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INEXTENSIBLE FLOWS OF CURVES WITH QUASI-FRAME IN 3-DIMENSIONAL GALILEAN SPACE G_3

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ABSTRACT. In this study we research inextensible flows of curves in 3-dimensional Galilean space G_3 with a new aspect. For this research we use a new adapted frame which called quasi-frame in 3-dimensional Galilean space G_3. From this perspective, inextensible curve flows are examined with the help of this frame then important characterizations and results are obtained.

Keywords: Galilean space, inextensible flows of curves, quasi frame.

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1. Introduction

The theory of curves is one of the most intriguing and thoroughly studied topics in differential geometry. Additionally, curve flows, which determine the evolution of curves or surfaces, are crucial to this theory. In this case, the curve's flow can be used to analyze the change in the curve. It is argued that a curve has an inextensible flow if the arc length is preserved. In addition to structural mechanics [19], computer vision [9, 14], and computer animation [3] all use the inextensible flows of curves and surfaces. The techniques researched in this paper are produced by Gage and Hamilton [7] and Grayson [8]. Kwon and Park offered a thorough description of the differences between heat flows and inextensible flows of planar curves [12]. Furthermore, in $\mathbb{R}^3$, Kwon et al. reveals a general formulation for developable surfaces and

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inextensible flows of curves [11]. Latifi and Razavi examined inextensible flows of curves in Minkowski 3-space [13]. Inextensible flows of curves were analyzed by Ögrenmis and Yeneroğlu [15] in the three-dimensional Galilean space G_3 and by Öztekin and Gün Bozok [16] in the four-dimensional Galilean space.

In the literature, computations have often been performed using the Frenet frame. Nevertheless, in certain situations, the Frenet frame has drawbacks. For instance, it is impossible to define the Frenet frame when the second derivative is zero. An alternative frame can therefore be defined in this situation. The frame known as quasi frame or q-frame is one of these alternative frames. Using the quasi-normal vector established by Coquillart in 1987 [1], a q-frame was obtained. This frame's principal concept is that the projection and tangent vectors are multiplied to obtain the vector known as the quasi-normal vector. Using a quasi-normal vector along a space curve, Dede et al. defined a new frame known as the q-frame [2]. With the help of these definitions, the quasi frame has been examined for different curves in many different spaces [5, 6, 10, 18].

In this research paper, with the help of the quasi frame inextensible flows of curves are researched in 3-dimensional Galilean space G_3. In this regards, new characterizations and important results have been obtained for inextensible curve flows.

2. Preliminaries

The Galilean space is one of the Cayley-Klein spaces with the projective metric with the signature (0,0,+,+). In 3-dimensional Galilean space denoted as G_3 the scalar product is descripted by

$$\langle w_1, w_2\rangle = \begin{cases} x_1x_2 & , if\ x_1 \neq 0 \vee x_2 \neq 0 \\ y_1y_2 + z_1z_2 & , if\ x_1 = 0 \wedge x_2 = 0 \end{cases} \tag{2.1}$$

where $w_1 = (x_1, y_1, z_1)$ and $w_2 = (x_2, y_2, z_2)$. Consideringly for a vector $w = (x, y, z)$ the Galilean norm can be expressed by

$$\|w\| = \begin{cases} x & , \ if\, x \neq 0 \\ \sqrt{y^2 + z^2} & , if\, x = 0 \end{cases} . \tag{2.2}$$

For an admissible curve C of the class C^r $(r \geq 3)$ in G_3 the following characterization can be defined

$$r = r\left(s, y\left(s\right), z\left(s\right)\right), \tag{2.3}$$

here s is the arc length on C. Also for this curve the curvature and torsion can be represented as

$$\kappa(s)=\sqrt{y''^2+z''^2} \text{ and } \tau(s)=\frac{1}{\kappa^2(s)}\det\left(r'(s),r''(s),r'''(s)\right). \tag{2.4}$$

