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ON SUPRA e^*-OPEN SETS AND SUPRA e^*-CONTINUOUS FUNCTIONS

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Abstract. In the present study, we introduced a novel type of generalized supra open sets called supra e^*-open sets via supra δ-closure operator which we define. Through this new concept, we defined and studied supra e^*-continuous functions, supra e^*-open functions and supra e^*-closed functions. Also, we investigated relationships between supra e^*-continuous functions and different generalized types of supra continuity.

Keywords: Supra regular open set, Supra δ-closure operator, Supra e^*-open set, Supra e^*-continuous function, Supra e^*-open function.

2020 Mathematics Subject Classification: 54A10, 54A20, 54C10, 54C08.

1. Introduction

In the last decades, numerous investigators have worked on generalized types of open sets such as preopen [17], semi-open [16], α-open [16], b-open [9], β-open [1], e-open [12], e^*-open [13]. Some studies conducted with the help of generalized open sets are as follows: The class of somewhere dense sets [3] are contained all α-open, preopen, semi-open, β-open and b-open sets except for the empty set. Also, the concept of ST_1-space is defined in the same paper and its various features are investigated. Al-Shami and Noiri continued to study more properties of somewhere dense sets in [4].

On the other hand, Mashhour et al. defined the notion of supra open sets [18] in supra topological spaces in 1983. Later, many researchers introduced and studied generalizations of supra open sets. In 2008, Devi et al. [11] explored a kind of sets and functions called

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supra α-open sets and supra α-continuous functions, subsequently. In 2010, Noiri and Sayed investigated supra pre-open sets [19] and supra b-open sets [20]. In 2013, Vidyarani [22] introduced supra regular open sets. In 2013, Jafari and Tahiliani [15] worked on supra β-open sets and supra β-continuous functions. In 2017, Al-Shami [2] studied supra semi-continuous functions and supra semi-open functions through supra semi-open sets.

The other studies referring to recent contributions in supra topology and their applications can be listed as follows: M. E. El-Shafei et al. [14] introduced strong supra regularly ordered spaces, strong supra normally ordered spaces and strong supra T_i-ordered spaces $(i = 0, 1, 2, 3, 4)$ on supra topological ordered spaces and investigate the main properties of them. B. A. Asaad et al. [10] studied the notion of an operator γ on a supra topological space and then this notion is utilized to analyze supra γ-open sets. T. M. Al-Shami and I. Alshammari [5] found new rough-approximation operators inspired by an abstract structure called supra topology. T. M. Al-Shami et al. [6] introduced new forms of limit points of a set and separation axioms on supra topological spaces via supra α-open sets (resp. supra β-open sets [7]) T. M. Al-Shami et al. [8] defined three types of supra compactness and three types of supra Lindelöfness using supra topological spaces via supra pre-open sets.

In this paper, first of all, we introduced the concept of supra δ-closure operator via supra regular open sets. Then, we defined supra e^*-openness with the help of this operator. Later, we developed the ideas of supra e^*-continuous functions and supra e^*-open functions via supra e^*-open sets. We also obtained several characterizations of supra e^*-continuity and revealed some of its basic features.

2. Preliminaries

In the whole of this study, unless explicitly stated topological spaces $(\Psi, \top)$ and $(\Phi, \bot)$ (or simply Ψ and Φ) always mean on which no separation axioms are supposed. Then the closure and interior of E are expressed by $cl(E)$ and $int(E)$, subsequently. The collection of all open (resp. closed) sets of Ψ are expressed by $O(\Psi)$(resp. $C(\Psi)$). Also, (Ψ, μ) and (Φ, η) represent supra topological spaces.

A point $x \in \Psi$ is referred to as δ-cluster point [21] of E if $int(cl(O)) \cap E \neq \emptyset$ for every open neighborhood O of x. The set of all δ-cluster points of O is called the δ-closure [21] of E and is expressed by $cl_\delta(E)$. If $E = cl_\delta(E)$, then E is called δ-closed [21] and the complementary of a δ-closed set is called δ-open [21]. The set $int_\delta(E) := \{x | (\exists O \in O(\Psi, x))(int(cl(O)) \subseteq E)\}$ is called the δ-interior of E.

