeds of wave propagation. We refer the interested readers to [3, 5, 7, 10, 16, 18, 19, 20, 21, 22, 25, 27, 33, 34, 40, 41] and references therein for details discussion on the subject.

According to these observations and results above, one can ask the following questions:

1) Is it possible to stabilize system (1.5) with nonlinear damping and nonlinear distributed delay (nonlinearity case) sebjected in the first equation? If so, does the stabilization of the system depend on a relationship between the coefficients of the system?

2) What assumptions can be made about h_1 and h_2 to ensure the stabilization of the system?

In the present work, we shall give an answers to these questions by considering (1.1) under appropriate assumptions on the weight of the delay and without imposing any restrictive growth assumption on the damping term at the origin, we establish an energy decay rate by using a perturbed energy method and some properties of convex functions. These arguments of convexity were introduced and developed by Cavalcanti et al. [12], Daoulatli et al. [14], Lasiecka [23], and used by Liu and Zuazua [26] and others. We obtain our results regardless of the wave speeds χ of the system which was mentioned in [4, 37]. Our result then extends some previous results to nonlinearity case.

2. Preliminaries

In this section, we shall present some definitions and preliminaries for well study of our problem (1.1). Throughout this paper, c or c_i, $i = 1, 2$ represent a positive constant and C_p is used to denote the Poincaré-type constant.

Concerning the delay term, we introduce the following variable:

$$u_t(x, t - rs) := \vartheta(x, r, t, s),\ x \in (0, 1),\ r \in (0, 1),\ s \in (\tau_1, \tau_2),\ t > 0,$$

which satisfies

$$s\vartheta_t(x, r, t, s) + \vartheta_r(x, r, t, s) = 0,\ x \in (0, 1), r \in (0, 1), s \in (\tau_1, \tau_2), t > 0.$$

Consequently, system (1.4) becomes

$$\begin{cases} \rho u_{tt} - \mu u_{xx} - b\varphi_x + \gamma h_1(u_t) + \int_{\tau_1}^{\tau_2} \mu(s) h_2(\vartheta(x, 1, t, s)) ds = 0,\ \text{in } (0, 1) \times \mathbb{R}_+, \\ J\varphi_{tt} - \delta\varphi_{xx} + k_1\omega_x + bu_x + \xi\varphi = 0,\ \text{in } (0, 1) \times \mathbb{R}_+, \\ \tau w_t + k_2 w + k_1\varphi_{tx} - k_3 w_{xx} = 0,\ \text{in } (0, 1) \times \mathbb{R}_+, \\ s\vartheta_t(x, r, t, s) + \vartheta_r(x, r, t, s) = 0,\ \text{in } (0, 1) \times (0, 1) \times \mathbb{R}_+ \times (\tau_1, \tau_2), \\ u(x, 0) = u_0(x), u_t(x, 0) = u_1(x), \varphi(x, 0) = \varphi_0(x), \varphi_t(x, 0) = \varphi_1(x), \\ u_t(x, t - \rho s) := \vartheta(x, r, t, s),\ r \in (0, 1), s \in (\tau_1, \tau_2), \\ \vartheta(x, r, 0, s) = f_0(x, -rs),\ x \in (0, 1), s \in (\tau_1, \tau_2). \end{cases} \tag{2.8}$$

With the mixed boundary conditions

$$u(0, t) = \varphi(1, t) = w(0, t) = u_x(1, t) = \varphi_x(0, t) = w_x(1, t) = 0. \tag{2.9}$$

Next, we suppose that h_1 and h_2 satisfy the following assumptions:

$(\boldsymbol{A}1)$. $h_1 : \mathbb{R} \to \mathbb{R}$ is a continuous and non-decreasing function with $h(0) = 0$ such that there exist positive constants k_1, k_2 and l and a convex, continuous and increasing function

$h:\mathbb{R}_+\to\mathbb{R}_+$ of class $C^1(\mathbb{R}_+)\cap C^2(]0,+\infty[)$ satisfying: $h''=0$ on $[0,l]$ or ($h'(0)=0$ and $h''>0$ on $(0,l]$) such that

$$\begin{aligned} &h\left(s^2+h_1^2(s)\right)\leq h_1(s)\,s \quad \text{for } |s|\leq l,\\ &k_1s^2\leq h_1(s)\,s\leq k_2s^2 \qquad \text{for } |s|>l. \end{aligned} \tag{2.10}$$

($\boldsymbol{A}2$). $h_2:\mathbb{R}\to\mathbb{R}$ is an odd non-decreasing Lipschitz function such that there exist positive constants α_1,α_2,c satisfying

$$\alpha_1 sh_2(s)\leq\Gamma_2(s)\leq\alpha_2 sh_1(s), \tag{2.11}$$

where $\Gamma_2(s)=\int_0^s h_2(r)dr$.

In addition, for address the case of nonlinear delay, we assume there exists a positive constant σ such that

$$\|\mu(s)\|_\infty\frac{1-\alpha_1}{\alpha_1}<\sigma \text{ and } 0<\gamma-\sigma\alpha_2(\tau_2-\tau_1)-\alpha_2\int_{\tau_1}^{\tau_2}\mu(s)ds. \tag{2.12}$$

Remark 2.1. *1) Hypothesis $(A1)$ implies that $sh_1(s)>0$, for all $s\neq 0$.*

2) The hypothesis $(A1)$ with $l=1$ was first introduced by Lasiecka and Tataru [24].

3) By the mean value Theorem for integrals and the monotonicity of h_2, it follows that

$$\Gamma_2(s)=\int_0^s h_2(r)dr\leq sh_2(s), \tag{2.13}$$

consequently, $\alpha_1\leq\alpha_2\leq 1$.

Let us now give an example for functions h_1 and h_2.

Example 2.1. *Let the function $h_1(r)=r^\kappa, r\in(0,1]$ (i.e $l=1$), and $\kappa\geqslant 1$. $h_1'(r)=\kappa r^{\kappa-1}$ which is strictly positive. In the neighborhood of 0 , let us set the function h defined by*

$$h(r)=c_\kappa r^{\frac{\kappa+1}{2}},$$

where $c_\kappa=(2\kappa)^{-\frac{\kappa+1}{2}}$. So, for $\kappa=1, h$ is linear on $[0,1]$, otherwise strictly convex on $(0,1]$, $h'(0)=0$ and $h''>0$ on $(0,1]$. In addition, we have

$$h^{-1}(r)=2\kappa r^{\frac{2}{\kappa+1}}.$$

Now, let r be near 0, (2.10) can be deduced from fact that $r^\kappa+r^{2\kappa}\leqslant 2\kappa r^2$. Next, suppose we set the non-decreasing odd function $h_2(r)=3^{-\kappa}r^3\,(h_2'\geqslant 0)$, then $rh_2(r)\leqslant rh_1(r)$ on $(0,1]$. Then, (2.13) follows automatically, that is $\Gamma_2(r)=\frac{3^{-\kappa}}{4}r^4\leqslant rh_2(r)=3^{-\kappa}r^4$, since $\Gamma_2(r)\leqslant rh_1(r)$, taking $\alpha_1\leqslant\frac{1}{4}$ and $\alpha_2\geqslant\alpha_1$, (2.11) is deduced.

