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ON RULED SURFACES BY SMARANDACHE GEOMETRY IN E^3

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Abstract. The paper introduces a series of new ruled surfaces by following the idea of Smarandache geometry according to Frenet frame by taking into account all the possible linear combinations of the frame vectors. The metric properties of each defined ruled surface is examined by computing the 1^{st} and 2^{nd} fundamental forms as well as the curvatures of Gaussian and the mean expressed by the harmonic curvature function. Therefore, the conditions for each surface to be minimal or developable are provided. Moreover, the constraints for the characteristics of the base curve are discussed whether it is geodesic, asymptotic or a curvature line on the generated ruled surface. Finally, the graphical illustrations are presented for each ruled surface with a given appropriate example.

Keywords: Smarandache geometry, Ruled surfaces, Fundamental forms, Principal curvatures, Developable and minimal surfaces, Geodesic, Asymptotic and curvature lines.

2020 Mathematics Subject Classification: 53A04, 53A05.

1. Introduction

Ruled surfaces are widely recognized as the most fundamental and extensively employed objects in the geometric modeling. Researchers utilize this type of surface in various grounds such as computer graphics, architecture, arts, sculpture, manufacturing, etc. The basic definition of a ruled surface is the image of lines' motion on and along a given curve. Therefore,

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a ruled surface is also called the surface of infinitely many lines. Interested readers can refer the main sources [1, 4, 19] to gain a deep insight into about ruled surfaces. Following the importance of this surface kind, researchers defined new ruled surfaces and examined their characterizations. For example, the ruled surfaces were re-visited by means of geodesic curvature and the 2^{nd} fundamental form by [14]. [12] conducted a study on the characteristics of ruled surfaces along the striction curves of a non-cylindrical ruled surface according to Frenet frame. The ruled surfaces according to Bishop frame and their characteristics was examined by [20] and [8]. Further, [13] studied the characteristics for the ruled surfaces with respect to the alternative frame. Moreover, the ruled surfaces generated by rotation minimizing frame (RMF) are investigated in [6] while those by Sannia frame are defined in [5].

Recently, [11] put forth a new way of generating ruled surfaces by taking the advantage of the idea of Smarandache curves which was defined by [21] for Minkowski space, and by [2] for Euclidean space E^3. The method relies on assigning one of the Smarandache curves as a base curve of the surface and utilizes other vector elements of the Frenet frame as the generator line. Thus, she named these newly constructed ruled surfaces as Smarandache ruled surfaces. However, prior to her study, [24] had also discussed the idea of constructing such ruled surfaces. They specifically worked on geodesic conditions of the tangent and normal surfaces with TN-Smarandache curve as a base curve [23]. In addition, the authors examined the geodesics of the binormal surface in [25]. By considering different frames such as alternative, Darboux, Flc (by [3]) and successor frame, Smarandache ruled surfaces were re-defined in [9], [10], [15] and [22], respectively. Moreover, this way of generating such ruled surfaces were benefited in [16, 17, 18] from different point of view most likely by incorporating the Darboux vector. Therefore, in this paper, with the motivation of the given studies, we extend our investigations for the studies in which Smarandache ruled surfaces according to Frenet frame were used. That is we consider all possible combinations of Frenet vectors to construct new ruled surfaces and examine their main characteristics in a more broader perspective.

2. Preliminaries

This section is to recall the primary concepts which we will be using through out the paper. Let $\gamma : s \in I \subset \Re \to E^3$ be a regular unit speed curve in three dimensional Euclidean space E^3 and denote $\{T, N, B, \kappa, \tau\}$ as its Frenet elements. Then, the definitions of Frenet vectors and the well-known Frenet formulas are given as

$$\begin{aligned} T&=\frac{\gamma'}{\|\gamma'\|}, \quad N=B\times T, \quad B=\frac{\gamma'\times\gamma''}{\|\gamma'\times\gamma''\|},\\ \kappa&=\frac{\|\gamma'\times\gamma''\|}{\|\gamma'\|^3}, \qquad \tau=\frac{\langle\gamma'\times\gamma'',\gamma'''\rangle}{\|\gamma'\times\gamma''\|^2},\\ T'&=\kappa\nu N, \quad N'=-\kappa\nu T+\tau\nu B, \quad B'=-\tau\nu N, \end{aligned} \tag{2.1}$$

where $\nu=\|\gamma'\|$, and $\langle\ ,\ \rangle$, $\times$ and $\|\ \|$ denote standard inner product, vector product and norm, respectively. κ is the curvature and τ is the torsion of the curve $\gamma(s)$ [1, 4]. Further, a surface χ is ruled if it is formed with the motion of line $X(s)$ on and along a given curve $\gamma(s)$. Thus, a parametrization for a ruled surface is given as follows