A subclass $\mu \subseteq 2^{\Psi}$ is referred to as a supra topology on Ψ [18] if Ψ is an element of μ and μ is closed under arbitrary union. (Ψ, μ) is referred to as a supra topological space (briefly, supra space) [18]. The members of μ are called supra open sets (briefly, s.o.) [18]. The complementary of supra open set is referred to as a supra closed set [18]. The intersection (resp. union) of all supra closed (resp. supra open) sets of Ψ containing (resp. contained in) E is called the supra closure [18] (resp. supra interior [18]) of E and is expressed by $cl^{\mu}(E)$ (resp. $int^{\mu}(E)$).

Definition 2.1. *Let (Ψ, μ) be a supra topological space. A subset E of Ψ is referred to as:*
(ι_1) *supra regular open* [22] *(briefly, s.r.o.) if $E = int^{\mu}(cl^{\mu}(E))$.*
(ι_2) *supra α-open* [11] *(briefly, s.α.o.) if $E \subseteq int^{\mu}(cl^{\mu}(int^{\mu}(E)))$.*
(ι_3) *supra semi-open* [3] *(briefly, s.s.o.) if $E \subseteq cl^{\mu}(int^{\mu}(E))$.*
(ι_4) *supra preopen* [19] *(briefly, s.p.o.) if $E \subseteq int^{\mu}(cl^{\mu}(E))$.*
(ι_5) *supra b-open* [20] *(briefly, s.b.o.) if $E \subseteq int^{\mu}(cl^{\mu}(E)) \cup cl^{\mu}(int^{\mu}(E))$.*
(ι_6) *supra β-open* [15] *(briefly, s.β.o.) if $E \subseteq cl^{\mu}(int^{\mu}(cl^{\mu}(E)))$.*

The complementary of a supra regular open (resp. supra α-open, supra semi-open, supra preopen, supra b-open, supra β-open) set is called supra regular closed [22] (resp. supra α-closed [11], supra semi-closed [3], supra preclosed [19], supra b-closed [20], supra β-closed [15]).

The collection of all supra open (resp. supra regular open, supra α-open, supra semi-open, supra preopen, supra b-open, supra β-open, supra closed, supra regular closed, supra α-closed, supra semi-closed, supra preclosed, supra b-closed, supra β-closed) sets in (Ψ, μ) is expressed by $\mu(\Psi)$ (resp. $R\mu(\Psi)$, $\alpha\mu(\Psi)$, $S\mu(\Psi)$, $P\mu(\Psi)$, $b\mu(\Psi)$, $\beta\mu(\Psi)$, $\mu^c(\Psi)$, $R\mu^c(\Psi)$, $\alpha\mu^c(\Psi)$, $S\mu^c(\Psi)$, $P\mu^c(\Psi)$, $b\mu^c(\Psi)$, $\beta\mu^c(\Psi)$).

Definition 2.2. [18] *Let Ψ be a topological space. If $\top \subseteq \mu$, then μ is called a supra topology associated with $\top$.*

Definition 2.3. *Let Ψ and Φ be two topological spaces and μ be a supra topology associated with $\top$. A function $\Delta : \Psi \to \Phi$ is called supra continuous (resp. supra α-continuous* [11]*, supra precontinuous* [19]*, supra semi-continuous* [3]*, supra b-continuous* [20]*, supra β-continuous* [15]*) if for all open set L of Φ, $\Delta^{-1}[L]$ is s.o. (resp. s.α.o., s.p.o., s.s.o., s.b.o., s.β.o.) in Ψ.*

Definition 2.4. *A function $\Delta : \Psi \to \Phi$ is referred to as e^*-continuous* [13] *if for every open set E of Φ, $\Delta^{-1}[E]$ is e^*-open in Ψ.*

3. Supra δ-closure Operator and Supra e^*-open Sets

In this part of the study, we define supra e^*-open sets via supra δ-closure operator and investigate some of its basic features.

Definition 3.1. *Let μ be a supra topology on Ψ, then the supra δ-closure of $E \subseteq \Psi$ is expressed as follows:*

$$cl_\delta^\mu(E) := \bigcap\{G|(E \subseteq G)(G \in R\mu^c(\Psi))\}.$$

and the supra δ-interior of $A \subseteq \Psi$ is expressed as follows:

$$int_\delta^\mu(E) := \bigcup\{V|(V \subseteq E)(V \in R\mu(\Psi))\}.$$

Definition 3.2. *Let (Ψ, μ) be a supra topological space and $E \subseteq \Psi$. The set E is called a supra δ-closed (briefly, s.δ.c.) set if $E = cl_\delta^\mu(E)$. The complementary of a supra δ-closed set is referred to as supra δ-open (briefly, s.δ.o.).*

Theorem 3.1.