3. Well possedness

In this section, we shall study the well-posedness of solutions to problem (1.1)-(1.2). We give existence and uniqueness results for our system using the semigroup theory. First, let us denote by $\vartheta(.)$ to $\vartheta(x,r,t,s)$, $\vartheta(1)$ to $\vartheta(x,1,t,s)$ and $\vartheta(0)$ to $\vartheta(x,0,t,s)$. Next, if we denote $U=(u,v,\varphi,\psi,w,\vartheta)^T$, where $v=u_t$, and $\psi=\varphi_t$, then, system can be rewritten as follows:

$$\begin{cases} U_t+\mathcal{A}U=0, t>0, \\ U(x,0)=U_0(x)=(u_0,u_1,\varphi_0,\varphi_1,w_0,f_0)^T. \end{cases}$$

The operators $\mathcal{A}:D(\mathcal{A})\subset\mathcal{H}\to\mathcal{H}$ is given by

$$\mathcal{A}U=\begin{pmatrix} -v \\ -\frac{\mu}{\rho}u_{xx}-\frac{b}{\rho}\varphi_x+\frac{\gamma}{\rho}h_1(u_t)+\frac{1}{\rho}\int_{\tau_1}^{\tau_2}\mu(s)h_2(\vartheta(1))ds \\ -\psi \\ -\frac{\delta}{J}\varphi_{xx}+\frac{b}{J}u_x+\frac{\xi}{J}\varphi+\frac{k_1}{J}w_x \\ -\frac{k_3}{\tau}w_{xx}+\frac{k_2}{\tau}w+\frac{k_1}{\tau}\psi_x \\ \frac{1}{s}\vartheta_r \end{pmatrix},$$

and $\mathcal{H}$ is the energy space given by

$$\mathcal{H}=H^1_*(0,1)\times L^2(0,1)\times H^1_\diamond(0,1)\times L^2(0,1)\times H^1_*(0,1)\times L^2((0,1)\times(0,1)\times(\tau_1,\tau_2)),$$

such that

$$\begin{aligned} H^1_*(0,1) &= \left\{u\in H^1(0,1): u(0)=0\right\}, \\ H^1_\diamond(0,1) &= \left\{\varphi\in H^1(0,1): \varphi(1)=0\right\}. \end{aligned}$$

The domain of $\mathcal{A}$ is given by

$$D(\mathcal{A})=\left\{\begin{array}{l} U=(u,v,\varphi,\psi,w,\vartheta) \mid \\ u,w\in H^2(0,1)\cap H^1_*(0,1), v\in H^1_*(0,1), \\ \varphi\in H^2(0,1)\cap H^1_\diamond(0,1), \psi\in H^1_\diamond(0,1), \\ \vartheta\in H^1_0((0,1);H^1(0,1)), \\ u_x(1)=w_x(1)=\varphi_x(0)=0 \end{array}\right\}.$$

For any $U=(u,v,\varphi,\psi,w,\vartheta)^T\in\mathcal{H},\tilde{U}=(\tilde{u},\tilde{v},\tilde{\varphi},\tilde{\psi},\tilde{w},\widetilde{\vartheta})^T\in\mathcal{H}$, we equip $\mathcal{H}$ with the inner product defined by

$$\begin{aligned}\langle U,\tilde{U}\rangle_{\mathcal{H}} &= \mu\int_0^1 u_x\tilde{u}_x dx+\rho\int_0^1 v\tilde{v}dx+\xi\int_0^1\varphi\tilde{\varphi}dx+\delta\int_0^1\varphi_x\tilde{\varphi}_x dx\\ &+b\int_0^1\left(u_x\tilde{\varphi}+\tilde{u}_x\varphi\right)dx+J\int_0^1\psi\tilde{\psi}dx+\tau\int_0^1 w\tilde{w}dx\\ &+\sigma\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s\left|\mu(s)\right|\vartheta(.)\widetilde{\vartheta}(.)dsdrdx. \qquad (3.14)\end{aligned}$$

Remark 3.1. *Under the condition $\alpha\xi>b^2$, it is easy to see that (3.14) defines an inner product.*

Note that

$$\xi\varphi^2+2bu_x\varphi+\mu u_x^2=\frac{1}{\mu}\left[(\mu\xi-b^2)\varphi^2+(\mu u_x+b\varphi)^2\right]\geq 0. \qquad (3.15)$$

For any $U,\tilde{U}\in D(\mathcal{A})$, we have

$$\begin{aligned}&(\mathcal{A}+m\mathcal{I})\,U-(\mathcal{A}+m\mathcal{I})\,\tilde{U}\\ &=\begin{pmatrix} -\left(v-\widetilde{v}\right)+m\left(u-\widetilde{u}\right)\\ \begin{pmatrix} -\frac{\mu}{\rho}\left(u_{xx}-\widetilde{u}_{xx}\right)-\frac{b}{\rho}\left(\varphi_x-\widetilde{\varphi}_x\right)+\frac{\gamma}{\rho}\left(h_1(v)-h_1(\widetilde{v})\right)\\ +m\left(v-\widetilde{v}\right)+\frac{1}{\rho}\int_{\tau_1}^{\tau_2}\mu(s)\left[h_2(\vartheta(1))-h_2(\widetilde{\vartheta}(1))\right]ds\end{pmatrix}\\ -\left(\psi-\widetilde{\psi}\right)+m\left(\varphi-\widetilde{\varphi}\right)\\ -\frac{\delta}{J}\left(\varphi_{xx}-\widetilde{\varphi}_{xx}\right)+\frac{b}{J}\left(u_x-\widetilde{u}_x\right)+\frac{\xi}{J}\left(\varphi-\widetilde{\varphi}\right)+\frac{k_1}{J}\left(w_x-\widetilde{w}_x\right)+m\left(\psi-\widetilde{\psi}\right)\\ -\frac{k_3}{\tau}\left(w_{xx}-\widetilde{w}_{xx}\right)+\frac{k_2}{\tau}\left(w-\widetilde{w}\right)+\frac{k_1}{\tau}\left(\psi_x-\widetilde{\psi}_x\right)+m\left(w-\widetilde{w}\right)\\ \frac{1}{s}\left(\vartheta_r-\widetilde{\vartheta}_r\right)+m\left(\vartheta-\widetilde{\vartheta}\right)\end{pmatrix},\end{aligned}$$

then, using (3.15) and the fact that h_1 is a non-decreasing function, we get for $L_{h_2}$the Lipschitz constant for h_2,

$$\begin{aligned}&\langle(\mathcal{A}+m\mathcal{I})\,U-(\mathcal{A}+m\mathcal{I})\,\tilde{U},\left(U-\tilde{U}\right)\rangle_{\mathcal{H}}\\ &= m\int_0^1\left(\underbrace{\mu\left(u_x-\widetilde{u}_x\right)^2+\xi\left(\varphi-\widetilde{\varphi}\right)^2+2b\left(u_x-\widetilde{u}_x\right)\left(\varphi-\widetilde{\varphi}\right)}_{\geq 0}\right)dx\\ &+m\int_0^1\left(J\left(\psi-\widetilde{\psi}\right)^2+\xi\left(\varphi-\widetilde{\varphi}\right)^2+\delta\xi\left(\varphi_x-\widetilde{\varphi_x}\right)^2\right)dx\\ &+\int_0^1\left(k_3\left(w_x-\widetilde{w}_x\right)^2+\left(\tau m+k_2\right)\left(w-\widetilde{w}\right)^2\right)dx\\ &+\left(m-\frac{\sigma}{2}\int_{\tau_1}^{\tau_2}\mu(s)ds\right)\int_0^1\left(v-\widetilde{v}\right)^2dx\end{aligned}$$

$$
\begin{aligned}
&+\gamma \int_0^1 \underbrace{\left(h_1(v)-h_1(\widetilde{v})\right)(v-\widetilde{v})}_{\geq 0} dx \\
&+\sigma \int_0^1 (v-\widetilde{v}) \int_{\tau_1}^{\tau_2} \mu(s)\left(h_2(\vartheta(1))-h_2\left(\widetilde{\vartheta}(1)\right)\right) dsdx \\
&+\frac{\sigma}{2} \int_0^1 \int_{\tau_1}^{\tau_2} \mu(s)\left(\vartheta(1)-\widetilde{\vartheta}(1)\right)^2 dsdx \\
&+\sigma m \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s\mu(s)\left(\vartheta-\widetilde{\vartheta}\right)^2 dsdrdx.
\end{aligned}
$$

