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NULL HYPERSURFACE NORMALIZED BY THE STRUCTURE VECTOR FIELD IN A PARASASAKIAN MANIFOLD

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Abstract. We examine the geometry of a null hypersurface M of a para-Sasakian manifold $(\overline{M}, \overline{\phi}, K, \overline{\eta}, \overline{g})$ transversal to the structure vector field K. The later is then a rigging ζ for M, and M is called K-normalized null hypersurface. We characterize the geometry of such a null hypersurface and prove under some conditions that there exist leaves of an integrable distribution of the screen distribution admitting an almost para complex structure. Also, we derive certain non-existence results and discuss some properties of semi-symmetric(resp. locally symmetric) K-normalized null hypersurfaces of para-Sasakian manifolds, for instance, we demonstrate that any para-Sasakian manifold admitting a semi-symmetric totally geodesic K-normalized null hypersurface is of constant negative curvature along the null hypersurface.

Keywords: Almost paracontact manifold, Para-Sasakian manifold, K-Normalized null hypersurface, Rigging vector field, Structure vector field.

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1. Introduction

A submanifold M of a semi-Riemannian manifold is null if the induced metric tensor is degenerate on M. Null hypersurfaces are specifically essential because of their applications

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in physics and mainly in general relativity. The principal differences between null and non-degenerate hypersurfaces stand up because of the absence of natural projections on the former. This prevents the usual geometric objects from being induced on null hypersurfaces.

The rigging technique introduced in [6] has shown to be an efficient tool to study a null hypersurface. Briefly, the main idea consists of choosing a vector field ζ, called a rigging, such that $\zeta_p \notin T_pM$ for all $p \in M$. From this unique arbitrary choice, we derive all the geometric objects needed to handle a null hypersurface: a null section of $\mathrm{Rad}(TM)$, a screen distribution in TM, a transversal null section, and all the associated tensors.

Several authors have studied the geometry of null submanifolds of para-Sasakian manifolds tangent to the structure vector field [1, 9,]. In [8], the authors considered the case where the null hypersurface is transversal to the structure vector field K. The later is then a rigging ζ for M, and M will be called K-normalized null hypersurface in this work. The question now arise of knowing wether it is always possible to select a structure vector field with specific geometric properties (closedness, quasi-conformality, etc.) but also with prescribed geometric properties for the null hypersurface (curvature condition, umbilicity, geodesibility, etc.). The goal of this paper is to provide a few answers to the above questions by studying the geometry of K-normalized null hypersurfaces in para-Sasakian manifolds.

The organization of this paper is the following. Section 2 contains all the preliminaries needed. In Section 3, we give an example, characterize the underlined null hypersurface (Theorem 3.1), and prove under some condition that there exist leaves of an integrable distribution of the screen distribution admitting an almost para-complex structure (Theorem 3.2). We establish sufficient conditions to guarantee that the Ricci type tensor Ric is an induced symmetric Ricci tensor of M (Theorem 3.3). We also show that there is no screen invariant K-normalized null hypersurface in an almost para-contact metric manifold $\overline{M}$ (Theorem 3.4) , and we establish obstruction results involving the geometric conditions on the structure vector field (Theorem 3.5, (Theorem 3.6 and Theorem 3.7). In Section 4, we discuss some properties of a semi-symmetric (resp. locally symmetric) normalized null hypersurfaces of para-Sasakian manifolds. We show that a K-normalized null hypersurface is totally geodesic if and only if it is locally symmetric (Theorem 4.1) and that any para-Sasakian manifold admitting a semi-symmetric totally geodesic K-normalized null hypersurface is of constant negative curvature along the null hypersurface (Theorem 4.2).

2. Preliminaries

In this section, we give a brief review about rigging tectnique and Para-sasakian manifolds.

2.1. **Rigging technique for null hypersurface.**

Let $(\overline{M}^{n+2}, \overline{g})$ be a Lorentzian manifold and (M, g) a null hypersurface of $(\overline{M}, \overline{g})$. Due to Gutiérrez and Olea; see [6]. A *rigging* for M is a vector field ζ defined on some open set of $\overline{M}$ containing M such that for each $p \in M$ $\zeta_p \notin T_pM$. Given a rigging ζ for M, we set $\overline{\omega} = \overline{g}(\zeta, \cdot)$, $\omega = i^*\overline{\omega}$, $\breve{g} = \overline{g} + \overline{\omega} \otimes \overline{\omega}$ and $\widetilde{g} = i^* \breve{g}$, where $i : M \hookrightarrow \overline{M}$ is the canonical inclusion map. It is well known that $\widetilde{g}$ is a Riemannian metric on M. The *rigged vector field* on M is the unique null vector field ξ given by $\tilde{g}(\xi, .) = \omega$ and it satisfies $\overline{g}(\zeta, \xi) = 1$. A rigging ζ defines a screen distribution $\mathscr{S}(\zeta)$ given by $\mathscr{S}(\zeta) = TM \cap \zeta^{\perp} = \ker \omega$. The null transversal vector field on M is

$$N = \zeta - \frac{1}{2}\overline{g}(\zeta, \zeta)\xi, \tag{2.1}$$

which is the unique null vector field such that $\overline{g}(N, \xi) = 1$. Moreover, it is worth noting that $T\overline{M}$ admits the following splitting

$$\begin{aligned} T\overline{M}|_M &= TM \oplus span(N) \\ &= \{\mathscr{S}(\zeta) \oplus span(\xi)\} \oplus span(N). \end{aligned} \tag{2.2}$$

According to the decomposition (2.2), the Gauss and Weingarten equations of M and $\mathscr{S}(\zeta)$ are the following (see[4, p. 82-85]):

$$\overline{\nabla}_X Y = \nabla_X Y + B(X, Y)N, \qquad \nabla_X PY = \overset{\star}{\nabla}_X PY + C(X, PY)\xi, \tag{2.3}$$

$$\overline{\nabla}_X N = -A_N X + \tau(X)N, \qquad \nabla_X \xi = -\overset{\star}{A}_\xi X - \tau(X)\xi, \quad \tau(X) = \overline{g}(\overline{\nabla}_X N, \xi), \tag{2.4}$$

$\forall X, Y$ tangent to M. Here, ∇ and $\overset{\star}{\nabla}$ are induced linear connction on TM and $\mathscr{S}(TM)$, respectively, B and C are the second fundamental forms on TM and $\mathscr{S}(\zeta)$ respectively. Moreover, A_N and $\overset{\star}{A}_\xi$ are the shape operators on TM and $\mathscr{S}(TM)$, respectively, connected with the second fundamental forms by $B(X, Y) = g(\overset{\star}{A}_\xi X, Y)$ and $C(X, PY) = g(A_N X, PY)$, and τ is a 1-form on TM. The induced linear connection ∇ is not a metric connection. In fact, using the fact that $\overline{\nabla}\overline{g} = 0$, we have

$$(\nabla_X g)(Y, Z) = B(X, Y)\omega(Z) + B(X, Z)\omega(Y), (2.16) \tag{2.5}$$

$\forall X, Y, Z \in \Gamma(TM)$. Also C is not symmetric since

$$C(X, Y) - C(Y, X) = g(\nabla_X Y - \nabla_Y X, N) = \omega([X, Y]), \forall X, Y \in \mathscr{S}(\zeta). \tag{2.6}$$

Let us denote by $\overline{R}$ and R the Riemannian curvature tensors of $\overline{\nabla}$ and ∇, respectively. Using (2.3) and (2.4), we get the so called Gauss-Codazzi equations [4]

$$\overline{g}(\overline{R}(X,Y)Z,\xi) = (\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) + \tau(X)B(Y,Z) - \tau(Y)B(X,Z), \quad (2.7)$$

$$\overline{g}(\overline{R}(X,Y)Z,PW) = \overline{g}(R(X,Y)Z,PW) + B(X,Z)C(Y,PW) - B(Y,Z)C(X,PW), \quad (2.8)$$

$$\overline{g}(\overline{R}(X,Y)\xi,N) = \overline{g}(R(X,Y)\xi,N) = C(Y,\overset{\star}{A}_\xi X) - C(\overset{\star}{A}_\xi Y,X) - 2d\tau(X,Y), \quad (2.9)$$

$\forall X,Y,Z$ and $W \in \Gamma(TM)$.