(ι_1) *Every s.r.o. set is a s.δ.o. set.*

(ι_2) *Every s.δ.o. set is a s.o. set.*

Proof. The proofs are clear from Definition 3.2. □

Remark 3.1. *In the subsequent example demonstrated that a s.o. set need not be a s.δ.o. set.*

Example 3.1. *Let $\Psi = \{\eth_1, \eth_2, \eth_3\}$. Define a supra topology $\mu = \{\Psi, \emptyset, \{\eth_1\}, \{\eth_1, \eth_3\}, \{\eth_2, \eth_3\}\}$ on Ψ. Then the set $\{\eth_1, \eth_3\}$ is a s.o. set, however, it is not s.δ.o.*

Question: Is there any s.δ.o. set which is not s.r.o.?

Definition 3.3. *Let (Ψ, μ) be a supra topological space and $E \subseteq \Psi$. The set E is called a supra e^*-open (briefly, s.e^*.o.) set if $E \subseteq cl^\mu(int^\mu(cl_\delta^\mu(E)))$. The complementary of a supra e^*-open set is referred to as supra e^*-closed. The collection of all supra e^*-open (resp. supra e^*-closed) set is expressed by $e^*\mu(\Psi)(e^*\mu^c(\Psi))$.*

Theorem 3.2. *Every s.β.o. set is a s.e^*.o. set.*

Proof. Let $E \in \beta\mu(\Psi)$. Thus, $E \subseteq cl^{\mu}(int^{\mu}(cl^{\mu}(E)))$. On the other hand, we have always $cl^{\mu}(E) \subseteq cl^{\mu}_{\delta}(E)$, then we get that $E \in e^{*}\mu(\Psi)$. □

Remark 3.2. *A s.e^*.o. set need not be s.β.o. as indicated by the subsequent example.*

Example 3.2. *Consider the supra topology in Example 3.1. Then the set $\{\eth_2\}$ is a s.e^*.o. set, however, it is not s.β.o.*

Remark 3.3. *From the above discussions and Theorem 3.2, we obtain the following diagram. However, the opposites of these implications don't hold always correct. Also, counterexamples of the other implications are shown in* [3], [11], [15] *and* [19].

$$\begin{array}{ccccccccccccc} & & & & & & & & \text{s.}p\text{.o.} & & & & \\ & & & & & & & \nearrow & & \searrow & & & \\ \text{s.}r\text{.o.} & \to & \text{s.}\delta\text{.o.} & \to & \text{s.o.} & \to & \text{s.}\alpha\text{.o.} & & & & \text{s.}b\text{.o.} & \to & \text{s.}\beta\text{.o.} \;\to\; \text{s.}e^{*}\text{.o.} \\ & & & & & & & \searrow & & \nearrow & & & \\ & & & & & & & & \text{s.}s\text{.o.} & & & & \end{array}$$

Theorem 3.3. *Let (Ψ, μ) be a supra topological space, then the following properties hold:*
(ι_1) If $\mathcal{A} \subseteq e^{*}\mu(\Psi)$, then $\cup\mathcal{A} \in e^{*}\mu(\Psi)$.
(ι_2) The intersection of two s.e^*.o. sets is not necessarily s.e^*.o.
(ι_3) $\Psi \in e^{*}\mu(\Psi)$.

Proof. (ι_1) : Let $\mathcal{A}$ be a collection of s.e^*.o. sets in Ψ.

$$\begin{aligned} E \in \mathcal{A} \;&\Rightarrow\; \subseteq cl^{\mu}(int^{\mu}(cl^{\mu}_{\delta}(E))) \subseteq \cup\mathcal{A} \\ &\Rightarrow\; E \subseteq cl^{\mu}(int^{\mu}(cl^{\mu}_{\delta}(\cup\mathcal{A}))) \\ &\Rightarrow\; \cup\mathcal{A} \subseteq cl^{\mu}(int^{\mu}(cl^{\mu}_{\delta}(\cup\mathcal{A}))). \end{aligned}$$