Then, by using Young's inequality, then for small ϵ and sufficiently large m, we obtain

$$
\begin{aligned}
& \langle(\mathcal{A}+m\mathcal{I})U-(\mathcal{A}+m\mathcal{I})\tilde{U},\left(U-\tilde{U}\right)\rangle_{\mathcal{H}} \\
\geq\; & m\int_0^1\left(\underbrace{\mu\left(u_x-\widetilde{u}_x\right)^2+\xi(\varphi-\widetilde{\varphi})^2+2b\left(u_x-\widetilde{u}_x\right)(\varphi-\widetilde{\varphi})}_{\geq 0}\right) dx \\
& +m\int_0^1\left(J\left(\psi-\widetilde{\psi}\right)^2+\xi(\varphi-\widetilde{\varphi})^2+\delta\xi\left(\varphi_x-\widetilde{\varphi_x}\right)^2\right) dx \\
& +\int_0^1\left(k_3\left(w_x-\widetilde{w}_x\right)^2+\left(\tau m+k_2\right)(w-\widetilde{w})^2\right) dx \\
& +\left(m-\frac{\sigma}{4\epsilon}-\frac{\sigma}{2}\int_{\tau_1}^{\tau_2}\mu(s)ds\right)\int_0^1(v-\widetilde{v})^2 dx \\
& +\gamma\int_0^1\underbrace{\left(h_1(v)-h_1(\widetilde{v})\right)(v-\widetilde{v})}_{\geq 0} dx \\
& +\left(\frac{\sigma}{2}-\epsilon L_{h_2}^2\right)\int_0^1\int_{\tau_1}^{\tau_2}\mu(s)\left(\vartheta(1)-\widetilde{\vartheta}(1)\right)^2 dsdx \\
& +\sigma m\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s\mu(s)\left(\vartheta-\widetilde{\vartheta}\right)^2 dsdrdx \\
\geq\; & 0,
\end{aligned}
$$

which implies that $\mathcal{A}$ is a m-accretive operator. One can prove that $\mathcal{A}+mI$ is a maximal monotone operator. for this latter, it is sufficient to demonstrate that $R(\lambda I+\mathcal{A})=\mathcal{H}$ for a large constant λ. From the fact that $D(\mathcal{A})$ is dense in $\mathcal{H}$ (see Proposition 7.1 in [11]) and the nonlinear semigroup theory [8, 9, 39], we can give the following well-posedness result.

Proposition 3.1. *Assume (***A**1*)-(***A**2*) hold and let $U_0 \in \mathcal{H}$, then there exists a unique solution $U \in C\left(\mathbb{R}_+, \mathcal{H}\right)$ of problem (2.8). Moreover, if $U_0 \in D(\mathcal{A})$, then*

$$
U \in C\left(\mathbb{R}_+, D(\mathcal{A})\right) \cap C^1\left(\mathbb{R}_+, \mathcal{H}\right).
$$

Now, let us recall few of some known algebraic and integral inequalities.

Lemma 3.1. *([11], Hölder's Inequality) Let $1 \leq p \leq \infty$, assume that $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$ then, $fg \in L^1(\Omega)$ and*

$$\int_\Omega |fg|\, dx \leq \|f\|_p \|g\|_q . \tag{3.16}$$

Lemma 3.2. [11] *(Poincaré's inequality) Suppose I is a bounded interval. Then there exists a constant C (depending on $|I| < \infty$) such that*

$$\|u\|_{W^{1,p}(I)} \leq C \|u'\|_{L^P(I)}, \text{ for all } u \in W_0^{1,p}(I). \tag{3.17}$$

Lemma 3.3. *([11], Cauchy-Schwarz Inequality) Every inner product satisfies the Cauchy-Schwarz inequality*

$$\langle x_1, x_2 \rangle \leq \|x_1\| \|x_2\| . \tag{3.18}$$

The equality sign holds if and only if x_1 and x_2 are dependent.

Lemma 3.4. [11]*(Young's Inequality) For all $a, b \in \mathbb{R}^+$, we have*

$$ab \leq \epsilon a^2 + \frac{b^2}{4\epsilon}, \tag{3.19}$$

where ϵ is any positive constant.

Next, Let us denote by h^* the conjugate function in the sense of Young of a convex function h (see [6], p. 64), that is,

$$h^*(p) = \sup_{t \in \mathbb{R}_+} (pt - h(t)).$$

Assume that $h'' > 0$, then for $p \geq 0$ a given number, h^* is the Legendre transform of h (see Liu and Zuazua [26]), which is given by

$$h^*(p) := p\left[h'\right]^{-1}(p) - h(\left[h'\right]^{-1}(p)), \tag{3.20}$$

and which satisfies the following inequality

Lemma 3.5. [31]*(Young's Inequality for the convex functions) Let h a convex function, h^* its conjugate in the sense of Young, we have*

$$px \leq h(x) + h^*(p) \quad \forall p, x \geq 0. \tag{3.21}$$

Remark 3.2. *Thanks to (3.20), along with (2.11), we write*

$$\begin{aligned} h^*(h_2(\vartheta(1))) &= \vartheta(1)h_2(\vartheta(1) - h(\vartheta(1)) \\ &\leq (1-\alpha_1)\vartheta(1)h_2(\vartheta(1). \end{aligned} \tag{3.22}$$

Next, for $\epsilon_0 > 0$ we define the functions J and K as below

$$J(t) := \begin{cases} t, & \text{if } h'' = 0 \text{ on } [0,l] \\ th'(\epsilon_0 t), & \text{if } h'(0) = 0 \text{ and } h'' > 0 \text{ on } (0,l] \end{cases} \tag{3.23}$$

and

$$K(t) = \int_t^1 \frac{1}{J(s)}\, ds \tag{3.24}$$

respectively.

Remark 3.3. *The relation (3.20) and the fact that $h(0) = 0$ and $(h')^{-1}$, h are increasing functions yield*

$$h^*(p) \leq p\left[h'\right]^{-1}(p) \quad \forall p \geq 0. \tag{3.25}$$

4. General decay

In this section, we give some lemmas allow us to prove the stability result o the solution. We define the functional energy of solutions of problem (1.1)-(1.2) as follows:

$$\begin{aligned} E(t) \ :&= \frac{1}{2}\int_0^1 \left(\rho {u_t}^2 + J\varphi_t^2 + \delta\varphi_x^2 + \underbrace{\xi\varphi^2 + 2bu_x\varphi + \mu u_x^2}_{\geq 0} + \tau w^2\right) dx \\ &+ \frac{\sigma}{2}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s\mu(s)\Gamma_2(\vartheta(.))dsdrdx. \end{aligned} \tag{4.26}$$

Lemma 4.1. *Let (u, φ, w) be a solution of (2.8)-(2.9). Then the energy functional $E(t)$, satisfies*

$$\begin{aligned} \mathrm{E}'(t) &\leq -k_2\int_0^1 w^2 dx - k_3\int_0^1 w_x^2 dx \\ &\quad -c\left[\int_0^1 u_t h_1(u_t)\, dx + \int_0^1\int_{\tau_1}^{\tau_2} \mu(s)\vartheta(1)h_2(\vartheta(1))dsdx\right] \\ &\leq 0. \end{aligned} \tag{4.27}$$