The orthonormal trihedron is expressed by

$$\begin{aligned} T(s) &= \left(1,y'(s),z'(s)\right), \\ N(s) &= \frac{1}{\kappa(s)}\left(0,y''(s),z''(s)\right), \\ B(s) &= \frac{1}{\kappa(s)}\left(0,-z''(s),y''(s)\right), \end{aligned} \tag{2.5}$$

where t,n,b are the tangent, principal normal and binormal vectors, respectively. Moreover, the Frenet formulas can be given by,

$$\begin{aligned} T'(s) &= \kappa(s)N(s), \\ N'(s) &= \tau(s)B(s), \\ B'(s) &= -\tau(s)N(s). \end{aligned} \tag{2.6}$$

For detailed information about Galilean space we refer to [20, 17].

In 3-dimensional Galilean space the quasi-frame, which is crucial to a variety of geometric computations is derived from Frenet-Serret frame of the curve and can be described by $\{T(s),N_q(s),B_q(s)\}$ as;

$$T=\frac{\alpha'}{\|\alpha'\|},\quad N_q=\frac{T\times z}{\|T\times z\|},\quad B_q=T\times N_q, \tag{2.7}$$

where z is the projection vector given by either $(1,0,0)$, $(0,1,0)$ or $(0,0,1)$. The parallelism respect to unit tangent vector T determines the choice of the projection vector z. Here it is selected $z=(1,0,0)$. Let $\theta(s)$ is an angle between N and N_q then the quasi-frame, known as $\{T(s),N_q(s),B_q(s)\}$ can be written

$$N_q=\cos\theta N+\sin\theta B, \tag{2.8}$$

$$B_q=-\sin\theta N+\cos\theta B, \tag{2.9}$$

and

$$N=\cos\theta N_q-\sin\theta B_q, \tag{2.10}$$

$$B=\sin\theta N_q+\cos\theta B_q. \tag{2.11}$$

Also using the equations (2.6) and (2.10), it is obtained that

$$T' = \kappa N = \kappa \cos\theta N_q - \kappa \sin\theta B_q. \tag{2.12}$$

By using the replacement $K_1 = \kappa \cos\theta$ and $K_2 = \kappa \sin\theta$, the following equation can be found,

$$T' = K_1 N_q - K_2 B_q. \tag{2.13}$$

In the same way considering the equations (2.8) and (2.9), it is determined that

$$N_q' = K_3 B_q, \quad B_q' = -K_3 N_q, \tag{2.14}$$

where $\theta' + \tau = K_3$. Therefore, the quasi-formulas are given by

$$\begin{aligned} T' &= K_1 N_q - K_2 B_q, \\ N_q' &= K_3 B_q, \\ B_q' &= -K_3 N_q. \end{aligned} \tag{2.15}$$

Consequently the quasi-curvatures K_1, K_2 and K_3 can be represented as

$$K_1 = \kappa \cos\theta, \quad K_2 = \kappa \sin\theta, \quad K_3 = \theta' + \tau, \tag{2.16}$$

where κ, τ are curvature and torsion, respectively [4].

Corollary 2.1. *Let $\alpha(s)$ be a curve in G_3. The quasi-curvatures K_1, K_2 and K_3 can be given, respectively, by* [4]

$$K_1 = g(T', N_q), K_2 = -g(T', B_q), K_3 = g(N_q', B_q) = -g(B_q', N_q). \tag{2.17}$$

Corollary 2.2. *The quasi-frame in the value of G_3, is a generalization of the Frenet frame. To be more precise, the quasi-frame and the Frenet frame are equal when K_2 equals zero* [4].