(ι_2) : Let $\Psi = \{\eth_1, \eth_2, \eth_3\}$ and let $\mu = \{\emptyset, \{\eth_1\}, \{\eth_1, \eth_2\}, \{\eth_2, \eth_3\}, \Psi\}$ be a supra topological space on Ψ. Although the subsets $\{\eth_1, \eth_3\}$ and $\{\eth_2, \eth_3\}$ are s.e^*.o. in Ψ, which is the intersection set $\{\eth_3\}$ is not s.e^*.o. in Ψ.
(ι_3) : Obvious. □

Theorem 3.4. *Let (Ψ, μ) be a supra topological space, then the following properties hold:*
(ι_1) If $\mathcal{A} \subseteq e^{*}\mu^{c}(\Psi)$, then $\cap\mathcal{A} \in e^{*}\mu^{c}(\Psi)$.
(ι_2) The union of two s.e^*.c. sets is not necessarily s.e^*.c.

Proof. It is obvious from the proof of Theorem 3.3. □

Definition 3.4. *The supra e^*-closure (resp. supra e^*-interior) of a set E is the intersection (resp. union) of the supra e^*-closed (resp. supra e^*-open) sets including (resp. included in) E, which is expressed by $cl^{\mu}_{e^*}(E)$ (resp. $int^{\mu}_{e^*}(E)$).*

Remark 3.4. *It is obvious from the above definition that $int^{\mu}_{e^*}(E) \in e^*\mu(\Psi)$ and $cl^{\mu}_{e^*}(E) \in e^*\mu^c(\Psi)$.*

Theorem 3.5. *The following properties hold for the supra e^*-interior and supra e^*-closure of subsets F and G of a space Ψ.*
(ι_1) $int^{\mu}_{e^*}(F) \subseteq F$ and $F \subseteq cl^{\mu}_{e^*}(F)$.
(ι_2) $int^{\mu}_{e^*}(F) = A$ iff F is a s.e*.o. set and $cl^{\mu}_{e^*}(F) = A$ iff A is a s.e*.c. set.
(ι_3) $int^{\mu}_{e^*}(\Psi \setminus F) = \Psi \setminus cl^{\mu}_{e^*}(F)$ and $cl^{\mu}_{e^*}(\Psi \setminus F) = \Psi \setminus int^{\mu}_{e^*}(F)$.
(ι_4) If $F \subseteq G$, then $int^{\mu}_{e^*}(F) \subseteq int^{\mu}_{e^*}(G)$ and $cl^{\mu}_{e^*}(F) \subseteq cl^{\mu}_{e^*}(G)$.

Proof. Straightforward. □

Theorem 3.6. *Let A and B be any subsets of a space Ψ, then the following properties hold:*
(ι_1) $int^{\mu}_{e^*}(A) \cup int^{\mu}_{e^*}(B) \subseteq int^{\mu}_{e^*}(A \cup B)$.
(ι_2) $cl^{\mu}_{e^*}(A \cap B) \subseteq cl^{\mu}_{e^*}(A) \cap cl^{\mu}_{e^*}(B)$.

Proof. Straightforward. □

Remark 3.5. *The inclusions in (ι_1) and (ι_2) in Theorem 3.6 can not replaced by equalities by as can be seen from the following examples.*

Example 3.3. *Let $\Psi = \{\eth_1, \eth_2, \eth_3\}$ and $\mu = \{\emptyset, \Psi, \{\eth_1\}, \{\eth_1, \eth_2\}, \{\eth_2, \eth_3\}\}$ be a supra topology on Ψ. Where, if $A = \{\eth_2\}$ and $B = \{\eth_3\}$, then $int^{\mu}_{e^*}(A) = int^{\mu}_{e^*}(B) = \emptyset$ and $int^{\mu}_{e^*}(A \cup B) = int^{\mu}_{e^*}(\{\eth_2, \eth_3\}) = \{\eth_2, \eth_3\}$.*

Example 3.4. *Let μ be the same supra topology on Ψ as given in the above example. If $C = \{\eth_1, \eth_2\}$ and $D = \{\eth_1, \eth_3\}$, then $cl^{\mu}_{e^*}(C) = cl^{\mu}_{e^*}(D) = \Psi$ and $cl^{\mu}_{e^*}(C \cap D) = cl^{\mu}_{e^*}(\{\eth_1\}) = \{\eth_1\}$.*

4. Supra e^*-continuous Functions

In this part of the study, we define a novel form of continuous functions called supra e^*-continuous. Also, we obtain several characterizations and investigate some of its fundamental properties.