$$\chi:\ \psi(s,v)=\gamma(s)+vX(s). \tag{2.2}$$

Here the curve $\gamma(s)$ is called as the base while $X(s)$ is the ruling. The normal vector field for the surface χ is computed by

$$\vec{n}_\psi=\frac{\psi_s\times\psi_v}{\|\psi_s\times\psi_v\|}. \tag{2.3}$$

The 1^{st} and 2^{nd} fundamental forms and the curvatures of Gaussian and mean for the given ruled surface χ are given by

$$I=Eds^2+2Fdsdv+dv^2, \qquad II=Lds^2+2Mdsdv, \tag{2.4}$$

$$K=\frac{-M^2}{E-F^2}, \qquad H=\frac{L-2FM}{2(E-F^2)}, \tag{2.5}$$

where the given coefficients are obtained by the following expressions

$$\begin{aligned} E&=\langle\psi_s,\psi_s\rangle, \qquad F=\langle\psi_s,\psi_v\rangle, \qquad G=\langle\psi_v,\psi_v\rangle,\\ L&=\langle\psi_{ss},\vec{n}_\psi\rangle, \quad M=\langle\psi_{sv},\vec{n}_\psi\rangle, \quad N=\langle\psi_{vv},\vec{n}_\psi\rangle. \end{aligned} \tag{2.6}$$

Note that for a ruled surfaces having the ruling of a unit vector, it is always valid that $G=1$ and $N=0$. Therefore, the fundamental forms and the curvatures are expressed in their simplified forms as in the Equation (2.4) and (2.5). Regarding to the given invariants of the surface χ, there exist following definitions:

Definition 2.1. *[1, 4, 19] The surface χ is developable (resp. minimal) if Gaussian (resp. mean) curvature vanishes that is $K=0$ (resp. $H=0$) .*

Moreover, the normal curvature κ_n, the geodesic curvature κ_g and the geodesic torsion τ_g of the surface χ are defined as follows

$$\kappa_n=\langle\gamma'',n_\psi\rangle, \qquad \kappa_g=\langle n_\psi\times T,T'\rangle, \qquad \tau_g=\langle n_\psi\times n_\psi',T'\rangle. \tag{2.7}$$

There also exist following definitions for given expressions above

Definition 2.2. *[1, 4, 19]*

- $\gamma(s)$ *is asymptotic on the surface* $\chi \iff \kappa_n = 0$,
- $\gamma(s)$ *is geodesic on the surface* $\chi \iff \kappa_g = 0$,
- $\gamma(s)$ *is principal line on the surface* $\chi \iff \tau_g = 0$.

Additionally, we recall the following theorems considered to the specific cases of which the curve $\gamma(s)$ is general or slant helix.

Theorem 2.1. *(Lancret's theorem) A curve is called a general helix* $\iff h = const.$, *where* $h = \dfrac{\tau}{\kappa}$ *is the harmonic curvature function [19, 26].*

Theorem 2.2. *A curve is called a slant helix* $\iff \sigma = const.$ *where* $\sigma = \dfrac{h'}{\kappa\left(1+h^2\right)^{\frac{3}{2}}}$, *[7].*

3. The Smarandache Based Ruled Surfaces according to Frenet Frame in E^3

In this section of the paper, it is of interest for us to extend the study of [11] and examine ruled surfaces formed by other possible combinations of Frenet vectors. In addition, the conditions for the base curve to be asymptotic, geodesic and curvature line on the constructed surface are provided, as well. Some special cases are discussed regarding to that the main curve is a general or a slant helix. Thus, twelve of new ruled surfaces all formed by the vectors of Frenet frame are studied by following the idea of Smarandache geometry.

3.1. **Ruled Surfaces with the Base TN− Smarandache Curve.**

Definition 3.1. *Let* $\gamma : s \in I \subset \Re \to E^3$ *be a regular unit speed curve of* C^2 *class and* $\{T,\ N,\ B\}$ *denotes the set of its Frenet vectors. The original definition of* $TN-$ *Smarandache ruled surface introduced in [11] is given as following*

$$ {}^{TN}_{B}\psi(s,v) = \frac{T(s)+N(s)}{\sqrt{2}} + vB(s). \tag{3.8}$$

However, in this study, other two ruled surfaces having the same $TN-$ *curve as a base curve with other Frenet vector as a ruling are considered, which are parameterized in the following*

$$\begin{aligned} {}^{TN}_{T}\psi(s,v) =& \frac{T(s)+N(s)}{\sqrt{2}} + vT(s), \\ {}^{TN}_{N}\psi(s,v) =& \frac{T(s)+N(s)}{\sqrt{2}} + vN(s). \end{aligned} \tag{3.9}$$

In addition, we also examine the conditions for the base curve to be asymptotic, geodesic and curvature line on each ruled surface.