Example 2.1. *Let $\beta : I \longrightarrow G_3$ be a curve defined as*

$$\beta(s) = (s, \sin 3s, \cos 3s).$$

Then the quasi frame of β is

$$\begin{aligned} T &= (1, 3\cos 3s, -3\sin 3s), \\ N_q &= (0, -\sin 3s, -\cos 3s), \\ B_q &= (0, \cos 3s, -\sin 3s), \end{aligned}$$

and quasi-curvatures are

$$K_1 = 9 \ K_2 = 0 \ K_3 = -3.$$

3. Inextensible Flows of Curves according to Quasi-Frame in G_3

According to this research, $\beta : [0,l] \times [0,w) \longrightarrow G_3$ is considered as a one parameter family of smooth curves in 3-dimensional Galilean space G_3 where l is the arc length of the initial curve. Moreover, u is the curve parameterization variable where $0 \leq u \leq l$. If the speed of the curve β is denoted as $v = \left|\frac{\partial \beta}{\partial u}\right|$, then the arc length of β can be represented as

$$s(u) = \int_0^u \left|\frac{\partial \beta}{\partial u}\right| du\,, \tag{3.18}$$

and $\frac{\partial}{\partial s}$ can be expressed by

$$\frac{\partial}{\partial s} = \frac{1}{v}\frac{\partial}{\partial u}\,,$$

here the arc length parameter is $ds = vdu$. For any flow in G_3 the following equation can be written

$$\frac{\partial \beta}{\partial t} = f_1 T + f_2 N_q + f_3 B_q, \tag{3.19}$$

where $\{T, N_q, B_q\}$ is quasi-frame in G_3. The arc length variation is given by

$$s(u,t) = \int_0^u vdu\ .$$

In G_3, the requirement that the curve not be subject to either elongation or compression can be given as the following condition

$$\frac{\partial}{\partial t} s(u,t) = \int_0^u \frac{\partial v}{\partial t} du = 0\,, \tag{3.20}$$

for $u \in [0,l]$ [15].

Definition 3.1. *A curve evolution $\beta(u,t)$ and its flow $\frac{\partial \beta}{\partial t}$ in G_3 are called inextensible if the following equation is satisfied* [15],

$$\frac{\partial}{\partial t}\left|\frac{\partial \beta}{\partial u}\right| = 0.$$

Lemma 3.1. *Let $\frac{\partial \beta}{\partial t} = f_1 T + f_2 N_q + f_3 B_q$ be a smooth flow of the curve β where $\{T, N_q, B_q\}$ is a quasi-frame in G_3. The flow is inextensible then*

$$\frac{\partial v}{\partial t} = \frac{\partial f_1}{\partial u}. \tag{3.21}$$

Proof. Let $\frac{\partial \beta}{\partial t}$ be a smooth flow of the curve β in G_3. Using the definition of β, we reach

$$v^2 = \left\langle \frac{\partial \beta}{\partial u}, \frac{\partial \beta}{\partial u} \right\rangle. \tag{3.22}$$

Since $\dfrac{\partial}{\partial u}$ and $\dfrac{\partial}{\partial t}$ are commute, we get

$$v\frac{\partial v}{\partial t}=\left\langle \frac{\partial \beta}{\partial u},\frac{\partial}{\partial u}\left(f_1T+f_2N_q+f_3B_q\right)\right\rangle. \tag{3.23}$$

Substituting (2.15) in (3.23) we reach

$$\begin{aligned}\frac{\partial v}{\partial t} &= \left\langle T,\left(\frac{\partial f_1}{\partial u}\right)T+\left(\frac{\partial f_2}{\partial u}+f_1vK_1-f_3vK_3\right)N_q\right.\\ &+ \left.\left(\frac{\partial f_3}{\partial u}-f_1vK_2+f_2vK_3\right)B_q\right\rangle .\end{aligned}$$

If necessary calculations are done then the equation (3.21) can be obtained easily. □

Theorem 3.1. *Let* $\dfrac{\partial \beta}{\partial t}=f_1T+f_2N_q+f_3B_q$ *be a smooth flow of the curve* β *in* G_3*. The flow is inextensible if and only if*

$$\frac{\partial f_1}{\partial s}=0. \tag{3.24}$$

Proof. Considering the equations (3.20) and (3.21) we have

$$\frac{\partial}{\partial t}s\left(u,t\right)=\int_0^u\frac{\partial v}{\partial t}du=\int_0^u\frac{\partial f_1}{\partial u}=0. \tag{3.25}$$