Definition 4.1. *Let Ψ and Φ be two topological spaces. Let μ be an associated supra topology with $\top$. A function $\Delta : \Psi \to \Phi$ is called supra e^*-continuous if for each open set V of Φ, $\Delta^{-1}[V]$ is supra e^*-open in Ψ.*

Theorem 4.1. *Every continuous function is a supra e^*-continuous function.*

Proof. Let $V \in O(\Phi)$ and μ be an associated supra topology with $\top$.

$$\left.\begin{array}{r} \left.\begin{array}{r} V \in O(\Phi) \\ \Delta \text{ is continuous} \end{array}\right\} \Rightarrow \ \Delta^{-1}[V] \in O(\Psi) \\ O(\Psi) \subseteq \mu(\Psi) \end{array}\right\} \Rightarrow \Delta^{-1}[V] \in \mu(\Psi) \subseteq e^*\mu(\Psi).$$

That means Δ is supra e^*-continuous. □

Remark 4.1. *A supra e^*-continuous function need not be neither continuous nor supra β-continuous as shown by the following examples.*

Example 4.1. *Let $\Psi = \{\eth_1, \eth_2, \eth_3\}$ and $\top = \{\emptyset, \Psi, \{\eth_3\}\}$ be a topology on Ψ and the supra topology μ is expressed as $\mu = \{\emptyset, \Psi, \{\eth_1\}, \{\eth_1, \eth_2\}\}$. Let $\Delta : (\Psi, \top) \to (\Psi, \top)$ be a function expressed as $\Delta := \{(\eth_1, \eth_1), (\eth_2, \eth_3), (\eth_3, \eth_2)\}$. The pre-image of the open set $\{\eth_3\}$ is $\{\eth_2\}$. In that case $\{\eth_2\} \in e^*\mu(\Psi)$ and $\{\eth_2\} \notin O(\Psi)$. Thus, Δ is supra e^*-continuous, however, it is not continuous.*

Example 4.2. *Let $\Psi = \{\eth_1, \eth_2, \eth_3\}$ and $\top = \{\emptyset, \Psi, \{\eth_2\}\}$ be a topology on Ψ. Consider the supra topology in Example 3.1 on Ψ. Then the identity function $\Delta : (\Psi, \top) \to (\Psi, \top)$ is supra e^*-continuous. However, it is not supra β-continuous.*

Remark 4.2. *From Remark 3.3 and Examples 4.1 and 4.2, we have the following diagram. However, the opposites of the requirements are not always true. Also, counterexamples of the other requirements are shown in* [15], [19] *and* [20].

$$\begin{array}{ccccccc} & & & & \text{supra pre-cont.} & & \\ & & & \nearrow & \downarrow & & \\ \text{supra cont.} & \to & \text{supra } \alpha\text{-cont.} & & \text{supra } b\text{-cont.} & \to & \text{supra } \beta\text{-cont.} \\ \uparrow & & & \searrow & \uparrow & & \downarrow \\ \text{cont.} & & & & \text{supra semi-cont.} & & \text{supra } e^*\text{-cont.} \end{array}$$

Theorem 4.2. *Let $\Delta : \Psi \to \Phi$ be a function and μ be an associated supra topology with $\top$. Then the following properties are equivalent:*

(ι_1) *Δ is s.e^*.c.;*

(ι_2) *If for each closed set F of Φ is $\Delta^{-1}[F]$ supra e^*-closed in Ψ;*

(ι_3) $cl^{\mu}_{e^*}\left(\Delta^{-1}[L]\right) \subseteq \Delta^{-1}[cl(L)]$ *for each* $L \subseteq \Phi$;

(ι_4) $\Delta\left[cl^{\mu}_{e^*}(E)\right] \subseteq cl(\Delta[E])$ *for each* $E \subseteq \Psi$;

(ι_5) $\Delta^{-1}[int(L)] \subseteq int^{\mu}_{e^*}\left(\Delta^{-1}[L]\right)$ *for each* $L \subseteq \Phi$.