We say that the rigging vector field ζ has a quasi-conformal screen distribution if there exists ϕ and σ in $C^\infty(M)$ such that

$$A_N X = \rho \overset{\star}{A}_\xi X + \sigma PX, \quad (2.10)$$

for any $X \in \Gamma(TM)$. For $\sigma = 0$, we simply say that ζ has conformal screen distribution. We say that the rigging vector field is distinguished if the one-form τ vanishes. A null hypersuface M is said to be totally umbilical (resp. totally geodesic) in $\overline{M}$ if there exists a smooth function k on M such that

$$B(X,Y) = kg(X,Y) \quad (2.11)$$

(resp. B vanishes identically on M). Remembering that $\overset{\star}{A}_\xi \xi = 0$, M is totally umbilical (resp. totally geodesic) in $\overline{M}$ if $\overset{\star}{A}_\xi X = kX$ for any $X \in \Gamma(\mathscr{S}(TM))$ (resp. $\overset{\star}{A}_\xi = 0$).

Also the screen distribution $\mathscr{S}(\zeta)$ is totally umbilical (resp. totally geodesic) in M if there is a smooth function λ such that $C(X,PY) = \lambda g(X,Y)$ for all $X,Y \in \Gamma(TM)$ (resp. C vanishes identically)([4],[2]).

2.2. **Para-Sasakian Manifolds.** A $(2n+1)$ dimensional manifold $\overline{M}^{2n+1}$ is said to be an almost paracontact metric manifold, if it admits a tensor field $\overline{\phi}$ of type $(1,1)$, a structure vector field K, a 1-form $\overline{\eta}$ and a pseudo-Riemannian metric $\overline{g}$ satisfying the following conditions[7][11]:

$$\overline{\phi}^2 = I - \overline{\eta} \otimes K, \quad \overline{\eta}(K) = 1, \quad \overline{\phi}(K) = 0, \quad \overline{\eta} \circ \overline{\phi} = 0 \quad (2.12)$$

$$\overline{g}(\overline{\phi}X,\overline{\phi}Y) = -\overline{g}(X,Y) + \overline{\eta}(X)\overline{\eta}(Y), X,Y \in \Gamma(T\bar{M}), \quad (2.13)$$

where I denotes the identity transformation. From (2.13), we deduce

$$\overline{g}(\overline{X},\overline{\phi}Y) = -\overline{g}(\overline{\phi}X,Y) \quad (2.14)$$

$$\overline{g}(X,K) = \overline{\eta}(X), \quad (2.15)$$

for $X, Y \in \Gamma(TM)$. From (2.15), we get

$$\overline{g}(K, K) = 1.$$

An almost para-contact metric manifold $(\overline{M}, \overline{\phi}, K, \overline{\eta}, \overline{g})$ is called a para-Sasakian manifold if [11]

$$(\overline{\nabla}_X \overline{\phi})Y = -\bar{g}(X, Y)K + \overline{\eta}(Y)X, \ \forall X, Y \in \Gamma(T\bar{M}). \tag{2.16}$$

From (2.16), we have

$$\overline{\nabla}_X K = -\overline{\phi}X. \tag{2.17}$$

Example 2.1. [1] *Let $\overline{M} = \mathbb{R}^{2n+1}$ be the $(2n+1)$-dimensional real space with standard coordinate system $(x_1, y_1, x_2, y_2, ..., x_n, y_n, z)$. Defining*

$$\begin{aligned} &\bar{\phi}\frac{\partial}{\partial x_\alpha} = \frac{\partial}{\partial y_\alpha}, \quad \bar{\phi}\frac{\partial}{\partial y_\alpha} = \frac{\partial}{\partial x_\alpha}, \quad \bar{\phi}\frac{\partial}{\partial z} = 0, \\ &\overline{\eta} = dz, \quad K = \frac{\partial}{\partial z}, \\ &\bar{g} = \overline{\eta} \otimes \overline{\eta} + \sum_{\alpha=1}^{n} dx_\alpha \otimes dx_\alpha - \sum_{\alpha=1}^{n} dy_\alpha \otimes dy_\alpha, \end{aligned}$$

where $\alpha = 1, 2, ..., n$. The set $(\mathbb{R}^{2n+1}_n, \overline{\phi}, K, \eta, \overline{g})$ is an almost paracontact metric manifold.

If the paraholomorphic sectional curvature denoted by c is constant on the para-Sasakian manifold $(\overline{M}, \overline{\phi}, K, \overline{\eta}, \overline{g})$, then the later is a para-Sasakian space form. Moreover, the curvature tensor $\overline{R}$ of $\overline{M}$ satisfies [9, Theorem 2.2]

$$\begin{aligned} \overline{g}(\overline{R}(X, Y)Z, W) = &\frac{c-3}{4}\{\overline{g}(Y, Z)\overline{g}(X, W) - \overline{g}(X, Z)\overline{g}(Y, W)\} \\ &+ \frac{(c+1)}{4}\Big\{\overline{\eta}(X)\overline{\eta}(Z)\overline{g}(Y, W) - \overline{\eta}(Y)\overline{\eta}(Z)\overline{g}(X, W) \\ &+ \overline{g}(X, Z)\overline{\eta}(Y)\overline{\eta}(W) - \overline{g}(Y, Z)\overline{\eta}(X)\overline{\eta}(W) \\ &+ \overline{g}(\overline{\phi}Y, Z)\overline{g}(\overline{\phi}X, W) - \overline{g}(\overline{\phi}X, Z)\overline{g}(\overline{\phi}Y, W) - 2\overline{g}(\overline{\phi}X, Y)\overline{g}(\overline{\phi}Z, W)\Big\}, \end{aligned} \tag{2.18}$$

$\forall \ X, Y, Z \in \Gamma(TM)$. We refer to $\overline{M}(c)$ as a para-Sasakian space form.

3. Normalized null hypersurfaces of a para-Sasakian manifold

Let (M, ζ) be a normalized null hypersurface of a para-Sasakian manifold $(\overline{M}, \overline{\phi}, K, \overline{\eta}, \overline{g})$. K has the following pointwise decomposition along M:

$$K = K_{\mathscr{S}} + \gamma\xi + \beta N,$$

where γ and β are smooth functions on $\overline{M}$ defined by $\beta = \overline{\eta}(\xi)$, $\gamma = \overline{\eta}(N)$, and $K_{\mathscr{S}} \in \Gamma(\mathscr{S}(\zeta))$. Consider a global vector field U on $\mathscr{S}(\zeta)$ and its corresponding 1-form μ defined by

$$U = -\overline{\phi}\xi, \;\; \mu(X) = \overline{g}(X, U), \forall X \in \Gamma(TM). \tag{3.19}$$

From (2.2), we have

$$\overline{\phi}X = \phi X + \mu(X)N, \forall X \in \Gamma(TM), \tag{3.20}$$

where ϕ is a (1, 1)-tensor field on M. From (2.17) and (3.20), we obtain the following result, which is similar to the one given in [3, Proposition 3.1]

Proposition 3.1. *Let* $(\overline{M}^{(2n+1)}, \overline{\phi}, K, \overline{\eta}, \overline{g})$ *be a para-Sasakian manifold and* M *a null hypersurface of* $\overline{M}$. *Then,* $\forall X \in \Gamma(TM)$ *we get*

$$\begin{aligned} \overset{\star}{\nabla}_X K_{\mathscr{S}} &= \gamma \overset{\star}{A}_\xi X + \beta A_N X - P(\phi(X)), \\ X \cdot \gamma &= \gamma\tau(X) - C(X, K_{\mathscr{S}}) - \omega(\phi(X)), \\ X \cdot \beta &= -\beta\tau(X) - B(X, K_{\mathscr{S}}) - \mu(X), \end{aligned} \tag{3.21}$$

where P *is the projection morphism of* $\Gamma(TM)$ *onto* $\Gamma(\mathscr{S}(\zeta))$ *associated to the decomposition*(2.2).