Proof. We multiply $(2.8)_1$ by u_t, $(2.8)_2$ by φ_t, and $(2.8)_3$ by w and then integrate over $(0,1)$, we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_0^1\left[\rho {u_t}^2+J\varphi_t^2+\delta\varphi_x^2+\underbrace{\xi\varphi^2+2bu_x\varphi+\mu u_x^2}_{\geq 0}+\tau w^2\right]dx \\ &+\gamma\int_0^1 u_t h_1(u_t)\,dx+\int_0^1\int_{\tau_1}^{\tau_2}\mu(s)u_t h_2(\vartheta(1))dsdx \\ = &-k_2\int_0^1 w^2dx-k_3\int_0^1 w_x^2dx. \end{aligned} \tag{4.28}$$

Now, by multiplying the fourth equation in (2.8) by $\sigma\,|\mu(s)|\,h_2(\vartheta(x,r,t,s))$, and integrating over $(0,1)\times(\tau_1,\tau_2)\times(0,1)$, using integration by parts, the definition of Γ_2, and the boundary conditions, gives

$$\begin{aligned} \sigma\int_0^1\int_{\tau_1}^{\tau_2}\int_0^1 s\mu(s)\vartheta_t h_2(\vartheta(.))drdsdx &= -\sigma\int_0^1\int_{\tau_1}^{\tau_2}\int_0^1 \mu(s)\vartheta_r h_2(\vartheta(.))drdsdx \\ \sigma\frac{d}{dt}\int_0^1\int_{\tau_1}^{\tau_2}\int_0^1 s\mu(s)\Gamma_2(\vartheta(.))drdsdx &= -\sigma\int_0^1\int_{\tau_1}^{\tau_2}\mu(s)\int_0^1\frac{\partial}{\partial r}\Gamma_2(\vartheta(.))drdsdx \\ &= -\sigma\int_0^1\int_{\tau_1}^{\tau_2}\mu(s)\Gamma_2(\vartheta(1))dsdx \\ &\quad +\sigma\int_0^1\int_{\tau_1}^{\tau_2}\mu(s)\Gamma_2(\vartheta(0))dsdx \\ &= -\sigma\int_0^1\int_{\tau_1}^{\tau_2}\mu(s)\Gamma_2(\vartheta(1))dsdx \\ &\quad +\sigma\int_0^1\int_{\tau_1}^{\tau_2}\mu(s)\Gamma_2(u_t)\,dsdx. \end{aligned} \tag{4.29}$$

The combination of (4.28)- (4.29) gives us

$$\begin{aligned} E'(t) &= -\gamma\int_0^1 u_t h_1(u_t)\,dx-\int_0^1\int_{\tau_1}^{\tau_2}\mu(s)u_t h_2(\vartheta(1))dsdx \\ &\quad -\sigma\int_0^1\int_{\tau_1}^{\tau_2}\mu(s)\Gamma_2(\vartheta(1))dsdx-k_3\int_0^1 w_x^2dx \\ &\quad +\sigma\int_0^1\int_{\tau_1}^{\tau_2}\mu(s)\Gamma_2(u_t)\,dsdx-k_2\int_0^1 w^2dx. \end{aligned} \tag{4.30}$$

So, by return to the convex conjugate of Γ_2, taking $p=h_2(\vartheta(1))$ and $x=u_t$, we get

$$u_t h_2(\vartheta(1)) \leq \Gamma_2^*(h_2(\vartheta(1)))+\Gamma_2(u_t), \tag{4.31}$$

$$\text{where, } \Gamma_2^*(h_2(\vartheta(1))) = \vartheta(1)h_2(\vartheta(1))-\Gamma_2(\vartheta(1)). \tag{4.32}$$

Using (4.30)- (4.32) and $(A2)$, we obtain

$$\begin{aligned} E'(t) &\leq -\gamma \int_0^1 u_t h_1(u_t)\,dx + \int_0^1 \int_{\tau_1}^{\tau_2} \mu(s)\vartheta(1)h_2(\vartheta(1))dsdx \\ &\quad + (\sigma+1)\int_0^1 \int_{\tau_1}^{\tau_2} \mu(s)\Gamma_2(u_t)\,dsdx - (\sigma+1)\int_0^1 \int_{\tau_1}^{\tau_2} \mu(s)\Gamma_2(\vartheta(1))dsdx \\ &\quad -k_2 \int_0^1 w^2 dx - k_3 \int_0^1 w_x^2 dx \\ &\leq -\left[\gamma - \alpha_2\sigma(\tau_2-\tau_1) - \alpha_2 \int_{\tau_1}^{\tau_2} \mu(s)ds\right] \int_0^1 u_t h_1(u_t)\,dx \\ &\quad - \int_0^1 \int_{\tau_1}^{\tau_2} [\frac{\alpha_1\sigma}{\|\mu(s)\|_\infty} - (1-\alpha_1)]\mu(s)\vartheta(1)h_2(\vartheta(1))dsdx \\ &\quad -k_2 \int_0^1 w^2 dx - k_3 \int_0^1 w_x^2 dx. \end{aligned}$$

Finally, by using (2.12) we obtain (4.27). □

Lemma 4.2. *Let (u,φ,w) be a solution of (2.8)-(2.9). Then, the functional*

$$I_1(t) = \frac{d}{4}\int_0^1 u_t u dx - \tau \int_0^1 w\left(\int_0^x u_t(y)dy\right)dx, t \geq 0,$$

satisfies, for $\eta, d > 0$ and $\forall t \geq 0$

$$\begin{aligned} I_1'(t) &\leq \left(c\eta - \frac{\mu d}{8\rho}\right)\int_0^1 u_x^2 dx + d\int_0^1 u_t^2 dx + (3c_1 + 2c\eta)\int_0^1 w^2 dx \\ &\quad + c_1 \int_0^1 \left(\varphi_t^2 + \varphi^2 + w_x^2\right)dx + \frac{c}{\eta}\int_0^1 \int_{\tau_1}^{\tau_2} \mu(s)\vartheta(1)h_2(\vartheta(1))dsdx \\ &\quad + \frac{c}{2\eta}\int_0^1 h_1^2(u_t)\,dx. \end{aligned} \tag{4.33}$$

Proof. Differentiating $I_1(t)$ and integrating by parts, we get

$$\begin{aligned} I_1'(t) &= \frac{d}{4}\int_0^1 u_t^2 dx - \frac{\mu d}{4\rho}\int_0^1 u_x^2 dx - \frac{bd}{4\rho}\int_0^1 \varphi u_x dx \\ &\quad - \frac{\gamma d}{4\rho}\int_0^1 h_1(u_t)\,u dx - \frac{d}{4\rho}\int_0^1 u \int_{\tau_1}^{\tau_2} \mu(s)h_2(\vartheta(1))dsdx \\ &\quad + k_3 \int_0^1 w_x u_t dx + k_2 \int_0^1 w\left(\int_0^x u_t(y)dy\right)dx - k_1 \int_0^1 \varphi_t u_t dx \\ &\quad + \frac{\tau\mu}{\rho}\int_0^1 w u_x dx + \frac{\tau b}{\rho}\int_0^1 w\varphi dx \\ &\quad + \frac{\tau\gamma}{\rho}\int_0^1 w\left(\int_0^x h_1(u_t)\,dy\right)dx + \frac{\tau}{\rho}\int_0^1 w \int_{\tau_1}^{\tau_2} \mu(s)\left(\int_0^x h_2(\vartheta(y,1))dy\right)dsdx. \end{aligned} \tag{4.34}$$