The proof can be finished by reversing the argument to demonstrate sufficiency. Therefore, the desired result is obtained. □

We now limit ourselves to parametrized curves with arc length. In other words, $v=1$ and the local coordinate u corresponds to s which is the arc length of the curve. Therefore the following lemma can be given;

Lemma 3.2. *Let* $\dfrac{\partial \beta}{\partial t}=f_1T+f_2N_q+f_3B_q$ *be a smooth flow of the curve* β *in* G_3*. If the flow is inextensible then,*

$$\begin{aligned}\frac{\partial T}{\partial t} &= \left(\frac{\partial f_2}{\partial s}+f_1K_1-f_3K_3\right)N_q+\left(\frac{\partial f_3}{\partial s}+f_2K_3-f_1K_2\right)B_q,\\ \frac{\partial N_q}{\partial t} &= \psi B_q,\\ \frac{\partial B_q}{\partial t} &= -\psi N_q,\end{aligned} \tag{3.26}$$

where $\psi=\left\langle \frac{\partial N_q}{\partial t},B_q\right\rangle$.

Proof. Since $\dfrac{\partial}{\partial t}$ and $\dfrac{\partial}{\partial s}$ are commute we get

$$\frac{\partial T}{\partial t}=\frac{\partial}{\partial t}\frac{\partial \beta}{\partial s}=\frac{\partial}{\partial s}\left(f_1T+f_2N_q+f_3B_q\right).$$

Thus, it can be seen that

$$\begin{aligned} \frac{\partial T}{\partial t} &= \left(\frac{\partial f_1}{\partial s}\right) T + \left(\frac{\partial f_2}{\partial s} + f_1 K_1 - f_3 K_3\right) N_q \\ &+ \left(\frac{\partial f_3}{\partial s} + f_2 K_3 - f_1 K_2\right) B_q. \end{aligned} \tag{3.27}$$

Substitute the equation (3.24) in (3.27), we find

$$\frac{\partial T}{\partial t} = \left(\frac{\partial f_2}{\partial s} + f_1 K_1 - f_3 K_3\right) N_q + \left(\frac{\partial f_3}{\partial s} + f_2 K_3 - f_1 K_2\right) B_q.$$

Let us differentiate the quasi-frame with respect to t as

$$\begin{aligned} 0 &= \frac{\partial}{\partial t} \langle T, N_q \rangle = \left(\frac{\partial f_2}{\partial s} + f_1 K_1 - f_3 K_3\right) + \left\langle T, \frac{\partial N_q}{\partial t} \right\rangle, \\ 0 &= \frac{\partial}{\partial t} \langle T, B_q \rangle = \left(\frac{\partial f_3}{\partial s} + f_2 K_3 - f_1 K_2\right) + \left\langle T, \frac{\partial B_q}{\partial t} \right\rangle, \\ 0 &= \frac{\partial}{\partial t} \langle N_q, B_q \rangle = \left\langle \frac{\partial N_q}{\partial t}, B_q \right\rangle + \left\langle N_q, \frac{\partial B_q}{\partial t} \right\rangle. \end{aligned}$$

Considering the above equation and the following equations

$$\left\langle \frac{\partial N_q}{\partial t}, N_q \right\rangle = \left\langle \frac{\partial B_q}{\partial t}, B_q \right\rangle = 0,$$

then, we obtain

$$\begin{aligned} \frac{\partial N_q}{\partial t} &= \psi B_q, \\ \frac{\partial B_q}{\partial t} &= -\psi N_q, \end{aligned}$$

where $\psi = \left\langle \frac{\partial N_q}{\partial t}, B_q \right\rangle$. □