Proof. On one hand, $\forall X \in \Gamma(TM)$, we get

$$\begin{aligned} \overline{\nabla}_X K &\overset{(2.17)}{=} -\overline{\phi}X \\ &= -\phi X - \mu(X)N = -P(\phi X) - \omega(\phi X)\xi - \mu(X))N. \end{aligned}$$

On the other hand, $\forall X \in \Gamma(TM)$, we get

$$\begin{aligned} \overline{\nabla}_X K = \overset{\star}{\nabla}_X K_{\mathscr{S}} &- \gamma\overset{\star}{A}_\xi X - \beta A_N X \\ &+ \left(C(X, K_{\mathscr{S}}) + X \cdot \gamma - \gamma\tau(X)\right)\xi \\ &+ \left(X \cdot \beta + \beta\tau(X) + B(X, K_{\mathscr{S}})\right)N. \end{aligned}$$

Matching the tangential, radical and transversal components of the expressions above we get the result. □

Now, we suppose that the structure vector field K never belongs to the tangent space of the null hypersurface M. In this case K can be taken as a rigging ζ for M. Thus, we have the following.

Definition 3.1. *A null hypersurface M of an almost para-contact metric manifold $(\overline{M}, \overline{\phi}, K, \overline{\eta}, \overline{g})$ such that the structure vector field K is a rigging for M is said to be* ***K-normalized***.

This leads to the following direct consequence of our Proposition 3.1.

Corollary 3.1. *Let $(\overline{M}^{(2n+1)}, \overline{\phi}, K, \overline{\eta}, \overline{g})$ be a para-Sasakian manifold and M a K-normalized null hypersurface of $\overline{M}$. Then for all $X \in \Gamma(TM)$, we have*

$$A_N X = -\frac{1}{2}\overset{\star}{A}_\xi X + P(\phi(X)), \tag{3.22}$$

$$\tau(X) = 2\omega(\phi(X)) = -\mu(X). \tag{3.23}$$

Proof. Since $K = \zeta$, then Eq.(2.1) leads to $\beta = 1$, $\gamma = \frac{1}{2}\overline{g}(\zeta, \zeta) = \frac{1}{2}$. Using this in (3.21) together with the fact that $\zeta_{\mathscr{S}} = 0$, we get the result. □

Let $(\overline{M}, \overline{\phi}, K, \overline{\eta}, \overline{g})$ be a (2n+1)-dimensional para-Sasakian manifold and M a K-normalized null hypersurface of $\overline{M}$. It is worth noting that

$$\overline{\eta}(\xi) = 1, \ \overline{\eta}(N) = \frac{1}{2}. \tag{3.24}$$

Applying $\overline{\phi}$ to the first equation of (3.19), we get

$$\overline{\phi} U \overset{(2.12)}{=} -N + \frac{1}{2}\xi. \tag{3.25}$$

Also, from (2.14), it is obvious that

$$\overline{g}(\overline{\phi}N, \xi) = -\frac{1}{2}\overline{g}(\overline{\phi}\xi, \xi) = 0, \quad \overline{g}(\overline{\phi}N, N) = -\frac{1}{2}\overline{g}(\overline{\phi}\xi, N) = 0, \tag{3.26}$$

Then,

$$2\overline{\phi}(N) = -\overline{\phi}(\xi) = U \in \mathscr{S}(\zeta) \tag{3.27}$$

since the components of both $\overline{\phi}N$ and $\overline{\phi}\xi$ with respect to ξ and N vanish.

From (3.23) and (3.26), the following Corollary holds:

Corollary 3.2. *Let $(\overline{M}^{(2n+1)}, \overline{\phi}, K, \overline{\eta}, \overline{g})$ be a para-Sasakian manifold and M a K-normalized null hypersurface of $\overline{M}$. Setting $W = N - \frac{1}{2}\xi$, we have*

$$\tau(\xi) = -\mu(\xi) = 2\omega(\phi\xi) = 0, \tag{3.28}$$

$$\overline{\phi}X = P(\phi X) + \tau(X)W, \ \forall X, Y \in \Gamma(TM). \tag{3.29}$$

Definition 3.2. [1] *Let* $(\overline{M}^{2n+1}, \overline{\phi}, K, \overline{\eta}, \overline{g})$ *be an almost paracontact metric manifold and* M *a null hypersurface of* $\overline{M}$. M *is said to be screen invariant (resp screen semi-invariant) if* $\overline{\phi}(X)$ *(resp. both* $\overline{\phi}N$ *and* $\overline{\phi}\xi$*) belong(s) to the screen distribution for all* $X \in \mathscr{S}(\zeta)$.

From equation (3.27), we get the following proposition given in [8, Proposition 3.1].

Proposition 3.2. [8] *A* K*-normalized null hypersurface of an almost para-contact metric manifold* $\overline{M}$ *is rather a screen semi-invariant null hypersurface of* $\overline{M}$.

The following Theorem proves the converse of this proposition. Namely, a rigged screen semi-invariant null hypersurface in an almost paracontact metric manifold is transversal to the structure vector field.

Theorem 3.1. *Let* M *be a null hypersurface of an almost paracontact metric manifold* $(\overline{M}, \overline{\phi}, K, \overline{\eta}, \overline{g})$ *and* (M, ζ) *a normalized null hypersurface. Let* ξ *and* N_ζ *be the rigged vector field and the null transversal vector field associated to* ζ. *If* $\overline{\phi}(span\{\xi\}) = \overline{\phi}(span\{N_\zeta\})$ *then* K *is transversal to* M. *If in addition* $\overline{g}(\zeta,\zeta) = 1$ *and* $\overline{\phi}N = -\frac{1}{2}\overline{\phi}\xi$, *then* M *is a* K*-normaized null hypersurface.*