Proof. $(\iota_1) \Rightarrow (\iota_2)$: Let F be a closed set in Φ.

$$\left.\begin{array}{r} F \in C(\Phi) \Rightarrow \Phi \setminus F \in O(\Phi) \\ \text{Hypothesis} \end{array}\right\} \Rightarrow \Delta^{-1}[\Phi \setminus F] = \Psi \setminus \Delta^{-1}[F] \in e^*\mu(\Psi) \Rightarrow \Delta^{-1}[F] \in e^*\mu^c(\Psi).$$

$(\iota_2) \Rightarrow (\iota_3)$: Let L be any subset of Φ.

$$\left.\begin{array}{r} L \subseteq \Phi \Rightarrow cl(L) \in C(\Phi) \\ \text{Hypothesis} \end{array}\right\} \Rightarrow \Delta^{-1}[cl(L)] \in e^*\mu^c(\Psi) \Rightarrow cl^{\mu}_{e^*}\left(\Delta^{-1}[L]\right) \subseteq cl^{\mu}_{e^*}\left(\Delta^{-1}[cl(L)]\right) = \Delta^{-1}[cl(L)].$$

$(\iota_3) \Rightarrow (\iota_4)$: Let E be any subset of Ψ.

$$\left.\begin{array}{r} E \subseteq \Psi \Rightarrow \Delta[E] \subseteq \Phi \\ \text{Hypothesis} \end{array}\right\} \Rightarrow cl^{\mu}_{e^*}(E) \subseteq cl^{\mu}_{e^*}\left(\Delta^{-1}[\Delta[E]]\right) \subseteq \Delta^{-1}[cl(\Delta[E])]$$

$$\Rightarrow \Delta\left[cl^{\mu}_{e^*}(E)\right] \subseteq \Delta[\Delta^{-1}[cl(\Delta[E])]] \subseteq cl(\Delta[E]).$$

$(\iota_4) \Rightarrow (\iota_5)$: Let L be any subset of Φ.

$$\left.\begin{array}{r} L \subseteq \Phi \Rightarrow \Psi \setminus \Delta^{-1}[L] \subseteq \Psi \\ \text{Hypothesis} \end{array}\right\} \Rightarrow \Delta\left[cl^{\mu}_{e^*}\left(\Psi \setminus \Delta^{-1}[L]\right)\right] \subseteq cl\left(\Delta\left[\Psi \setminus \Delta^{-1}[L]\right]\right)$$

$$\Rightarrow \Delta\left[\Psi \setminus int^{\mu}_{e^*}\left(\Delta^{-1}[L]\right)\right] \subseteq cl(\Phi \setminus L) = \Phi \setminus int(L)$$

$$\Rightarrow \Psi \setminus int^{\mu}_{e^*}\left(\Delta^{-1}[L]\right) \subseteq \Delta^{-1}[\Phi \setminus int(L)]$$

$$\Rightarrow \Delta^{-1}[int(L)] \subseteq int^{\mu}_{e^*}\left(\Delta^{-1}[L]\right).$$

$(\iota_5) \Rightarrow (\iota_1)$: Let O be an open set in Φ.

$$\left.\begin{array}{r} O \in O(\Phi) \\ \text{Hypothesis} \end{array}\right\} \Rightarrow \Delta^{-1}[O] \subseteq \Delta^{-1}[int(O)] \subseteq int^{\mu}_{e^*}(\Delta^{-1}[O])$$

$$\Rightarrow \Delta^{-1}[O] \in e^*\mu(\Psi)$$

Thus, Δ is supra e^*-continuous. □

Theorem 4.3. *If $\Delta : \Psi \to \Phi$ is supra e^*-continuous and $\Gamma : \Phi \to \zeta$ is continuous, then the composition $\Gamma \circ \Delta : \Psi \to \zeta$ is supra e^*-continuous.*

Proof. Let $V \in O(\zeta)$.

$$\left.\begin{array}{r} \left.\begin{array}{l} V \in O(\zeta) \\ \Gamma \text{ is cont.} \end{array}\right\} \Rightarrow \Gamma^{-1}[V] \in O(\Phi) \\ \Delta \text{ is supra } e^*\text{-cont.} \end{array}\right\} \Rightarrow \Delta^{-1}[\Gamma^{-1}[V]] = (\Gamma \circ \Delta)^{-1}[V] \in e^*\mu(\Psi)$$

Thus, $\Gamma \circ \Delta$ is supra e^*-continuous. □

Theorem 4.4. *Let $\Delta : \Psi \to \Phi$ be a function and μ and η be the associated supra topologies with $\top$ and $\bot$, subsequently. Afterwards Δ is supra e^*-continuous if one of the following holds:*

(ι_1) $\Delta^{-1}\left[int^{\mu}_{e^*}(L)\right] \subseteq int\left(\Delta^{-1}[L]\right)$ *for each* $L \subseteq \Phi$.