By using Cauchy-Schwarz inequality (3.18), we obtain

$$\begin{aligned}
\left(\int_0^x u_t(y)dy\right)^2 &\leq \left(\int_0^1 u_t\ dx\right)^2 \leq \int_0^1 u_t^2\ dx,\\
\left(\int_0^x h_1\left(u_t\right)dy\right)^2 &\leq \left(\int_0^1 h_1\left(u_t\right)\ dx\right)^2 dx \leq \int_0^1 h_1^2\left(u_t\right)\ dx,\\
\left(\int_0^x h_2(\vartheta(y,1))dy\right)^2 &\leq \left(\int_0^1 h_2(\vartheta(1))\ dx\right)^2 dx \leq \int_0^1 h_2^2(\vartheta(1))dx.
\end{aligned}$$

Next, On account of (2.13), (3.21), and (3.22), we obtain

$$h_2^2(\vartheta(x,1,t,s)) \leq 2\vartheta(x,1,t,s)h_2(\vartheta(x,1,t,s)). \tag{4.35}$$

Then, using Young's inequality (3.19), Cauchy–Schwarz inequality and (4.35), we get

$$-\frac{bd}{4\rho}\int_0^1 \varphi u_x dx \leq \frac{d\mu}{16\rho}\int_0^1 u_x^2 dx + \frac{c_1}{2}\int_0^1 \varphi^2 dx, \tag{4.36}$$

$$\frac{\tau\mu}{\rho}\int_0^1 w u_x dx \leq \frac{d\mu}{16\rho}\int_0^1 u_x^2 dx + c_1\int_0^1 w^2 dx, \tag{4.37}$$

$$k_2\int_0^1 w\left(\int_0^x u_t(y)dy\right)dx \leq \frac{d}{4}\int_0^1 u_t^2 dx + c_1\int_0^1 w^2 dx, \tag{4.38}$$

$$-k_1\int_0^1 \varphi_t u_t dx \leq \frac{d}{4}\int_0^1 u_t^2 dx + c_1\int_0^1 \varphi_t^2 dx, \tag{4.39}$$

$$k_3\int_0^1 w_x u_t dx \leq \frac{d}{4}\int_0^1 u_t^2 dx + c_1\int_0^1 w_x^2 dx, \tag{4.40}$$

$$\left|-\frac{\gamma d}{4\rho}\int_0^1 h_1\left(u_t\right)u dx\right| \leq \frac{c}{4\eta}\int_0^1 h_1^2\left(u_t\right)dx + c\eta\int_0^1 u_x^2 dx, \tag{4.41}$$

$$\begin{aligned}
\left|-\frac{d}{4\rho}\int_0^1 u\int_{\tau_1}^{\tau_2}\mu(s)h_2(\vartheta(1)dsdx\right| &\leq \eta c_p\int_{\tau_1}^{\tau_2}\mu(s)ds\int_0^1 u_x^2 dx\\
&\quad +\frac{1}{4\eta}\int_0^1\int_{\tau_1}^{\tau_2}\mu(s)h_2^2(\vartheta(1))dsdx,\\
&\leq +\frac{c}{2\eta}\int_0^1\int_{\tau_1}^{\tau_2}\mu(s)\vartheta(1)h_2(\vartheta(1))dsdx\\
&\quad +c\eta\int_0^1 u_x^2 dx,
\end{aligned} \tag{4.42}$$

$$\frac{\tau b}{\rho}\int_0^1 w\varphi dx \leq c_1\int_0^1 w^2 dx + \frac{c_1}{2}\int_0^1 \varphi^2 dx, \tag{4.43}$$

$$\frac{\tau\gamma}{\rho}\int_0^1 w\left(\int_0^x h_1\left(u_t\right)dy\right)dx \leq \frac{c}{4\eta}\int_0^1 h_1^2\left(u_t\right)dx + c\eta\int_0^1 w^2 dx, \tag{4.44}$$

$$\begin{aligned}
\frac{\tau}{\rho}\int_0^1 w\int_{\tau_1}^{\tau_2}\mu(s)\left(\int_0^x h_2(\vartheta(y,1))dy\right)dsdx &\leq \frac{c}{2\eta}\int_0^1\int_{\tau_1}^{\tau_2}\mu(s)\vartheta(1)h_2(\vartheta(1))dsdx\\
&\quad +c\eta\int_0^1 w^2 dx.
\end{aligned} \tag{4.45}$$

By substituting (4.36)-(4.45) into (4.34), we get (4.33). □

Lemma 4.3. *Let (u, φ, w) be a solution of (2.8)-(2.9). Then the functional*

$$I_2(t) := J\int_0^1 \varphi_t\varphi dx - \frac{b\rho}{\mu}\int_0^1 u_t\left(\int_0^x \varphi(y)dy\right)dx, t \geq 0,$$

satisfies, for any $\varepsilon_1 > 0$, the following estimate

$$\begin{aligned} I_2'(t) \leq & -\frac{\delta}{2}\int_0^1 \varphi_x^2 dx - \lambda\int_0^1 \varphi^2 dx + \varepsilon_1\int_0^1 u_t^2 dx + c_2\int_0^1 w_x^2 dx \\ & + \left(J + \frac{3c_2}{\varepsilon_1}\right)\int_0^1 \varphi_t^2 dx + \varepsilon_1\int_0^1 h_1^2(u_t)dx \\ & + 2\varepsilon_1\int_0^1\int_{\tau_1}^{\tau_2} \mu(s)\vartheta(1)h_2(\vartheta(1))dsdx, \end{aligned} \tag{4.46}$$

where $\lambda = \left(\xi - \frac{b^2}{\mu}\right)$.

Proof. By differentiating $I_2(t)$, we obtain

$$\begin{aligned} I_2'(t) = & J\int_0^1 \varphi_{tt}\varphi dx + J\int_0^1 \varphi_t^2 dx - \frac{b\rho}{\mu}\int_0^1 u_t\left(\int_0^x \varphi_t(y)dy\right)dx \\ & - \frac{b\rho}{\mu}\int_0^1 u_{tt}\left(\int_0^x \varphi(y)dy\right)dx. \end{aligned}$$

Next, using integrating by parts together with the boundary conditions, we get

$$\begin{aligned} I_2'(t) = & -\delta\int_0^1 \varphi_x^2 dx - \left(\xi - \frac{b^2}{\mu}\right)\int_0^1 \varphi^2 dx + J\int_0^1 \varphi_t^2 dx \\ & - k_1\int_0^1 w_x\varphi dx - \frac{b\rho}{\mu}\int_0^1 u_t\left(\int_0^x \varphi_t(y)dy\right)dx \\ & - \frac{\gamma_1 b}{\mu}\int_0^1 h_1(u_t)\left(\int_0^x \varphi(y)dy\right)dx \\ & - \frac{\gamma_2 b}{\mu}\int_0^1\left(\int_0^x \varphi(y)dy\right)\int_{\tau_1}^{\tau_2} \mu(s)h_2(\vartheta(1))dsdx. \end{aligned}$$

Thanks to Young's, Poincaré (3.17) and Cauchy-Schwarz's inequalities and (4.35), we obtain

$$\begin{aligned} -k_1\int_0^1 w_x\varphi dx \leq & \frac{\delta}{2}\int_0^1 \varphi_x^2 dx + c_2\int_0^1 w_x^2 dx, \\ -\frac{b\rho}{\mu}\int_0^1 u_t\left(\int_0^x \varphi_t(y)dy\right)dx \leq & \varepsilon_1\int_0^1 u_t^2 dx + \frac{c_2}{\varepsilon_1}\int_0^1 \varphi_t^2 dx, \end{aligned}$$

$$-\frac{\gamma_1 b}{\mu}\int_0^1 h_1(u_t)\left(\int_0^x \varphi_t(y)dy\right)dx \leq \varepsilon_1\int_0^1 h_1^2(u_t)\,dx+\frac{c_2}{\varepsilon_1}\int_0^1\varphi_t^2dx,$$

$$\begin{aligned}-\frac{\gamma_1 b}{\mu}\int_0^1\int_{\tau_1}^{\tau_2}\mu(s)h_2(\vartheta(1))\left(\int_0^x \varphi_t(y)dy\right)dsdx &\leq \varepsilon_1\int_0^1\int_{\tau_1}^{\tau_2}\mu(s)h_2^2(\vartheta(1))dsdx\\ &\quad+\frac{c_2}{\varepsilon_1}\int_0^1\varphi_t^2dx\\ &\leq 2\varepsilon_1\int_0^1\int_{\tau_1}^{\tau_2}\mu(s)\vartheta(1)h_2(\vartheta(1))dsdx\\ &\quad+\frac{c_2}{\varepsilon_1}\int_0^1\varphi_t^2dx.\end{aligned}$$