Proof. In this proof, N stands for N_ζ. Assume that $\overline{\phi}span\{\xi\} = \overline{\phi}span\{N_\zeta\}$. Then, there exists a non vanishing function θ such that $\overline{\phi}\xi = \theta\overline{\phi}N$. The inner product of this relation with respect to $\overline{\phi}\xi$ and $\overline{\phi}N$ give $(\overline{\eta}(\xi))^2 = \theta(-1+\overline{\eta}(\xi)\overline{\eta}(N))$ and $-1+\overline{\eta}(\xi)\overline{\eta}(N) = \theta(\overline{\eta}(N))^2$. Since $\theta \neq 0$, we get $\overline{\eta}(\xi) \neq 0$ and $\overline{\eta}(N) \neq 0$ and $(\overline{\eta}(\xi))^2 = (\theta\overline{\eta}(N))^2$. The later gives $\overline{\eta}(\xi) = \pm\theta\overline{\eta}(N)$. The case $\overline{\eta}(\xi) = \theta\overline{\eta}(N)$ implies that $\overline{\eta}(\xi)^2 = \theta\overline{\eta}(\xi)\overline{\eta}(N) = -\theta + \theta\overline{\eta}(\xi)\overline{\eta}(N)$, which is a contradiction. Thus $\overline{\eta}(\xi) = -\theta\overline{\eta}(N)$, from which $\overline{\eta}(\xi)^2 = -\theta\overline{\eta}(\xi)\overline{\eta}(N) = -\theta + \theta\overline{\eta}(\xi)\overline{\eta}(N)$, that is

$$\overline{\eta}(\xi)\overline{\eta}(N) = \frac{1}{2}. \tag{3.30}$$

Since $\theta = -\frac{\overline{\eta}(\xi)}{\overline{\eta}(N)} \neq 0$ and $\overline{\phi}\xi = \theta\overline{\phi}N$, it is worth noting that $\overline{\eta}(N)\overline{\phi}\xi + \overline{\eta}(\xi)\overline{\phi}N = 0$. Applying $\overline{\phi}$ to this equation to get $\overline{\eta}(N)\xi - \overline{\eta}(N)\overline{\eta}(\xi)K + \overline{\eta}(\xi)N - \overline{\eta}(\xi)\overline{\eta}(N)K = 0$. This together with (3.30) give $K = \overline{\eta}(N)\xi + \overline{\eta}(\xi)N = \gamma\xi + \beta N$. Thus K is transversal to M, which gives the first claim.

Now,

$$\overline{\phi}N = -\frac{1}{2}\overline{\phi}\xi \implies \overline{\phi}(N + \frac{1}{2}\xi) = 0 \implies \overline{\phi}\zeta = 0.$$

Operating $\overline{\phi}$ to the last equation of above relation and using the first equation of (2.13), we have $\zeta = \overline{\eta}(\zeta)K$. This leads to $\zeta = \pm K$ as $\overline{g}(\zeta, \zeta) = 1$. Therefore, K is a rigging for M and M is K-normalized. Which completes the proof.

□

Example 3.1. *Let $\overline{M} = \mathbb{R}^5$ be a 5-dimensional almost paracontact metric manifold with the structure $(\overline{\phi}, K, \overline{\eta}, \overline{g})$ given in Example 2.1.*

Consider a submanifold M of $(\mathbb{R}^5_2, \overline{\phi}, \zeta, \overline{\eta}, \overline{g})$ given by

$$M = \{(x_1, y_1, x_2, y_2, z) \in \mathbb{R}^5 | x_1 + x_2 - \sqrt{3}y_1 + z = 0\}.$$

It worth noting that TM is spanned by

$$\{V_1 = \frac{\partial}{\partial x_1} - \frac{\partial}{\partial z}, \quad V_2 = \frac{\partial}{\partial x_2} - \frac{\partial}{\partial z}, \quad V_3 = \frac{\partial}{\partial y_1} + \sqrt{3}\frac{\partial}{\partial z}, \quad V_4 = \frac{\partial}{\partial y_2}\}.$$

Since $K = \frac{\partial}{\partial z}$ is a spacelike vector field, then we may use it as a rigging ζ for M. Then, the corresponding rigged vector field is

$$\xi = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} + \sqrt{3}\frac{\partial}{\partial y_1} + \frac{\partial}{\partial z}.$$

The associated null transversal vector field is

$$N = K - \frac{1}{2}\xi = \frac{1}{2}(-\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2} - \sqrt{3}\frac{\partial}{\partial y_1} + \frac{\partial}{\partial z}).$$

The associated screen distribution is

$$\mathscr{S}(\zeta) = \{U_1 = \frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}, \quad U_2 = \frac{\partial}{\partial x_1} + 2\frac{\partial}{\partial x_2} + \sqrt{3}\frac{\partial}{\partial y_1}, \quad U_3 = \frac{\partial}{\partial y_2}\}.$$

Next,

$$\begin{aligned}\overline{\phi}\xi &= \frac{\partial}{\partial y_1} + \frac{\partial}{\partial y_2} + \sqrt{3}\frac{\partial}{\partial x_1} \\ &= \frac{\sqrt{3}}{3}(2U_1 + U_2) + U_3.\end{aligned}$$

From which we have $\overline{\phi}N = -\frac{1}{2}\overline{\phi}\xi \in \mathscr{S}(\zeta)$. Thus M is screen semi-invariant.

Now, using (3.19) in (2.13) leads to $\overline{g}(U, U) = 1$, thus the distribution $\overline{\phi}(\langle U \rangle)$ is nondegenerate. Then we are able to define the unique nondegenerate distribution D_0 by

Definition 3.3.

$$\mathscr{S}(\zeta) = D_0 \perp \langle U \rangle.$$

This ends in the subsequent decomposition:

$$TM = \{D_0 \perp \langle U\rangle\} \perp \langle \xi\rangle, \tag{3.31}$$

$$T\overline{M} = \{D_0 \perp \langle U\rangle\} \perp \{\langle \xi\rangle \oplus \langle N\rangle\}.$$

Proposition 3.3. [8] *D_0 is $\overline{\phi}$- invariant.*

Setting

$$D = \langle \xi\rangle \perp D_0 \quad \text{and} \quad D' = \langle U\rangle,$$

it follows that

$$TM = D \oplus D'.$$

Let M be a K-normalized null hypersurface, and S be the projection morphism of TM on D_0 with respect to the decomposition (3.31). From this, any vector field X on M is expressed as follows

$$X = SX + \omega(X)\xi + \mu(X)U. \tag{3.32}$$

Applying $\overline{\phi}$ to (3.32) and using (3.19), (3.25) and the fact $\overline{\eta}(X) = \omega(X)$, we have

$$\overline{\phi}X = \psi X - \frac{1}{2}\mu(X)\xi - \overline{\eta}(X)U + \mu(X)N = \psi X - \overline{\eta}(X)U + \mu(X)W, \tag{3.33}$$

where ψ is a globally defined tensor field of type (1, 1) on TM by

$$\psi X = \overline{\phi}SX, \forall X \in \Gamma(TM). \tag{3.34}$$

Applying $\overline{\phi}$ to (3.33) and using (2.12), (3.25), (3.19) and $\overline{\eta}(X) = \omega(X)$, we have

$$\begin{aligned} X - \frac{1}{2}\omega(X)\xi - \overline{\eta}(X)N = \overline{\phi}^2 X &= \overline{\phi}(\psi X) + \frac{1}{2}\mu(X)U + \frac{1}{2}\omega(X)\xi - \overline{\eta}(X)N + \frac{1}{2}\mu(X)U \\ &= \psi^2 X - \frac{1}{2}\mu(\psi X)\xi - \overline{\eta}(\psi X)U \\ &+ \mu(\psi X)N + \mu(X)U + \frac{1}{2}\omega(X)\xi - \overline{\eta}(X)N \\ &= \psi^2 X + \mu(X)U + \frac{1}{2}\omega(X)\xi - \overline{\eta}(X)N. \end{aligned}$$

This leads to

$$\psi^2 X = X - \omega(X)\xi - \mu(X)U, \tag{3.35}$$

which implies that

$$\psi^2 X = SX.$$

Substituting (3.33) into (2.13), we have

$$-g(X,Y)+\overline{\eta}(X)\overline{\eta}(Y)=\overline{g}(\overline{\phi}X,\overline{\phi}Y)=g(\psi X,\psi Y)-\mu(X)\mu(Y)-\overline{\eta}(X)\overline{\eta}(Y).$$