3.1.1. *The characteristics of the ruled surface* ${}_{B}^{TN}\psi(s,v)$.

[11] put forth the following corollaries for the surface ${}_{B}^{TN}\psi(s,v)$

Corollary 3.1. *[11]*

- ${}_{B}^{TN}\psi(s,v)$ *is developable when* $\gamma(s)$ *is a plane curve,*
- ${}_{B}^{TN}\psi(s,v)$ *is minimal if the following relation holds* $\kappa\tau^2\left(1-2v^2\right)+\left(\kappa'+2\kappa^2\right)\left(\sqrt{2}v\tau-\kappa\right)=0.$

Remark 3.1. *We note that in order for* ${}_{B}^{TN}\psi(s,v)$ *be minimal, the following relation should hold*

$$\kappa\tau^2\left(1-2v^2\right)+\sqrt{2}\tau v\left(2\kappa^2+\kappa'\right)-\kappa\left(\sqrt{2}\tau' v+2\kappa^2\right)=0.$$

Now, let us consider the characteristic of the base curve on the ${}_{B}^{TN}\psi(s,v)$, and examine the conditions for it be asymptotic, geodesic and curvature line by associating to $\gamma(s)$.

Theorem 3.1. *The normal curvature* ${}_{B}^{TN}\kappa_n$*, the geodesic curvature* ${}_{B}^{TN}\kappa_g$ *and the geodesic torsion* ${}_{B}^{TN}\tau_g$ *of the* ${}_{B}^{TN}\psi(s,v)$ *surface are given as follows*

$$\begin{aligned}
{}_{B}^{TN}\kappa_n &= \frac{\sqrt{2}hv\left(\kappa^2+\kappa'\right)-\tau^2-2\kappa^2}{\sqrt{2}\sqrt{\left(\sqrt{2}hv-1\right)^2+1}},\\
{}_{B}^{TN}\kappa_g &= \frac{\sqrt{2}h'\left(\sqrt{2}-hv\right)-\tau\left(h^2+2\right)\left(\sqrt{2}hv-1\right)}{\left(h^2+2\right)\sqrt{\left(\sqrt{2}hv-1\right)^2+1}},\\
{}_{B}^{TN}\tau_g &= \frac{2vh'\left(h'\sqrt{2}+2\tau hv\right)+\left(\left(\sqrt{2}hv-1\right)\left(h^2+2\right)\tau\kappa-2h'\kappa\right)\left(\sqrt{2}hv-h^2-2\right)}{\left(\left(\sqrt{2}hv-1\right)^2+1\right)\left(h^2+2\right)^{\frac{3}{2}}},
\end{aligned} \tag{3.10}$$

respectively.

Proof. By referring the Equation (2.1), the tangent T_{TN} of $TN-$ Smarandache curve, its derivative and the second order derivative of $TN-$ Smarandache curve are given as

$$\begin{aligned}
T_{TN} &= \frac{-T+N+hB}{\sqrt{h^2+2}},\\
T'_{TN} &= \frac{\left(h'h-\tau h-2\kappa\right)T-\left(h'h+2\kappa+\left(h^2+3\right)\tau h\right)N+\left(2h'+\tau\left(h^2+2\right)\right)B}{\left(h^2+2\right)^{\frac{3}{2}}},\\
\left(\frac{T+N}{\sqrt{2}}\right)'' &= -\frac{\left(\kappa'+\kappa^2\right)T-\left(\kappa'-\kappa^2-\tau^2\right)N-\left(\tau'+\kappa\tau\right)B}{\sqrt{2}}.
\end{aligned} \tag{3.11}$$

Moreover, the derivative of the normal vector of ${}_{B}^{TN}\phi(s,v)$ defined as ${}_{B}^{TN}n=\dfrac{\left(\kappa-v\tau\sqrt{2}\right)T+\kappa N}{\sqrt{2}\sqrt{\kappa^2-\sqrt{2}\kappa v\tau+v^2\tau^2}}$ in [11], but expressed by the harmonic curvature function as ${}^{TN}_{B}n = -\dfrac{\left(\sqrt{2}hv-1\right)T-N}{\sqrt{\left(\sqrt{2}hv-1\right)^2+1}}$ is given in the following

$$
\begin{aligned}
{}^{TN}_{B}n' = &-\left(\frac{h'v+v\tau\left(\sqrt{2}hv-2\right)+\sqrt{2}\kappa}{2\left(h^2v^2-\sqrt{2}hv+1\right)^{\frac{3}{2}}}\right)T \\
&-\left(\frac{h'v\left(\sqrt{2}hv-1\right)}{2\left(h^2v^2-\sqrt{2}hv+1\right)^{\frac{3}{2}}}+\frac{\kappa\left(2hv-\sqrt{2}\right)}{2\sqrt{h^2v^2-\sqrt{2}hv+1}}\right)N \\
&+\left(\frac{\tau\sqrt{2}}{2\sqrt{h^2v^2-\sqrt{2}hv+1}}\right)B.
\end{aligned}
$$

By substituting the relations given above into Equation 2.7, the proof is completed. ☐

Without the need for proof, from Definition 2.2 and the Equation 3.10 of Theorem 3.1, we have the following corollary:

Corollary 3.2.