Then, we find that

$$\begin{aligned}I_2'(t) &\leq -\frac{\delta}{2}\int_0^1\varphi_x^2dx-\lambda\int_0^1\varphi^2dx+\varepsilon_1\int_0^1 u_t^2dx+c_2\int_0^1 w_x^2dx\\ &\quad+\left(J+\frac{c_2}{\varepsilon_1}\right)\int_0^1\varphi_t^2dx+\varepsilon_1\int_0^1 h_1^2(u_t)\,dx\\ &\quad+2\varepsilon_1\int_0^1\int_{\tau_1}^{\tau_2}\mu(s)\vartheta(1)h_2(\vartheta(1))dsdx,\end{aligned}$$

which is (4.46) with $\lambda=\xi-\frac{b^2}{\mu}>0$. □

Lemma 4.4. *Let (u,φ,w) be a solution of (2.8)-(2.9). Then, the functional*

$$I_3(t)=-\tau\int_0^1 w\left(\int_0^x\varphi_t(y)dy\right)dx,$$

satisfies, for any $\varepsilon_2,\varepsilon_3,\varepsilon_4>0$, the following estimate

$$\begin{aligned}I_3'(t) &\leq (-k_1+2\varepsilon_2)\int_0^1\varphi_t^2dx+\varepsilon_2C_p\int_0^1 u_x^2dx+\varepsilon_4\int_0^1\varphi^2dx\\ &\quad+c_2\left(\frac{(b\tau)^2}{4\varepsilon_2}+\frac{\delta^2}{4\varepsilon_3}+\frac{\xi^2}{4\varepsilon_4}+\frac{k_2^2}{4\varepsilon_2}+k_1\right)\int_0^1 w^2dx\\ &\quad+\varepsilon_3\int_0^1\varphi_x^2dx.\end{aligned}\tag{4.47}$$

Proof. By differentiating $I_3(t)$, integrating by parts and using (2.8), we obtain

$$\begin{aligned}I_3'(t) &= -Jk_1\int_0^1\varphi_t^2dx+k_1\int_0^1 w^2dx+\tau b\int_0^1 wudx\\ &\quad-Jk_3\int_0^1 w_x\varphi_tdx-\xi\int_0^1 w\left(\int_0^x\varphi(y)dy\right)dx\\ &\quad+Jk_2\int_0^1 w\left(\int_0^x\varphi_t(y)dy\right)dx-\delta\int_0^1 w\varphi_xdx.\end{aligned}\tag{4.48}$$

Using Young's, Cauchy-Schwarz's and Poincaré inequalities, we find

$$\tau b\int_0^1 wudx \leq \varepsilon_2 C_p\int_0^1 u_x^2dx+\frac{(\tau b)^2}{4\varepsilon_2}\int_0^1 w^2dx, \quad (4.49)$$

$$-\delta\int_0^1 w\varphi_x dx \leq \varepsilon_3\int_0^1 \varphi_x^2dx+\frac{\delta^2}{4\varepsilon_3}\int_0^1 w^2dx, \quad (4.50)$$

$$-\xi\int_0^1 w\left(\int_0^x \varphi(y)dy\right)dx \leq \varepsilon_4\int_0^1 \varphi^2dx+\frac{\xi^2}{4\varepsilon_4}\int_0^1 w^2dx, \quad (4.51)$$

$$Jk_2\int_0^1 w\left(\int_0^x \varphi_t(y)dy\right)dx \leq \varepsilon_2\int_0^1 \varphi_t^2dx+\frac{k_2^2}{4\varepsilon_2}\int_0^1 w^2dx, \quad (4.52)$$

$$-Jk_3\int_0^1 w_x\varphi_t dx \leq \varepsilon_2\int_0^1 \varphi_t^2dx+\frac{k_3^2}{4\varepsilon_2}\int_0^1 w_x^2dx. \quad (4.53)$$

Estimate (4.47) follows by substituting (4.49)-(4.53) into(4.48). □

Lemma 4.5. *Let (u,φ,w) be a solution of (2.8)-(2.9), then, the functional*

$$I_4(t)=\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s\mu(s)e^{-sr}\Gamma_2(\vartheta(.))dsdrdx, \quad (4.54)$$

satisfies the estimate

$$\begin{aligned} I_4'(t) \leq\ & -\alpha_1 e^{-\tau_2}\int_0^1\int_{\tau_1}^{\tau_2}\mu(s)\vartheta(1)h_2(\vartheta(1))dsdx \\ & +\alpha_2\left(\int_{\tau_1}^{\tau_2}\mu(s)ds\right)\int_0^1 u_t h_1(u_t)\,dx \\ & -e^{-\tau_2}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s\mu(s)\Gamma_2(\vartheta(1))dsdrdx. \end{aligned} \quad (4.55)$$

Proof. Differentiating I_4, using the fourth equation in (2.8), $(A2)$, and the fact that $\vartheta(0)=u_t$, we obtain

$$\begin{aligned} I_4'(t) &= \int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s\mu(s)\vartheta_t(.)e^{-sr}h_2(\vartheta(.))dsdrdx \\ &= -\int_0^1\int_0^1\int_{\tau_1}^{\tau_2}\mu(s)\vartheta_r(.)e^{-sr}h_2(\vartheta(.))dsdrdx \\ &= -\int_0^1\int_{\tau_1}^{\tau_2}\int_0^1\mu(s)\frac{d}{dr}\left[e^{-sr}\Gamma_2(\vartheta(.))\right]drdsdx \\ &\quad -\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s\mu(s)e^{-sr}\Gamma_2(\vartheta(.))dsdrdx \\ &= -\int_0^1\int_{\tau_1}^{\tau_2} e^{-s}\mu(s)\Gamma_2(\vartheta(1))dsdx+\int_0^1\int_{\tau_1}^{\tau_2}\mu(s)\Gamma_2(u_t)\,dsdx \end{aligned}$$

$$
\begin{aligned}
& -\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s\mu(s)e^{-sr}\Gamma_2(\vartheta(.))dsdrdx \\
\leq\ & -\alpha_1 e^{-\tau_2}\int_0^1\int_{\tau_1}^{\tau_2}\mu(s)\vartheta(1)h_2(\vartheta(1))dsdx \\
& -e^{-\tau_2}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s\mu(s)\Gamma_2(\vartheta(.))dsdrdx \\
& +\alpha_2\int_{\tau_1}^{\tau_2}\mu(s)ds\int_0^1 u_t h_1(u_t)\,dx.
\end{aligned}
$$

Using the fact that $-e^{-sr} \leq -e^{-s}$ for all $r \in [0,1]$, we then obtain (4.55). □

Now, we define the Lyapunov functional $\mathcal{L}(t)$ by

$$
\mathcal{L}(t) = NE(t) + \sum_{i=1}^{3} N_i I_i(t) + I_4(t), \tag{4.56}
$$

here, N, N_1, N_2 and N_3 are positive constants.