This gives

$$g(\psi X,\psi Y)=-g(X,Y)+2\omega(X)\omega(Y)+\mu(X)\mu(Y). \tag{3.36}$$

Theorem 3.2. *Let $(\overline{M}^{(2n+1)},\overline{\phi},K,\overline{\eta},\overline{g})$ be a para-Sasakian manifold and M a K-normalized null hypersurface of $\overline{M}$ such that $C(X,Y)=C(Y,X)$, $B(X,\overline{\phi}Y)=B(\overline{\phi}X,Y)$, $\forall X,Y\in\Gamma(D_0)$. Then (M_0,g,ψ) is an almost paracomplex manifold, where M_0 is a leaf of the almost paracontact complex distribution D_0.*

Proof. Under the hypothesis together with [8, Theorem 6.2], we have that D_0 is integrable. From (3.35) and (3.36), we have

$$\psi^2X=X,\ g(\psi X,\psi Y)=-g(X,Y)\ \forall X,Y\in\Gamma(D_0). \tag{3.37}$$

From (3.37), the claim follows. □

Proposition 3.4. *If M is a K-normalized in $(\overline{M}^{2n+1},\overline{\phi},K,\overline{\eta},\overline{g})$, then for $X,Y\in\Gamma(TM)$, we have*

$$(\nabla_X\phi)Y=-\frac{1}{2}g(X,Y)\xi+\omega(Y)X+\mu(Y)A_NX+\frac{1}{2}B(X,Y)U, \tag{3.38}$$

$$(\nabla_X\mu)(Y)=-g(X,Y)-B(X,\phi Y)-\tau(X)\mu(Y),$$

$$\nabla_XU=-X-\tau(X)U+\phi(\overset{\star}{A}_\xi X), \tag{3.39}$$

$$B(X,U)=\mu(\overset{\star}{A}_\xi X), \tag{3.40}$$

$$\overset{\star}{\nabla}_{PX}U=-PX+\tau(X)\overline{\phi}\xi+P(\phi(\overset{\star}{A}_\xi X)),$$

$$C(X,U)=\omega(\phi(\overset{\star}{A}_\xi X))-\omega(X). \tag{3.41}$$

Proof. Let $X,Y\in\Gamma(TM)$, we get

$$\begin{aligned}-g(X,Y)\zeta+\overline{\eta}(Y)X&=(\overline{\nabla}_X\overline{\phi})Y=\overline{\nabla}_X\bar{\phi}Y-\bar{\phi}(\overline{\nabla}_XY)\\&=\overline{\nabla}_X(\phi Y+\mu(Y)N)-\bar{\phi}(\nabla_XY+B(X,Y)N)\\&=\overline{\nabla}_X\phi Y+\overline{\nabla}_X\mu(Y)N-\bar{\phi}(\nabla_XY)-B(X,Y)\bar{\phi}N\\&=(\nabla_X\phi)Y+(\nabla_X\mu)(Y)N+B(X,\phi Y)N-\mu(Y)A_NX\\&\quad+\mu(Y)\tau(X)N-B(X,Y)\bar{\phi}N.\end{aligned} \tag{3.42}$$

Also,

$$\begin{aligned}\nabla_X U + B(X,U)N &= \overline{\nabla}_X U \\ &= -(\overline{\nabla}_X \overline{\phi}\xi) \\ &= -\Big((\overline{\nabla}_X \overline{\phi})\xi + \overline{\phi}(\overline{\nabla}_X \xi)\Big) \\ &= -X + \tau(X)\overline{\phi}\xi + \overline{\phi}(\overset{\star}{A}_\xi X) \\ &= -X + \tau(X)\overline{\phi}\xi + \phi(\overset{\star}{A}_\xi X) + \mu(\overset{\star}{A}_\xi X)N. \qquad (3.43)\end{aligned}$$

Next, it is worth noting that

$$\begin{aligned}\overset{\star}{\nabla}_X U + C(X,U)\xi + B(X,U)N = -PX - \omega(X)\xi + \tau(X)\overline{\phi}\xi + P(\phi(\overset{\star}{A}_\xi X)) \\ + \omega(\phi(\overset{\star}{A}_\xi X))\xi + \mu(\overset{\star}{A}_\xi X)N. \qquad (3.44)\end{aligned}$$

When we equate tangential and normal parts in (3.42)(resp, (3.43), (3.44)), we get the result. □

The following result is a direct consequence of Proposition 3.4.

Corollary 3.3. *Let* $(\overline{M}^{(2n+1)}, \overline{\phi}, K, \overline{\eta}, \overline{g})$ *be a para-Sasakian manifold and* M *a null hypersurface of* $\overline{M}$*. Then there is no* K*-normalization such that* $\nabla\phi = 0$ *or* $\nabla U = 0$*.*

Proof. (i) Replacing X and Y with ξ in (3.38) and X by ξ in (3.39) give $(\nabla_\xi \phi)\xi = \xi$ and $\nabla_\xi U = \xi$, which completes the proof. □

Proposition 3.5. *Let* $(\overline{M}^{(2n+1)}, \overline{\phi}, K, \overline{\eta}, \overline{g})$ *be a para-Sasakian manifold and* M *a* K*-normalized null hypersurface of* $\overline{M}$*. Then* $\forall X \in \Gamma(TM)$*, we have*

$$\phi(A_N X) = -\phi(\overset{\star}{A}_\xi X) + 2X + 2\tau(X)U - \frac{1}{2}\omega(X)\xi.$$

Moreover, $\phi(A_N\xi)$ *and* ξ *are linearly related.*

Proof. Since $U = 2\overline{\phi}N$, we have

$$\begin{aligned}\nabla_X U + B(X,U)N = \overline{\nabla}_X U &= 2(\overline{\nabla}_X \overline{\phi}N) \\ &= 2\Big((\overline{\nabla}_X \overline{\phi})N + \overline{\phi}(\overline{\nabla}_X N)\Big) \\ &\overset{(2.3)-(2.16)}{=} -\omega(X)\xi - 2\omega(X)N + X - 2\overline{\phi}(A_N X) + 2\tau(X)\overline{\phi}N \\ &= -\omega(X)\xi + \tau(X)U - 2\phi(A_N X) - 2\mu(A_N X)N - 2\omega(X)N,\end{aligned}$$

that is

$$\nabla_X U = -\omega(X)\xi + \tau(X)U - 2\phi(A_N X) + X. \tag{3.45}$$

By equating (3.39) and (3.45), we have

$$\phi(A_N X) = -\frac{1}{2}\phi(\overset{\star}{A}_\xi X) + X + \tau(X)U - \frac{1}{2}\omega(X)\xi, \tag{3.46}$$

which gives the first claim. Now, setting $X = \xi$ in (3.46), we have $\phi(A_N\xi) = -\frac{1}{2}\xi$, which completes the proof. □

It is known that the Ricci type tensor Ric is an induced symmetric Ricci tensor of M if and only if the one form τ is closed on M [4, Theorem 3.2]. Using this, we get the following.