- *The $TN-$ Smarandache curve of $\gamma(s)$ cannot be asymptotic on ${}^{TN}_{B}\psi(s,v)$.*
- *If $\gamma(s)$ is plane curve, then its corresponding $TN-$ Smarandache curve lies both as geodesic and curvature line on the ruled surface ${}^{TN}_{B}\psi(s,v)$, while the normal curvature simplifies to ${}^{TN}_{B}\kappa_n = -\kappa^2$.*

3.1.2. *The characteristics of the ruled surface ${}^{TN}_{T}\psi(s,v)$.*

Theorem 3.2. *The 1^{st} and 2^{nd} fundamental forms, and the curvatures of Gaussian and mean for the ruled surface, ${}^{TN}_{T}\psi(s,v)$ are given as following:*

$$
\begin{aligned}
{}^{TN}_{T}I &= \left(\frac{\kappa^2+\tau^2}{2}+\kappa^2\left(v+\frac{1}{\sqrt{2}}\right)^2\right)ds^2-\kappa\sqrt{2}\,dsdv+dv^2, \\
{}^{TN}_{T}II &= \frac{\left(-2\tau\kappa^2\left(v^2+\sqrt{2}v+1\right)+\left(\sqrt{2}v+1\right)\left(\kappa'\tau-\tau'\kappa\right)-\tau^3\right)ds^2+2\kappa\tau\sqrt{2}dsdv}{\sqrt{2}\sqrt{\left(\kappa\left(\sqrt{2}v+1\right)\right)^2+\tau^2}}, \\
{}^{TN}_{T}K &= -2\left(\frac{\kappa\tau}{\left(\kappa\left(\sqrt{2}v+1\right)\right)^2+\tau^2}\right)^2, \\
{}^{TN}_{T}H &= \frac{-2\tau\kappa^2v\left(\sqrt{2}+v\right)+\left(\sqrt{2}v+1\right)\left(\kappa'\tau-\tau'\kappa\right)-\tau^3}{\sqrt{2}\left(\left(\kappa\left(\sqrt{2}v+1\right)\right)^2+\tau^2\right)^{\frac{3}{2}}},
\end{aligned}
$$

respectively.

Proof. The partial derivatives of ${}_{T}^{TN}\psi(s,v)$ are given as follows

$$\begin{aligned}
{}_{T}^{TN}\psi(s,v)_s &= \left(\frac{-\kappa}{\sqrt{2}}\right)T + \kappa\left(v + \frac{1}{\sqrt{2}}\right)N + \left(\frac{\tau}{\sqrt{2}}\right)B,\\
{}_{T}^{TN}\psi(s,v)_{ss} &= -\left(\frac{\kappa'}{\sqrt{2}} + \kappa^2\left(v+\frac{1}{\sqrt{2}}\right)\right)T + \left(-\frac{1}{\sqrt{2}}(\kappa^2+\tau^2-\kappa') + v\kappa'\right)N\\
&\quad + \left(\frac{1}{\sqrt{2}}\tau' + \left(v+\frac{1}{\sqrt{2}}\right)\kappa\tau\right)B,\\
{}_{T}^{TN}\psi(s,v)_v &= T, \qquad {}_{T}^{TN}\psi(s,v)_{sv} = \kappa N, \qquad {}_{T}^{TN}\psi(s,v)_{vv} = 0.
\end{aligned}$$

From Equation 2.3, the normal of ${}_{T}^{TN}\psi(s,v)$ is given as

$$ {}_{T}^{TN}n = \frac{\tau N - \kappa\left(\sqrt{2}v+1\right)B}{\sqrt{\left(\kappa\left(\tau^2+\sqrt{2}v+1\right)\right)^2}}.$$

By recalling Equation 2.6, the components for the fundamental forms can be obtained. Then, by substituting those in Equation 2.4 and 2.5, the proof is completed. □

By using the Definition 2.1 and the Theorem 3.2, the two corollaries given below are valid without the need for proof.

Corollary 3.3.