Lemma 4.6. *Let (u,φ,w) be a solution of (2.8)-(2.9). Then, there exist two positive constants μ_1 and μ_2 such that the Lyapunov functional (4.56) satisfies*

$$
\mu_1 \mathrm{E}(t) \leq \mathcal{L}(t) \leq \mu_2 \mathrm{E}(t), \forall t \geq 0. \tag{4.57}
$$

Proof. From (4.56), we have

$$
\begin{aligned}
|\mathcal{L}(t) - NE(t)| \leq\ & \frac{dN_1}{4}\int_0^1 |u_t u|\,dx + \tau N_1\int_0^1\left|w\left(\int_0^x u_t(y)dy\right)\right|dx \\
& + JN_2\int_0^1|\varphi_t\varphi|\,dx + \frac{b\rho N_2}{\mu}\int_0^1\left|u_t\left(\int_0^x \varphi(y)dy\right)\right|dx \\
& + N_3\tau J\int_0^1\left|w\left(\int_0^x\varphi_t(y)dy\right)\right|dx \\
& + \int_0^1\int_0^1\int_{\tau_1}^{\tau_2}\left|se^{-sr}\right|\Gamma_2(\vartheta(.))dsdrdx.
\end{aligned}
$$

By using Young's, Poincaré and Cauchy-Schwarz inequalities, we obtain

$$
|\mathcal{L}(t) - NE(t)| \leq \gamma E(t),
$$

which yields

$$
(N-\gamma)E(t) \leq \mathcal{L}(t) \leq (N+\gamma)E(t), \tag{4.58}
$$

this completes the proof. □

Theorem 4.1. *Let $(\varphi_0, \varphi_1, \psi_0, \psi_1, \theta_0, q_0, f_0)^T \in \mathcal{H}$ be given. Assume that $A1 - A2$ are satisfied, then there exist c_1,c_2,$c_3 > 0$ for which the (weak) solution of problem (2.8)-(2.9) satisfies*

$$E(t) \le c_1 K^{-1}(c_2 t + c_3), \ \forall t \ge 0. \tag{4.59}$$

Proof. By differentiating equation (4.56), then recalling Eqs. (4.27), (4.33), (4.46), (4.47) and (4.55), we have

$$\begin{aligned}
\mathcal{L}'(t) \le\ & (-k_3 N + c_1 N_1 + c_2 N_2)\int_0^1 w_x^2 dx + (-k_2 N + (3c_1 + 2c\eta) N_1)\int_0^1 w^2 dx \\
&+ N_3 c_2 \left(\frac{(b\tau)^2}{4\varepsilon_2} + \frac{\delta^2}{4\varepsilon_3} + \frac{\xi^2}{4\varepsilon_4} + \frac{k_2^2}{4\varepsilon_2} + k_1\right)\int_0^1 w^2 dx \\
&+(dN_1 + N_2\varepsilon_1)\int_0^1 u_t^2 dx + \left(\alpha_2 \int_{\tau_1}^{\tau_2}\mu(s)ds + \frac{c}{4\eta} - Nc_0\right)\int_0^1 u_t h_1(u_t)\, dx \\
&+ \left(\left(c\eta - \frac{\mu d}{8\rho}\right) N_1 + \varepsilon_2 C_p N_3\right)\int_0^1 u_x^2 dx + \left(\frac{N_1 c}{2\eta} + N_2\varepsilon_1\right)\int_0^1 h_1^2(u_t)\, dx \\
&+(-\lambda N_2 + c_1 N_1 + N_3\varepsilon_4)\int_0^1 \varphi^2 dx + \left(-\frac{\delta}{2}N_2 + N_3\varepsilon_3\right)\int_0^1 \varphi_x^2 dx \\
&+ \left(N_1 c_1 + \left(J + \frac{3c_2}{\varepsilon_1}\right) N_2 + N_3(-k_1 + 2\varepsilon_2)\right)\int_0^1 \varphi_t^2 dx \\
&+ \left(\left(2\varepsilon_1 N_2 + \frac{N_1 c}{\eta}\right) - Nc_0 - \alpha_1 e^{-\tau_2}\right)\int_0^1 \int_{\tau_1}^{\tau_2} \mu(s)\vartheta(1) h_2(\vartheta(1)) ds dx \\
&- e^{-\tau_2}\int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s\mu(s)\Gamma_2(\vartheta(.)) ds dr dx.
\end{aligned} \tag{4.60}$$

At this point, we set $\varepsilon_1 = 1$, $\varepsilon_2 = \frac{1}{N_3}$ and choose η small enough so that

$$\eta \le \frac{\mu d}{8c\rho}.$$

Next, take N_1 large enough so that,

$$\left(c\eta - \frac{\mu d}{8\rho}\right) N_1 + C_p < 0.$$

Let us fix N_1 and select $\varepsilon_3 = \varepsilon_4 = \frac{1}{N_3}$, choose N_2 large enough so that

$$-\frac{\delta}{2}N_2 + 1 < 0 \text{ and } -\lambda N_2 + c_1 N_1 + 1 < 0.$$

Fix N_2 and select ε_2 so small that

$$\varepsilon_2 < \frac{k_1}{2},$$

choose N_3 large enough so that

$$N_1 c_1 + (J + 3c_2) N_2 + N_3(-k_1 + 2\varepsilon_2) < 0 .$$

Finally, we choose N large enough so that

$$\begin{array}{l}
-k_3N+c_1N_1+c_2N_2<0,\\
-k_2N+(3c_1+2c\eta)\,N_1+N_3c_2\left(\frac{(b\tau)^2}{4\varepsilon_2}+\frac{\delta^2}{4\varepsilon_3}+\frac{\xi^2}{4\varepsilon_4}+\frac{k_2^2}{4\varepsilon_2}+k_1\right)<0,\\
\left(2\varepsilon_1N_2+\frac{N_1c}{\eta}\right)-Nc_0-\alpha_1e^{-\tau_2}<0,\\
\alpha_2\int_{\tau_1}^{\tau_2}\mu(s)ds+\frac{c}{4\eta}-Nc_0<0.
\end{array}$$

All these choices with the relation (4.60) leads to

$$\begin{array}{l}
\mathcal{L}'(t)\leq-\lambda_1\int_0^1\left(u_x^2+u_t^2+\varphi^2+\varphi_x^2+\varphi_t^2+w^2+\sigma\int_0^1\int_{\tau_1}^{\tau_2}s\mu(s)\Gamma_2(\vartheta(.))dsdr\right)dx\\
-\lambda_2\int_0^1\left(w_x^2+w^2\right)dx+c\left(\int_0^1u_t^2dx+\int_0^1h_1^2\left(u_t\right)dx\right),\ c,\lambda_1,\lambda_2>0.
\end{array}\tag{4.61}$$

On the other hand, from (4.26) and by using Young's inequality, we obtain

$$\begin{aligned}
E(t)\leq\ &\tfrac{1}{2}\int_0^1\left(\rho u_t^2+J\varphi_t^2+(\mu+|b|)u_x^2+\delta\varphi_x^2+(\xi+|b|)\varphi^2+\tau w^2\right)dx\\
&+\tfrac{\sigma}{2}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2}s\mu(s)\Gamma_2(\vartheta(.))dsdrdx\\
&\leq\upsilon_1\left(\int_0^1\left(u_t^2+\varphi_t^2+u_x^2+\varphi_x^2+\varphi^2+w^2\right)dx\right)\\
&+\upsilon_1\sigma\int_0^1\int_0^1\int_{\tau_1}^{\tau_2}s\mu(s)\Gamma_2(\vartheta(.))dsdrdx,\upsilon_1>0,
\end{aligned}$$

which implies that

$$\begin{aligned}
&-\int_0^1\left(u_t^2+\varphi_t^2+u_x^2+\varphi_x^2+\varphi^2+w^2+\sigma\int_0^1\int_{\tau_1}^{\tau_2}s\mu(s)\Gamma_2(\vartheta(.))dsdr\right)dx\\
\leq\ &-\upsilon_2E(t),\upsilon_2>0.
\end{aligned}\tag{4.62}$$