Theorem 3.3. *Let M be a K-normalized null hypersurface of a para-Sasakian manifold $\overline{M}$. Then Ric is an induced symmetric Ricci tensor of M if and only if $B(\phi X, Y) = B(X, \phi Y)$.*

Proof. From (3.23), we have $\tau(X) = -\mu(X)$. Differentiating this and using (3.43), we get

$$\begin{aligned} Y\tau(X) &= -\overline{g}(\overline{\nabla}_Y X, U) - \overline{g}(X, \overline{\nabla}_Y U) \\ &\overset{(2.3)-(3.43)}{=} -\mu(\nabla_Y X) + g(X,Y) + \tau(Y)\mu(X) - \overline{g}(X, \overline{\phi}(\overset{\star}{A}_\xi Y)) \\ &\overset{(3.23)(2.14)}{=} -\mu(\nabla_Y X) + g(X,Y) - \tau(Y)\tau(X) + \overline{g}(\overline{\phi}(X), \overset{\star}{A}_\xi Y) \\ &\overset{(3.20)}{=} -\mu(\nabla_Y X) + g(X,Y) - \tau(Y)\tau(X) + B(\phi(X), Y), \forall X, Y \in \Gamma(TM). \end{aligned} \tag{3.47}$$

Interchanging X and Y in (3.47), we get

$$X\tau(Y) - Y\tau(X) = -\mu([X,Y]) + B(\phi Y, X) - B(\phi X, Y). \tag{3.48}$$

On the other hand, (3.23) gives

$$\tau([X,Y]) = -\mu([X,Y]). \tag{3.49}$$

Then, by (3.48), (3.49) and the definition of $d\tau$, we have

$$\begin{aligned} 2d\tau(X,Y) &= [X\tau(Y) - Y\tau(X) - \tau([X,Y])] \\ &= B(\phi Y, X) - B(\phi X, Y). \end{aligned} \tag{3.50}$$

Thus $d\tau(X,Y) = 0$ if and only if $B(\phi X, Y) = B(X, \phi Y)$. Therefore, the claim follows from [4, Theorem 3.2]. □

Corollary 3.4. *Let M be a K-normalized null hypersurface of a para-Sasakian manifold $\overline{M}$. If M is totally geodesic, then the one form τ is closed. Moreover, Ric is an induced symmetric Ricci tensor of M.*

The presence of transversal structure vector field K in para-Sasakian manifolds prevents the existence of invariant null hypersurfaces. However, it is not the case when M is tangent to K (see [1, Theorem 10 and 14]). In the following, we obtain some non-existence results for K-normalized null hypersurfaces of a para-Sasakian manifolds.

Theorem 3.4. *There is no screen-invariant K-normalized null hypersurface in an almost para-contact metric manifold $\overline{M}$.*

Proof. If $\overline{\phi}(X) \in \mathscr{S}(\zeta)$, $\forall X \in \mathscr{S}(\zeta)$, then using (2.14) and (3.27), we will get $\overline{g}(\overline{\phi}\xi, \overline{\phi}\xi) = -\overline{g}(\xi, \overline{\phi}(\overline{\phi}\xi)) = 0$, which is absurd since from (2.13) $\overline{g}(\overline{\phi}\xi, \overline{\phi}\xi) = 1$. □

Theorem 3.5. *Let $(\overline{M}, \overline{\phi}, K, \overline{\eta}, \overline{g})$ be a para-Sasakian manifold and M a K-normalized null hypersurface of $\overline{M}$. Then,*

(i) *K cannot be screen quasi-conformal.*
(ii) *$\mathscr{S}(\zeta)$ cannot be totally umbilical in M.*
(iii) *M cannot be distinguished.*

Proof. Since from (3.28) of Corollary 3.2, $\tau(\xi) = -\mu(\xi) = 0$, then we have from (3.29) that $\overline{\phi}\xi = P(\overline{\phi}\xi)$.

(i) If K is screen quasi-conformal, then from (2.10), we will have $A_N\xi = 0$, which is a contracdition since from (3.22) $A_N\xi = P(\overline{\phi}\xi) = \overline{\phi}\xi \neq 0$. Indeed, if $\overline{\phi}\xi = 0$, then we will get from (2.13) that $0 = \overline{g}(\overline{\phi}\xi, \overline{\phi}\xi) = -g(\xi, \xi) + \overline{\eta}(\xi)\overline{\eta}(\xi) = 1$, which is absurd.

(ii) If $\mathscr{S}(\zeta)$ is totally umbilical, then using (3.41) of Proposition 3.4, we have $1 = C(\xi, U) = \gamma g(\xi, U) = 0$, which is absurd.

(iii) If $\tau(X) = 0\ \forall X \in \mathscr{S}(\zeta)$, we will have from (3.23) that $\mu(U) = 0$ as $U = -\overline{\phi}\xi \in \mathscr{S}(\zeta)$, which is absurd since from (2.13) $1 = \overline{g}(\overline{\phi}\xi, \overline{\phi}\xi) = \mu(U)$. This completes the proof. □

Now, $\forall X \in \Gamma(TM)$, $\forall Y \in \mathscr{S}(\zeta)$, we have

$$\begin{aligned} C(X,Y) &= \overline{g}(\nabla_X Y, N) \\ &= \overline{g}(\overline{\nabla}_X Y, N) \\ &= -\overline{g}(Y, \overline{\nabla}_X N). \end{aligned} \tag{3.51}$$

From item (*iii*) of Theorem 3.5, together with (3.51) and (3.51), we have the following corollary.

Corollary 3.5. *Let $(\overline{M}, \overline{\phi}, K, \overline{\eta}, \overline{g})$ be a para-Sasakian manifold and M a K-normalized null hypersurface of $\overline{M}$. Then,*

(a) *$\mathscr{S}(\zeta)$ cannot be a parallel distribution.*

(b) *N cannot be a closed conformal vector field.*

Theorem 3.6. *Let $(\overline{M}^{2n+1}, \overline{\phi}, K, \overline{\eta}, \overline{g})(n \geq 1)$ be a para-Sasakian manifold. Then the structure vector field K cannot be a closed normalization for any null hypersurface M.*

Proof. Suppose that $K = \zeta$ is a closed normalization for any null hypersurface M, then $\overline{\eta} = \overline{g}(\zeta, .)$ is a closed 1-form on M which implies that $\mathscr{S}(\zeta)$ is integrable. This implies from (2.6) that C is symmetric on $\mathscr{S}(\zeta)$. From (3.22), we have

$$-\overline{g}(\overline{\phi}X, Y) + \frac{1}{2}B(X, Y) + C(X, Y) = 0, \ \forall X, Y \in \mathscr{S}(\zeta). \tag{3.52}$$

Using (3.52) together with the fact that B and C are symmetric, we have

$$\overline{g}(\overline{\phi}X, Y) - \overline{g}(\overline{\phi}Y, X) = 0, \ \forall X, Y \in \mathscr{S}(\zeta). \tag{3.53}$$

Since $\overline{g}(\overline{\phi}X, Y) \overset{(2.14)}{=} -\overline{g}(X, \overline{\phi}Y)$, (3.53) leads to $\overline{g}(\overline{\phi}X, Y) = 0$. This together with (3.33) give

$$\overline{g}(\psi X, Y) = 0, \ \forall X, Y \in \mathscr{S}(\zeta). \tag{3.54}$$

From (3.54) together with the fact that $\mathscr{S}(\zeta)$ is non-degenerate, we have $\psi X = 0$ for all $X \in \mathscr{S}(\zeta)$. This and (3.35), give $X = 0$ for all $X \in \mathscr{S}(\zeta)$. Which is absurd since $\mathscr{S}(\zeta)$ is of rank $2n - 1$ with $n > 1$. Thus ζ cannot be closed. □

Theorem 3.7. *Let $(\overline{M}, \overline{\phi}, K, \overline{\eta}, \overline{g})$ be a para-Sasakian Lorentzian manifold . Then the structure vector field K cannot be a normalization for any flat null hypersurface M with parallel screen shape operator $\overset{\star}{A}_\xi$.*