- ${}_{T}^{TN}\psi(s,v)$ *is both developable and minimal when* $\gamma(s)$ *is a plane curve.*
- ${}_{T}^{TN}\psi(s,v)$ *is minimal if the following relation holds*

$$-2\tau\kappa^2 v\left(\sqrt{2}+v\right) + \left(\sqrt{2}v+1\right)\left(\kappa'\tau - \tau'\kappa\right) - \tau^3 = 0.$$

Theorem 3.3. *The normal curvature* ${}_{T}^{TN}\kappa_n$*, the geodesic curvature* ${}_{T}^{TN}\kappa_g$ *and the geodesic torsion* ${}_{T}^{TN}\tau_g$ *of the* ${}_{T}^{TN}\psi(s,v)$ *surface are given as follows*

$$\begin{aligned}
{}_{T}^{TN}\kappa_n &= -\frac{\kappa\tau\left(h^2+\sqrt{2}v+2\right) + h'\kappa\left(\sqrt{2}v+1\right) + \sqrt{2}hv\kappa'}{\sqrt{2}\sqrt{h^2+\left(\sqrt{2}v+1\right)^2}},\\
{}_{T}^{TN}\kappa_g &= \frac{hh' - (\tau h + 2\kappa)\left(\sqrt{2}v+1\right)}{(h^2+2)\sqrt{h^2+\left(\sqrt{2}v+1\right)^2}}, \qquad (3.12)\\
{}_{T}^{TN}\tau_g &= \frac{\left(h(h')^2 - 2h'\kappa - \left(h^2+\sqrt{2}v+2\right)\left(h^2+2\right)\tau\kappa\right)\left(\sqrt{2}v+1\right) + \left(h^2+2v^2+1\right)h'\tau h}{\left(h^2+\left(\sqrt{2}v+1\right)^2\right)(h^2+2)^{\frac{3}{2}}},
\end{aligned}$$

respectively.

Proof. Recall the Equation 3.11, since the base curve is still the same $TN-$ Smarandache curve. Moreover, the derivative of the normal of ${}_{T}^{TN}\psi(s,v)$ ruled surface expressed by the harmonic curvature function as ${}^{TN}_{T}n = \dfrac{hN - \left(\sqrt{2}v+1\right)B}{\sqrt{h^2+\left(\sqrt{2}v+1\right)^2}}$ is given in the following

$$\begin{aligned}
{}^{TN}_{T}n' = & \left(-\frac{\tau}{\sqrt{h^2+\left(\sqrt{2}v+1\right)^2}}\right)T \\
& + \left(\frac{h'\left(2v^2+2\sqrt{2}v+1\right)}{\left(h^2+\left(\sqrt{2}v+1\right)^2\right)^{\frac{3}{2}}} + \frac{\tau\left(\sqrt{2}v+1\right)}{\sqrt{h^2+\left(\sqrt{2}v+1\right)^2}}\right)N \\
& + \left(\frac{hh'\left(\sqrt{2}v+1\right)}{\left(h^2+\left(\sqrt{2}v+1\right)^2\right)^{\frac{3}{2}}} + \frac{h\tau}{\sqrt{h^2+\left(\sqrt{2}v+1\right)^2}}\right)B.
\end{aligned}$$

By substituting the relations given above into Equation 2.7, the proof is completed. □

Without the need for proof, from Definition 2.2 and the Equation 3.12 of Theorem 3.3, we have the following corollary:

Corollary 3.4.

- *The $TN-$ Smarandache curve of $\gamma(s)$ cannot be geodesic on ${}^{TN}_{T}\psi(s,v)$.*
- *If $\gamma(s)$ is a plane curve, then its corresponding $TN-$ Smarandache curve lies both as asymptotic and curvature line on ${}^{TN}_{T}\psi(s,v)$, while the geodesic curvature is expressed by ${}^{TN}_{T}\kappa_g = -\kappa$.*

3.1.3. *The characteristics of the ruled surface ${}^{TN}_{N}\psi(s,v)$.*

Theorem 3.4. *The 1^{st} and 2^{nd} fundamental forms, and the curvatures of Gaussian and mean for the ruled surface, ${}^{TN}_{N}\psi(s,v)$ are given as following:*

$$\begin{aligned}
{}^{TN}_{N}I &= \left(\left(\kappa^2+\tau^2\right)\left(v^2+v\sqrt{2}+1\right) - \frac{\tau^2}{2}\right)ds^2 + \kappa\sqrt{2}dsdv + dv^2, \\
{}^{TN}_{N}II &= \left(\frac{\left(\kappa'\tau-\kappa\tau'\right)\left|\sqrt{2}v+1\right|}{\sqrt{2}\sqrt{\kappa^2+\tau^2}}\right)ds^2, \\
{}^{TN}_{N}K &= 0, \\
{}^{TN}_{N}H &= \frac{\left(\kappa'\tau-\kappa\tau'\right)}{\left|2v+\sqrt{2}\right|\left(\kappa^2+\tau^2\right)^{\frac{3}{2}}},
\end{aligned}$$

respectively.