The combination of (4.61) and (4.62) gives

$$\begin{aligned}
\mathcal{L}'(t)\ \leq\ &-c_1\int_0^1\left(u_t^2+\varphi_t^2+w^2+(u_x+\varphi)^2+\varphi_x^2+\sigma\int_0^1\int_{\tau_1}^{\tau_2}s\mu(s)\Gamma_2(\vartheta(.))dsdr\right)dx\\
&+c\left(\int_0^1u_t^2dx+\int_0^1h_1^2\left(u_t\right)dx\right)\\
\leq\ &-c_1E(t)+c\left(\int_0^1\left(u_t^2+h_1^2\left(u_t\right)\right)dx\right).
\end{aligned}\tag{4.63}$$

Let us define the following sets

$$\Sigma_u=\left\{x\in(0,1):\left|u_t\left(x,t\right)\right|>l\right\}\ \text{ and }\Theta_u=(0,1)\setminus\Sigma_u.$$

Now, we estimate the last term in the right-hand side of (4.63). First, we have

$$\begin{aligned}
\int_0^1\left(u_t^2+h_1^2\left(u_t\right)\right)dx\ =\ &\int_{\Sigma_u}\left(u_t^2+h_1^2\left(u_t\right)\right)dx\\
&+\int_{\Theta u}\left(u_t^2+h_1^2\left(u_t\right)\right)dx.
\end{aligned}$$

Using $\boldsymbol{A1}$ and (4.27), we easily show that

$$\begin{aligned}\int_{\Sigma_u}\left(u_t^2+h_1^2\left(u_t\right)\right)dx &\leq \left(k_1^{-1}+k_2\right)\int_{\Sigma_\psi} u_t h_1\left(u_t\right)dx \\ &\leq \left(k_1^{-1}+k_2\right)\int_0^1 u_t h_1\left(u_t\right)dx \\ &\leq -cE'\left(t\right). \end{aligned} \tag{4.64}$$

If $h''=0$ on $[0,l]$: This implies that there exist $k_1', k_2'>0$ such that $k_1' s^2 \leq h_1(s)\,s \leq k_2' s^2$ for all $s\in\mathbb{R}_+$, and then (4.64) is also satisfied for $|u_t(x,t)|\leq l$, then on all $(0,1)$. From (4.63), (4.64), we arrive at

$$\left(\mathcal{L}(t)+cE(t)\right)' \leq -cJ\left(E(t)\right), \quad \forall t\geq 0, \tag{4.65}$$

where J is defined in (3.23).

If $h'(0)=0$ and $h''>0$ on $(0,l]$: Since h is convex and increasing, h^{-1} is concave and increasing, by using $\boldsymbol{A1}$, the reversed Jensen's inequality for concave function (see [35], p. 61), and (4.27), we obtain,

$$\begin{aligned}\int_{\Theta_\psi}\left(u_t^2+h_1^2\left(u_t\right)\right)dx &\leq \int_{\Theta_\psi} h^{-1}\left(u_t h_1\left(u_t\right)\right)dx \\ &\leq \int_{\Theta_\psi} h^{-1}\left(u_t h_1\left(u_t\right)\right)dx \\ &\leq \left|\Theta_\psi\right| h^{-1}\left(\int_{\Theta_\psi}\frac{1}{\left|\Theta_\psi\right|}u_t h_1\left(u_t\right)dx\right) \\ &\leq ch^{-1}\left(\int_{\Theta_\psi} u_t h_1\left(u_t\right)dx\right) \\ &\leq ch^{-1}\left(\int_0^1 u_t h_1\left(u_t\right)\right)dx \\ &\leq ch^{-1}\left(-cE'\left(t\right)\right). \end{aligned} \tag{4.66}$$

Therefore, from (4.63), (4.64) and (4.66), we find that

$$\mathcal{L}'(t) \leq -cE(t) + ch^{-1}(-cE'(t)) - cE'(t), \quad \forall t\geq 0.$$

By using Young's inequality (3.21), (3.25) and the fact that

$$E' \leq 0, \ \textit{and}\ h''>0,$$

we get for $\varepsilon_0 > 0$ small enough and $c_0 > 0$ large enough,

$$\begin{aligned}
& \left[h'\left(\varepsilon_0 E(t)\right)\left[\mathcal{L}(t) + cE(t)\right] + c_0 E(t)\right]' \\
= \ & \varepsilon_0' E(t)\, h''\left(\varepsilon_0 E(t)\right)\left[\mathcal{L}(t) + cE(t)\right] \\
& + h'\left(\varepsilon_0 E(t)\right)\left[\mathcal{L}'(t) + cE'(t)\right] + c_0' E(t) \\
\leq \ & -ch'\left(\varepsilon_0 E(t)\right) E(t) + ch'\left(\varepsilon_0 E(t)\right) h^{-1}(-cE'(t)) + c_0' E(t) \\
\leq \ & -ch'\left(\varepsilon_0 E(t)\right) E(t) + ch^*(h'\left(\varepsilon_0 E(t)\right)) - cE'(t) + c_0' E(t) \\
\leq \ & -ch'\left(\varepsilon_0 E(t)\right) E(t) + c\varepsilon_0 h'\left(\varepsilon_0 E(t)\right) E(t) \\
\leq \ & -ch'\left(\varepsilon_0 E(t)\right) E(t) = -cJ\left(E(t)\right).
\end{aligned} \tag{4.67}$$

Now, let us define the following functional:

$$\mathcal{G}(t) = \begin{cases} \mathcal{L}(t) + cE(t) & \text{if } h'' = 0 \text{ on } [0, l], \\ h'\left(\varepsilon_0 E(t)\right)\left[\mathcal{L}(t) + cE(t)\right] + c_0 E(t) & \text{if } h'(0) = 0 \text{ and } h'' > 0 \text{ on } (0, l]. \end{cases}$$

Using (4.57), we have

$$\mathcal{G} \sim E,$$

and exploiting (4.65) and (4.67), we easily deduce that

$$\mathcal{G}'(t) \leq -cJ\left(E(t)\right), \quad \forall t \geq 0.$$

Next, let us set

$$\mathcal{R}(t) = \varepsilon \mathcal{G}(t),$$

where $0 < \varepsilon < \bar{\varepsilon}$ and $\bar{\varepsilon}$ is a positive constant satisfying

$$\mathcal{G}(t) \leq \frac{1}{\bar{\varepsilon}} E(t), \quad \forall t \geq 0.$$

We also have

$$\mathcal{M} \sim E, \tag{4.68}$$

and for $t \geq 0$

$$\mathcal{M}'(t) \leq -c\varepsilon J\left(\mathcal{M}(t)\right). \tag{4.69}$$

Noting that $K' = -1/J$ (see (3.24)), we get from (4.69)

$$\mathcal{M}'(t)\, K'\left(\mathcal{M}(t)\right) \geq c\varepsilon, \quad \forall t \geq 0.$$

A simple integration over $(0, t)$ then yields

$$K\left(\mathcal{M}(t)\right) \geq K\left(\mathcal{M}(0)\right) + c\varepsilon t.$$

Then, since K^{-1} is decreasing , we deduce that

$$\begin{aligned} \mathcal{M}(t) &\leq K^{-1}\left(c\varepsilon t + K\left(\mathcal{M}(0)\right)\right) \\ &\leq K^{-1}\left(c_2 t + c_3\right). \end{aligned}$$

From this latter inequality and (4.68) we obtain easily (4.59). Then the proof is completed.

□

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