Proof. Let $X \in \mathcal{S}(\zeta)$, we have

$$\begin{aligned}
\overline{g}(R(X, \xi)\xi, X) &= \overline{g}(\nabla_X \nabla_\xi \xi, X) - \overline{g}(\nabla_\xi \nabla_X \xi, X) - \overline{g}(\nabla_{[X,\xi]}\xi, X) \\
&= \tau(\xi)\overline{g}(\overset{\star}{A}_\xi (X), X) - \overline{g}(\nabla_\xi(-\tau(X)\xi - \overset{\star}{A}_\xi (X)), X) + \overline{g}(\overset{\star}{A}_\xi ([X, \xi]), X) \\
&\overset{(3.28)}{=} \overline{g}(\nabla_\xi \overset{\star}{A}_\xi (X), X) + \overline{g}(\overset{\star}{A}_\xi (\nabla_X \xi) - \overline{g}(\overset{\star}{A}_\xi (\nabla_\xi X), X) \\
&= \overline{g}((\nabla_\xi \overset{\star}{A}_\xi)(X), X) - \overline{g}(\overset{\star}{A}_\xi (X), \overset{\star}{A}_\xi (X)).
\end{aligned} \tag{3.55}$$

If M were flat and $\overset{\star}{A}_\xi$ parallel then (3.55) will imply that

$$g(\overset{\star}{A}_\xi (X), \overset{\star}{A}_\xi (X)) = 0$$

which means that M is totally geodesic as the screen distribution is positive definite. By using this we will get

$\forall X, Y \in \Gamma(TM)$,

$$\begin{aligned} 0 = R(X,Y)U &= \nabla_X \nabla_Y U - \nabla_Y \nabla_X U - \nabla_{[X,Y]} U \\ &\overset{(3.39)}{=} -\nabla_X Y - X\tau(Y)U + \tau(Y)X + \tau(Y)\tau(X)U \\ &+ \nabla_Y X + Y\tau(X)U - \tau(X)Y - \tau(X)\tau(Y)U \\ &+ [X,Y] + \tau(Y,X)U \\ &= \tau(Y)X - \tau(X)Y + \frac{1}{2}d\tau(Y,X)U \\ &\overset{3.4}{=} \tau(Y)X - \tau(X)Y. \end{aligned} \tag{3.56}$$

Setting $X = \xi$ in (3.56) and using (3.28), we will have $0 = \tau(Y)\xi\ \forall X \in \Gamma(TM)$, that is $\tau(X) = 0\ \forall X \in \Gamma(TM)$, which contradicts item *(iii)* of Theorem 3.5 and completes the proof of the Theorem. □

Theorem 3.8. *Let M be a totally umbilical K-normalized null hypersurface of a para-Sasakian space form $(\overline{M}(c), \overline{\phi}, K, \overline{\eta}, \overline{g})$. Then the umbilical factor k satisfies the partial differential equation*

$$\xi(k) - k^2 - \frac{(c+1)}{8} = 0. \tag{3.57}$$

Moreover, if M is totally geodesic, then $c = -1$.

Setting $X = W = \xi$ in (2.18) together with the fact that $\overline{\eta}(\xi) = 1$, we have

$$\overline{g}(\overline{R}(\xi, Y)Z, \xi) = \frac{(c+1)}{4}\Big\{ -\overline{g}(Y,Z) + 3\mu(Z)\mu(Y)\Big\}. \tag{3.58}$$

From (2.11)-(2.5), it is worth noting that

$$\begin{aligned} (\nabla_X B)(Y,Z) &= XB(Y,Z) - B(\nabla_X Y, Z) - B(\nabla_X Z, Y) \\ &= X(k)g(Y,Z) + kXg(Y,Z) - kg(\nabla_X Y, Z) - kg(\nabla_X Z, Y) \\ &= X(k)g(Y,Z) + k(\nabla_X g)(Y,Z) \\ &= X(k)g(Y,Z) + k^2\{g(X,Y)\eta(Z) + g(X,Z)\eta(Y)\}. \end{aligned}$$

This leads to

$$(\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) + \tau(X)B(Y,Z) - \tau(Y)B(X,Z)$$
$$= \{X(k) - k^2\eta(X) + k\tau(X)\}g(Y,Z) - \{Y(k) - k^2\eta(Y) + k\eta(Y)\}g(X,Z).$$

Replacing X by ξ in this equation and using (2.7) and (3.58), we have

$$\frac{(c+1)}{4}\Big\{ - g(PY,PZ) + 3\mu(PZ)\mu(PX)\Big\} = g(PY,PZ)\Big\{\xi(k) - k^2 + k\tau(\xi)\Big\}.$$

Choosing $PX = PZ = U \in \mathscr{S}(\zeta)$ together with the fact that $\mu(U) = \overline{g}(U,U) = 1$ and $\tau(\xi) = 0$, the previous equation becomes

$$\xi(k) - k^2 - \frac{(c+1)}{8} = 0.$$

Which gives item (3.57). Setting $k = 0$ in this equation, we have $c = -1$, which completes the proof.

Theorem 3.9. *There is no K-normalized null hypersurface of para-Sasakian space form $\overline{M}(c)(c \neq -1)$ such that the second fundamental form B is parallel.*

Proof. From (2.7) and (3.58), we have

$$\frac{(c+1)}{4}\Big\{ - \overline{g}(Y,Z) + 3\mu(Z)\mu(Y)\Big\} = (\nabla_\xi B)(Y,Z) - (\nabla_Y B)(\xi,Z) + \tau(\xi)B(Y,Z).$$

Being B parallel, choosing $PY = PZ = U \in \mathscr{S}(\zeta)$ together with the fact that $\tau(\xi) = 0$, $\mu(U) = 1 = g(U,U)$, the previous equation becomes $\frac{(-c-1)}{8} = 0$, that is $c = -1$, which is a contradiction. Hence, the claim holds. □

4. K-Normalized null hypersurfaces with certain symmetries

This section deals with locally symmetric and semi-symmetric K-normalized null hypersurfaces of para-Sasakian space forms.

We say that a null hypersurface M is locally symmetric [5], if the following holds

$$(\nabla_W R)(X,Y)Z = 0 \quad \forall X,Y,Z,W \in \Gamma(TM).$$

Using Lemma 3.2 in [5], $\forall X,Y,Z,W\ T \in \mathscr{S}(\zeta)$, we have,

$$\begin{aligned}\overline{g}((\overline{\nabla}_W\overline{R})(X,Y)Z,T) &= \overline{g}((\nabla_W R)(X,Y)Z,T) + (\nabla_W B)(X,Z)C(Y,T)\\ &+ B(X,Z)g((\nabla_W A_N)Y,T) - (\nabla_W B)(Y,Z)C(X,T)\\ &- B(Y,Z)g((\nabla_W A_N)X,T) - B(Y,Z)\tau(X)C(W,T)\\ &+ (\nabla_Y B)(X,Z)C(W,T) - (\nabla_X B)(Y,Z)C(W,T),\end{aligned} \tag{4.59}$$

and

$$\begin{aligned}\overline{g}((\overline{\nabla}_W\overline{R})(X,Y)Z,N) &= g((\nabla_W R)(X,Y)Z,N) + B(X,Z)g((\nabla_W(A_N Y),N)\\ &- B(Y,Z)g((\nabla_W(A_N X),N) - B(W,X)\overline{R}(N,Y,Z,N)\\ &- B(W,Y)\overline{R}(X,N,Z,N).\end{aligned} \tag{4.60}$$

Lemma 4.1. *Let $\overline{M}(c)$ be a para-Sasakian space form and $\overline{R}$ the Riemannian curvature tensor of Levi-Civita connection $\overline{\nabla}$. Then we have for any $X,Y,Z,W \in \Gamma(TM)$,*

$$(\overline{\nabla}_W\overline{R})(X,Y)Z = \frac{c+1}{4}\begin{pmatrix}\overline{g}(Y,Z)\overline{g}(X,\overline{\phi}W)\zeta - \overline{g}(X,Z)\overline{g}(Y,\overline{\phi}W)\zeta + \overline{g}(Y,Z)\overline{\eta}(X)\overline{\phi}W\\ -\overline{g}(X,Z)\overline{\eta}(Y)\overline{\phi}W + \overline{g}(Y,\overline{\phi}W)\overline{\eta}(Z)X - \overline{g}(X,\overline{\phi}W)\overline{\eta}(Z)Y\\ +\overline{g}(Z,\overline{\phi}W)\overline{\eta}(Y)X - \overline{g}(Z,\overline{\phi}W)\overline{\eta}(X)Y\end{pmatrix}. \tag{4.61}$$

Proof. Proof of the Lemma 4.1 here [9]. □

The following result is a transversal version of Theorem 4.2 of [9], where it was assumed that the structure vector field is tangent to the null hypersurface.