(ι_2) $cl\left(\Delta^{-1}[L]\right) \subseteq \Delta^{-1}\left[cl^{\mu}_{e^*}(L)\right]$ *for each* $L \subseteq \Phi$.

(ι_3) $\Delta[cl(E)] \subseteq cl^{\mu}_{e^*}(\Delta[E])$ *for each* $E \subseteq \Psi$.

Proof. (ι_1) : Let $L \in O(\Phi)$.

$$\left.\begin{array}{r} L \in O(\Phi) \Rightarrow L \subseteq \Phi \\ \text{Hypothesis} \end{array}\right\} \Rightarrow \Delta^{-1}\left[int^{\mu}_{e^*}(L)\right] \subseteq int\left(\Delta^{-1}[L]\right) \Rightarrow \Delta^{-1}[L] \subseteq int\left(\Delta^{-1}[L]\right)$$

$$\left.\begin{array}{r} \Rightarrow \Delta^{-1}[L] \in O(\Psi) \\ O(\Psi) \subseteq e^*\mu(\Psi) \end{array}\right\} \Rightarrow \Delta^{-1}[L] \in e^*\mu(\Psi)$$

Thus, Δ is supra e^*-continuous.

(ι_2) : Let $L \in O(\Phi)$.

$$\left.\begin{array}{r} L \in O(\Phi) \Rightarrow \Phi \setminus L \subseteq \Phi \\ \text{Hypothesis} \end{array}\right\} \Rightarrow cl\left(\Delta^{-1}[\Phi \setminus L]\right) \subseteq \Delta^{-1}\left[cl^{\mu}_{e^*}(\Phi \setminus L)\right]$$

$$\Rightarrow \Psi \setminus int\left(\Delta^{-1}[L]\right) \subseteq \Psi \setminus \Delta^{-1}\left[int^{\mu}_{e^*}(L)\right]$$

$$\Rightarrow \Delta^{-1}\left[int^{\mu}_{e^*}(L)\right] \subseteq int\left(\Delta^{-1}[L]\right)$$

This condition is the same as (ι_1). Thus, Δ is supra e^*-continuous.

(ι_3) : Let $E \in O(\Phi)$.

$$\left.\begin{array}{r} E \in O(\Phi) \Rightarrow \Delta^{-1}[E] \subseteq \Psi \\ \text{Hypothesis} \end{array}\right\} \Rightarrow \Delta[cl(\Delta^{-1}[E])] \subseteq cl^{\mu}_{e^*}(\Delta[\Delta^{-1}[E]])$$

$$\Rightarrow \Delta[cl(\Delta^{-1}[E])] \subseteq cl^{\mu}_{e^*}(E)$$

$$\Rightarrow \Delta^{-1}[\Delta[cl(\Delta^{-1}[E])]] \subseteq \Delta^{-1}[cl^{\mu}_{e^*}(E)]$$

$$\Rightarrow cl(\Delta^{-1}[E]) \subseteq \Delta^{-1}[cl^{\mu}_{e^*}(E)]$$

This condition is the same as (ι_2). Thus, Δ is supra e^*-continuous. □

5. Supra e^*-open Functions and Supra e^*-closed Functions

Definition 5.1. *A function $\Delta : \Psi \to \Phi$ is called supra e^*-open (resp. supra e^*-closed) if for each open (resp. closed) set F of Ψ, $\Delta[F]$ is s.e^*.o. (resp. s.e^*.c.) in Φ.*

Theorem 5.1. *A function $\Delta : \Psi \to \Phi$ is supra e^*-open iff $\Delta[int(A)] \subseteq int^{\mu}_{e^*}(\Delta[A])$ for each set A in Ψ.*

Proof. Necessity. Let $A \subseteq \Psi$ and suppose that Δ is supra e^*-open.

$$\left.\begin{array}{r} A \subseteq \Psi \Rightarrow A \supseteq int(A) \in O(\Psi) \\ \Delta \text{ is supra } e^*\text{-open} \end{array}\right\} \Rightarrow \left.\begin{array}{r} \Delta[A] \supseteq \Delta[int(A)] \in e^*\mu(\Phi) \\ int^{\mu}_{e^*}(\Delta[A]) = \cup\{B|(B \subseteq \Delta[A])(B \in e^*\mu(\Psi))\} \end{array}\right\} \Rightarrow$$

$\Rightarrow \Delta[int(A)] \subseteq int^{\mu}_{e^*}(\Delta[A])$.