Proof. The partial derivatives of ${}^{TN}_{N}\psi(s,v)$ are given as follows

$$\begin{aligned}
{}^{TN}_{N}\psi(s,v)_s =& -\left(v+\frac{1}{\sqrt{2}}\right)\kappa T+\frac{1}{\sqrt{2}}\kappa N+\left(v+\frac{1}{\sqrt{2}}\right)\tau B,\\
{}^{TN}_{N}\psi(s,v)_{ss} =& -\left(\frac{1}{\sqrt{2}}\kappa^2+\left(v+\frac{1}{\sqrt{2}}\right)\kappa'\right)T\\
&-\left(\left(v+\frac{1}{\sqrt{2}}\right)\tau^2-\frac{1}{\sqrt{2}}\kappa'+\left(v+\frac{1}{\sqrt{2}}\right)\kappa^2\right)N\\
&+\left(\left(v+\frac{1}{\sqrt{2}}\right)\tau'+\frac{1}{\sqrt{2}}\kappa\tau\right)B,\\
{}^{TN}_{N}\psi(s,v)_v =& N, \qquad {}^{TN}_{N}\psi(s,v)_{sv} = -\kappa T+\tau B, \qquad {}^{TN}_{N}\psi(s,v)_{vv} = 0.
\end{aligned}$$

From Equation 2.3, the normal of ${}^{TN}_{N}\psi(s,v)$ is given as

$${}^{TN}_{N}n = -\epsilon_1\frac{\tau T+\kappa B}{\sqrt{\kappa^2+\tau^2}},$$

where $\epsilon_1 = sign(1+\sqrt{2}v)$. By recalling Equation 2.6, the components for the fundamental forms can be obtained. Then, by substituting those in Equation 2.4 and 2.5, the proof is completed. □

By using the Definition 2.1 and the Theorem 3.4, the following corollaries are clear without the need for proof.

Corollary 3.5.

- *The surface ${}^{TN}_{N}\psi(s,v)$ is always developable.*
- *${}^{TN}_{N}\psi(s,v)$ is minimal when $\gamma(s)$ is a general helix.*

Theorem 3.5. *The normal curvature ${}^{TN}_{N}\kappa_n$, the geodesic curvature ${}^{TN}_{N}\kappa_g$ and the geodesic torsion ${}^{TN}_{N}\tau_g$ of the ${}^{TN}_{N}\psi(s,v)$ surface are given as follows*

$$\begin{aligned}
{}^{TN}_{N}\kappa_n =& -\epsilon_1\frac{h'\kappa}{\sqrt{2}\sqrt{h^2+1}},\\
{}^{TN}_{N}\kappa_g =& -\epsilon_1\frac{\tau h\left(h^2+3\right)+h'h+2\kappa}{\left(h^2+2\right)\sqrt{h^2+1}},\\
{}^{TN}_{N}\tau_g =& -\frac{h'\left(\tau h\left(h^2+3\right)+h'h+2\kappa\right)}{\left(h^2+2\right)^{\frac{3}{2}}\left(h^2+1\right)},
\end{aligned} \tag{3.13}$$

respectively.

Proof. Recall again the Equation 3.11. Moreover, the derivative of the normal of ${}^{TN}_{N}\psi(s,v)$ expressed by the harmonic curvature function as ${}^{TN}_{N}n = -\epsilon_1 \dfrac{hT+B}{\sqrt{h^2+1}}$ is given in the following

$${}^{TN}_{N}n' = -\epsilon_1 \sigma \kappa \left(T - hB\right).$$

By substituting the relations given above into Equation 2.7, the proof is completed. □

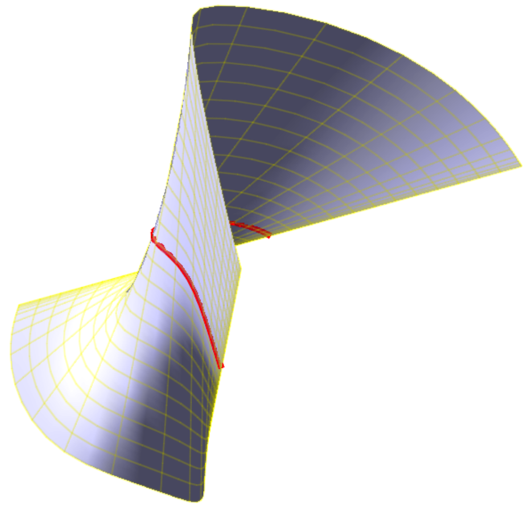
Without the need for proof, from Definition 2.2 and the Equation 3.13 of Theorem 3.5, we have the following remark:

Remark 3.2. *The given corollaries for* ${}^{TN}_{T}\psi(s,v)$ *are the same as of* ${}^{TN}_{N}\psi(s,v)$.