Theorem 4.1. *Let M be K-normalized null hypersurfaces of a para-Sasakian space form $\overline{M}(c)$. If M is locally symmetric, then $c = -1$. If $c = -1$, then M is locally symmetric if and only if it is totally geodesic.*

Proof. Let $\overline{M}(c)$ be a para-Sasakian space form and M a locally symmetric K-normalized null hypersurface of $\overline{M}(c)$. From (4.61), we have

$$\overline{g}((\overline{\nabla}_W\overline{R})(\xi,Y)\xi,N) = \frac{c+1}{4}\overline{g}(Y,\overline{\phi}W), \quad \forall W \in \Gamma(TM) \text{ and } Y \in \mathscr{S}(\zeta). \tag{4.62}$$

From (2.18), we have

$$\overline{g}(\overline{R}(\xi,N)\xi,N) \overset{(3.24)-(3.26)}{=} \frac{c-3}{4} - \frac{(c+1)}{4} = -1. \tag{4.63}$$

By taking $X = \xi$ and $Z = \xi$ in (4.60) and using (4.63)-(4.62), we obtain

$$B(W, Y) = \frac{c+1}{4}\overline{g}(Y, \overline{\phi}W), \tag{4.64}$$

for any $W \in \Gamma(TM)$ and $Y \in \mathscr{S}(\zeta)$.

Taking $Y = U$ and $W = \xi$ in this equation together with the fact that $\overline{g}(U, U) = 1$, we have $c = -1$. Hence, the first claim holds. Now, let (M, ζ) K-normalized in $\overline{M}(c)$ with $c = -1$. If M is locally symmetric, we get $B = 0$, due to (4.64). Conversely if (M, ζ) is totally geodesic, using (4.59), (4.60) and (3.48), we get

$$g((\nabla_W R)(X, Y)Z, PT) = 0 \quad \text{and} \quad g((\nabla_W R)(X, Y)Z, N) = 0.$$

Which completes the proof. □

Definition 4.1. [10] *We say that M is semi-symmetric if R satisfies $R(X, Y)R = 0\ \forall X, Y \in \Gamma(TM)$, where $R(X, Y)$ operates on R as a derivation of the tensor algebra at each point.*

Theorem 4.2. *Let $(\overline{M}, \overline{\phi}, K, \overline{\eta}, \overline{g})$ be a para-Sasakian manifold and M be a totally geodesic K-normalized null hypersurface of $\overline{M}$. If M is semi-symmetric, then $\overline{M}$ is of constant negative curvature along the null hypersurface.*

Proof. $\forall X, Y, Z \in \Gamma(TM)$, we have

$$\begin{aligned}(\nabla_Z R)(X, Y)U - R(X, Y)Z = \nabla_Z(R(X, Y)U) - R(\nabla_Z X, Y)U - R(X, \nabla_Z Y)U\\ - R(X, Y)\nabla_Z U - R(X, Y)Z.\end{aligned} \tag{4.65}$$

But, from (3.56), we have

$$\begin{aligned}\nabla_Z R(X, Y)U &= \nabla_Z(\tau(Y)X) - \nabla_Z(\tau(X)Y) \qquad (4.66)\\ &= (Z\cdot\tau(Y))X + \tau(Y)\nabla_Z X - (Z\cdot\tau(X))Y - \tau(X)\nabla_Z Y\\ &\overset{(3.39)}{=} -\mu(\nabla_Z Y)X + g(Z, Y)X - \tau(Z)\tau(Y)X + \tau(Y)\nabla_Z X\\ &+ \mu(\nabla_Z X)Y - g(Z, X)Y + \tau(Z)\tau(X)Y - \tau(X)\nabla_Z Y\\ -R(\nabla_Z X, Y)U - R(X, Y)\nabla_Z U &\overset{(3.56)}{=} -\tau(Y)\nabla_Z X + \tau(\nabla_Z X)Y - \tau(\nabla_Z Y)X + \tau(X)\nabla_Z Y\\ -R(X, Y)\nabla_Z U - R(X, Y)Z &\overset{(3.39)}{=} +R(X, Y)Z + \tau(Z)\tau(Y)X - \tau(Z)\tau(X)Y - R(X, Y)Z.\end{aligned}$$

Substituting the above equations in (4.65), we have

$$(\nabla_Z R)(X, Y)U - R(X, Y)Z = g(Y, Z)X - g(X, Z)Y. \tag{4.67}$$

Setting $Z=U$ in (4.67) and using (3.56), we get $\forall X,Y\in\Gamma(TM)$

$$(\nabla_U R)(X,Y)U = R(X,Y)U + g(Y,U)X - g(X,U)Y \overset{(3.23)-(3.39)}{=} 0. \tag{4.68}$$

Next, $\forall X,Y,Z,W\in\Gamma(TM)$

$$\begin{aligned}
0 &\overset{(3.56)}{=} (R(W,Z)R)(X,Y)U = R(W,Z)R(X,Y)U - R(X,Y)R(W,Z)U\\
&- R(R(W,Z)X,Y)U - R(X,R(W,Z)Y)U\\
&= \tau(Y)R(W,Z)X - \tau(X)R(W,Z)Y - \tau(Z)R(X,Y)W + \tau(W)R(X,Y)Z\\
&- \tau(Y)R(W,Z)X + \tau(R(W,Z)X)Y - \tau(R(W,Z)Y)X + \tau(X)R(W,Z)Y\\
&= -\tau(Z)R(X,Y)W + \tau(W)R(X,Y)Z + \tau(R(W,Z)X)Y - \tau(R(W,Z)Y)X\\
&\overset{(2.8)}{=} -\tau(Z)R(X,Y)W + \tau(W)R(X,Y)Z + \overline{g}(\overline{R}(W,Z)U,X)Y - \overline{g}(\overline{R}(W,Z)U,Y)X\\
&\overset{(3.43)}{=} \tau(W)\{g(Z,Y)X - g(Z,X)Y + R(X,Y)Z\}\\
&-\tau(Z)\{g(W,Y)X - g(W,X)Y + R(X,Y)W\}\\
&\overset{(4.65)}{=} \tau(W)(\nabla_W R)(X,Y)U - \tau(Z)(\nabla_Z R)(X,Y)U.
\end{aligned} \tag{4.69}$$

Setting $W=U$ in (4.69) and using (4.68), we have $(\nabla_Z R)(X,Y)U=0$. From this and (4.67), we have $R(X,Y)Z=-g(Y,Z)X+g(X,Z)Y$, for all $X,Y,Z\in\Gamma(TM)$. From this together with (2.7) and (2.8), we have $\overline{g}(\overline{R}(X,Y)Z,W) = -\{\overline{g}(Y,Z)\overline{g}(X,W)-\overline{g}(X,Z)\overline{g}(Y,W)\}$ for all $X,Y,Z,W\in\Gamma(TM)$. Which completes the proof. □

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