Theorem 3.2. *Let* $\dfrac{\partial \beta}{\partial t} = f_1 T + f_2 N_q + f_3 B_q$ *be a smooth flow of the curve* β *in* G_3. *If the flow is inextensible, the following partial differential equation holds:*

$$\begin{aligned} \frac{\partial K_1}{\partial t} &= \frac{\partial}{\partial s}\left(\frac{\partial f_2}{\partial s} + f_1 K_1 - f_3 K_3\right) - \left(\frac{\partial f_3}{\partial s} + f_2 K_3 - f_1 K_2\right) K_3 - K_2 \psi, \\ \frac{\partial K_2}{\partial t} &= K_1 \psi - \frac{\partial}{\partial s}\left(\frac{\partial f_3}{\partial s} + f_2 K_3 - f_1 K_2\right) K_3 - \left(\frac{\partial f_2}{\partial s} + f_1 K_1 - f_3 K_3\right) K_3, \\ \frac{\partial K_3}{\partial t} &= \frac{\partial \psi}{\partial s}, \end{aligned}$$

where $\psi = \left\langle \frac{\partial N_q}{\partial t}, B_q \right\rangle$.

Proof.

$$\begin{aligned}
\frac{\partial}{\partial s}\frac{\partial T}{\partial t} &= \frac{\partial}{\partial s}\left[\left(\frac{\partial f_2}{\partial s}+f_1K_1-f_3K_3\right)N_q+\left(\frac{\partial f_3}{\partial s}+f_2K_3-f_1K_2\right)B_q\right], \\
&= \frac{\partial}{\partial s}\left[\left(\frac{\partial f_2}{\partial s}+f_1K_1-f_3K_3\right)\right]N_q+\left(\frac{\partial f_2}{\partial s}+f_1K_1-f_3K_3\right)K_3B_q \\
&+ \frac{\partial}{\partial s}\left[\left(\frac{\partial f_3}{\partial s}+f_2K_3-f_1K_2\right)\right]B_q \\
&- \left(\frac{\partial f_3}{\partial s}+f_2K_3-f_1K_2\right)K_3N_q. \qquad (3.28)
\end{aligned}$$

Also, we have

$$\begin{aligned}
\frac{\partial}{\partial t}\frac{\partial T}{\partial s} &= \frac{\partial}{\partial t}\left(K_1N_q-K_2B_q\right), \qquad (3.29) \\
&= \frac{\partial K_1}{\partial t}N_q+K_1\psi B_q-\frac{\partial K_2}{\partial t}B_q+K_2\psi N_q,
\end{aligned}$$

where $\psi=\left\langle \frac{\partial N_q}{\partial t}, B_q\right\rangle$. Hence from (3.28) and (3.29), we get

$$\begin{aligned}
\frac{\partial K_1}{\partial t} &= \frac{\partial}{\partial s}\left(\frac{\partial f_2}{\partial s}+f_1K_1-f_3K_3\right)-\left(\frac{\partial f_3}{\partial s}+f_2K_3-f_1K_2\right)K_3-K_2\psi, \\
\frac{\partial K_2}{\partial t} &= K_1\psi-\frac{\partial}{\partial s}\left(\frac{\partial f_3}{\partial s}+f_2K_3-f_1K_2\right)-\left(\frac{\partial f_2}{\partial s}+f_1K_1-f_3K_3\right)K_3.
\end{aligned}$$

Also,

$$\begin{aligned}
\frac{\partial}{\partial s}\frac{\partial N_q}{\partial t} &= \frac{\partial}{\partial s}\left(\psi B_q\right) \\
&= \frac{\partial \psi}{\partial s}B_q-K_3\psi N_q. \qquad (3.30)
\end{aligned}$$

Also, we have

$$\begin{aligned}
\frac{\partial}{\partial t}\frac{\partial N_q}{\partial s} &= \frac{\partial}{\partial t}\left(K_3B_q\right) \\
&= \frac{\partial K_3}{\partial t}B_q-K_3\psi N_q. \qquad (3.31)
\end{aligned}$$

Hence from (3.30) and (3.31), we get

$$\frac{\partial K_3}{\partial t}=\frac{\partial \psi}{\partial s}. \qquad (3.32)$$

□

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