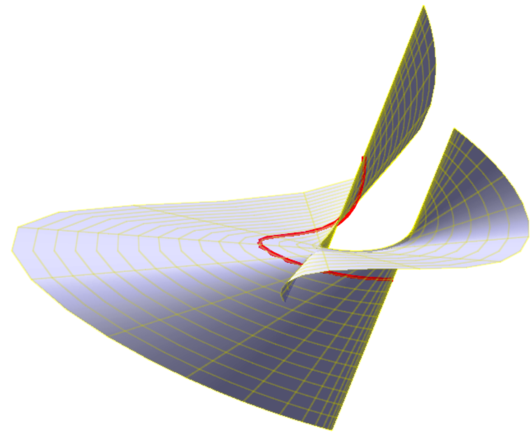
Example 3.1. *Let* $\gamma : [-2\pi, 2\pi] \to E^3$ *be a regular unit speed curve defined by the following parameterization* $\gamma(s) = (cosh(s), sinh(s), 2s)$. *Then, we compute its Frenet apparatus as follows*

$$\begin{aligned} T &= \frac{\left(\sinh\left(s\right), \cosh\left(s\right), 2\right)}{\sqrt{4+\cosh\left(2s\right)}}, \quad N = \frac{\left(5\cosh\left(s\right), 3\sinh\left(s\right), -2\sinh\left(2s\right)\right)}{\sqrt{\left(4+\cosh\left(2s\right)\right)\left(1+4\cosh\left(2s\right)\right)}}, \\ B &= \frac{\left(-2\sinh\left(s\right), 2\cosh\left(s\right), -1\right)}{\sqrt{1+4\cosh\left(2s\right)}}, \quad \kappa = \frac{\sqrt{1+4\cosh\left(2s\right)}}{\left(4+\cosh\left(2s\right)\right)^{\frac{3}{2}}}, \quad \tau = \frac{2}{1+4\cosh\left(2s\right)}. \end{aligned} \tag{3.14}$$

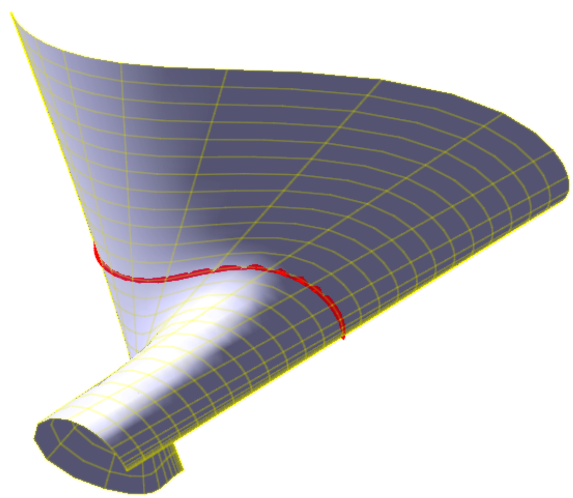
Hence, from the Equations 3.8, 3.9 and 3.14, the ruled surfaces ${}^{TN}_{T}\psi(s,v)$, ${}^{TN}_{N}\psi(s,v)$ *and* ${}^{TN}_{B}\psi(s,v)$ *can be easily obtained (see Fig. 1).*



(a) The ruled surface ${}^{TN}_{T}\psi(s,v)$



(b) The ruled surface ${}^{TN}_{N}\psi(s,v)$



(c) The ruled surface ${}^{TN}_{B}\psi(s,v)$

FIGURE 1. Ruled surfaces with base curve of TN-Smarandache curve (red) where $s \in [-2\pi, 2\pi]$ and $v \in [-2, 2]$

3.2. **Ruled Surfaces with the Base TB− Smarandache Curve.**

Definition 3.2. *Let $\gamma : s \in I \subset \Re \to E^3$ be a regular unit speed curve of C^2 class and $\{T,\ N,\ B\}$ denotes the set of its Frenet vectors. The original definition of $TB-$ Smarandache* *ruled surface introduced in [11] is as following:*

$$ {}^{TB}_{N}\psi(s,v) = \frac{T+B}{\sqrt{2}} + vN. \tag{3.15} $$

As noted before, the other two ruled surfaces with the base of $TB-$ Smarandache curve and with the ruling of other two Frenet vectors are discussed, which are parameterized in the following

$$ \begin{aligned} {}^{TB}_{T}\psi(s,v) =& \frac{T+B}{\sqrt{2}} + vT, \\ {}^{TB}_{B}\psi(s,v) =& \frac{T+B}{\sqrt{2}} + vB. \end{aligned} \tag{3.16} $$

The conditions for the base curve to be asymptotic, geodesic and curvature line on each ruled surface are examined, as well.