Sufficiency. Suppose that $\Delta[int(A)] \subseteq int^{\mu}_{e^*}(\Delta[A])$ for each set $A \in \Psi$.

$$\left.\begin{array}{c} A \in O(\Psi) \\ \text{Hypothesis} \end{array}\right\} \Rightarrow \Delta[A] \subseteq \Delta[int(A)] \subseteq int^{\mu}_{e^*}(\Delta[A]) \Rightarrow \Delta[A] \in e^*\mu(\Phi)$$

Hence, Δ is supra e^*-open. □

Theorem 5.2. *A function $\Delta : \Psi \to \Phi$ is supra e^*-closed iff $cl^{\mu}_{e^*}(\Delta[A]) \subseteq \Delta[cl(A)]$ for all set A in Ψ.*

Proof. It is obvious from the Theorem 5.1. □

Theorem 5.3. *Let $\Delta : \Psi \to \Phi$ and $\Gamma : \Phi \to \zeta$ be two functions. Then the following properties hold:*

(ι_1) *Whenever $\Gamma \circ \Delta$ is supra e^*-open and Δ is continuous surjective, afterwards Γ is supra e^*-open.*

(ι_2) *Whenever $\Gamma \circ \Delta$ is open and Γ is e^*-continuous injective, afterwards Δ is supra e^*-open.*

Proof. (ι_1) : Let $U \in O(\Phi)$.

$$\left.\begin{array}{r} U \in O(\Phi) \\ \Delta \text{ is continuous} \end{array}\right\} \Rightarrow \left.\begin{array}{r} \Delta^{-1}[U] \in O(\Psi) \\ \Gamma \circ \Delta \text{ is supra } e^*\text{-open} \end{array}\right\} \Rightarrow$$

$\Rightarrow (\Gamma \circ \Delta)[\Delta^{-1}[V]] = \Gamma[\Delta[\Delta^{-1}[V]]] \overset{\Delta \text{ is surj.}}{=} \Gamma[V] \in e^*\mu(\zeta)$

Hence, Γ is supra e^*-open.

(ι_2) : Let $U \in O(\Psi)$.

$$\left.\begin{array}{r} U \in O(\Psi) \\ \Gamma \circ \Delta \text{ is open} \end{array}\right\} \Rightarrow \left.\begin{array}{r} (\Gamma \circ \Delta)[U] \in O(\zeta) \\ \Gamma \text{ is } e^*\text{-continuous} \end{array}\right\} \Rightarrow$$

$\Rightarrow \Gamma^{-1}[(\Gamma \circ \Delta)[V]] = \Gamma^{-1}[\Gamma[\Delta[V]]] \overset{\Gamma \text{ is inj.}}{=} \Delta[V] \in e^*\mu(\Phi)$

Hence, Δ is supra e^*-open. □

Theorem 5.4. *Let $\Delta : \Psi \to \Phi$ be a bijection. Then the following functions are equivalent:*

(ι_1) *Δ is a s.e^*.o.;*

(ι_2) *Δ is a s.e*.c.;*

(ι_3) *Δ^{-1} is a s.e*.c.*

Proof. $(\iota_1) \Rightarrow (\iota_2)$: Obvious.

$(\iota_2) \Rightarrow (\iota_3)$: Let $F \in C(\Psi)$.

$$\left.\begin{array}{r} F \in C(\Psi) \\ \Delta \text{ is supra } e^*\text{-closed} \end{array}\right\} \Rightarrow \Delta[F] \overset{\Delta \text{ is bij.}}{=} (\Delta^{-1})^{-1}[F] \in e^*\mu^c(\Phi)$$

By Theorem 4.2(ι_2) Δ^{-1} is supra e^*-continuous.

$(\iota_3) \Rightarrow (\iota_1)$: Let $F \in O(\Psi)$.

$$\left.\begin{array}{r} F \in O(\Psi) \\ \Delta^{-1} \text{ is supra } e^*\text{-continuous} \end{array}\right\} \Rightarrow (\Delta^{-1})^{-1}[F] \overset{\Delta \text{ is bij.}}{=} \Delta[F] \in e^*\mu(\Phi)$$

Hence, Δ is supra e^*-open. □

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