3.2.1. *The characteristics of the ruled surface ${}^{TB}_{N}\psi(s,v)$.*

Ouarab, [11] obtained the following corollaries for the ruled surface ${}^{TB}_{N}\psi(s,v)$ as

Corollary 3.6. *[11]*

- ${}^{TB}_{N}\psi(s,v)$ *is always developable.*
- ${}^{TB}_{N}\psi(s,v)$ *is minimal when $\gamma(s)$ is a general helix.*

The base curve characteristics of the ${}^{TB}_{N}\psi(s,v)$ surface associated to $\gamma(s)$ is given with the following theorem.

Theorem 3.6. *The normal curvature ${}^{TB}_{N}\kappa_n$, the geodesic curvature ${}^{TB}_{N}\kappa_g$ and the geodesic torsion ${}^{TB}_{N}\tau_g$ of the ${}^{TB}_{N}\psi(s,v)$ surface are given as follows*

$$ {}^{TB}_{N}\kappa_n = 0, \quad {}^{TB}_{N}\kappa_g = -\epsilon_2\kappa\sqrt{h^2+1}, \quad {}^{TB}_{N}\tau_g = 0, \tag{3.17} $$

respectively, , where $\epsilon_2 = sign(v)$.

Proof. By utilizing the Equation (2.1), the tangent and its derivative, and the second order derivative of the $TB-$ Smarandache curve are given as

$$ \begin{aligned} T_{TB} =& \eta N, \qquad T'_{TB} = -\eta\kappa(T-hB), \\ \left(\frac{T+B}{\sqrt{2}}\right)'' =& \frac{\kappa^2(h-1)T - (\tau'-\kappa')N - h\kappa^2(h-1)B}{\sqrt{2}}, \end{aligned} \tag{3.18} $$

where $\eta = sign(\kappa-\tau)$. Moreover, the derivative of the normal of ${}^{TB}_{N}\psi(s,v)$ ruled surface defined as ${}^{TB}_{N}n = \pm\frac{\tau T + \kappa B}{\sqrt{\kappa^2+\tau^2}}$ in [11], but expressed by the harmonic curvature function as ${}^{TB}_{N}n = -\epsilon_2\frac{hT+B}{\sqrt{h^2+1}}$ is given in the following

$$ {}^{TB}_{N}n' = -\epsilon_2\sigma\kappa\,(T-hB). $$

By substituting the relations given above into Equation 2.7, the proof is completed. □

Without the need for proof, from Definition 2.2 and the Equation 3.17 of Theorem 3.6, we have the following corollary:

Corollary 3.7.

- *The $TB-$ Smarandache curve of $\gamma(s)$ is always asymptotic and curvature line on ${}^{TB}_{N}\psi(s,v)$, while the geodesic curvature simplifies to ${}^{TB}_{N}\kappa_g = -\epsilon_2\kappa$.*
- *The $TB-$ Smarandache curve of $\gamma(s)$ cannot be geodesic on ${}^{TB}_{N}\psi(s,v)$.*

3.2.2. *The characteristics of the ruled surface ${}^{TB}_{T}\psi(s,v)$.*

Theorem 3.7. *The 1^{st} and 2^{nd} fundamental forms, and the curvatures of Gaussian and mean for the ruled surface, ${}^{TB}_{T}\psi(s,v)$ are given as following:*

$$\begin{aligned}
{}^{TB}_{T}I &= \left(\left(v+\frac{1}{\sqrt{2}}\right)\kappa-\frac{1}{\sqrt{2}}\tau\right)^2 ds^2+dv^2,\\
{}^{TB}_{T}II &= -\,\epsilon_3\left(\left(v+\frac{1}{\sqrt{2}}\right)\kappa-\frac{1}{\sqrt{2}}\tau\right)\tau ds^2,\\
{}^{TB}_{T}K &= 0, \qquad {}^{TB}_{T}H = -\epsilon_3\frac{\tau}{\sqrt{2}\left(\kappa(\sqrt{2}v+1)-\tau\right)}\ ,
\end{aligned}$$

respectively, where $\epsilon_3 = sign(\kappa\left(1+\sqrt{2}v\right)-\tau)$.

Proof. The partial derivatives of ${}^{TB}_{T}\psi(s,v)$ are given as follows

$$\begin{aligned}
{}^{TB}_{T}\psi(s,v)_s &= \left(\left(v+\frac{1}{\sqrt{2}}\right)\kappa-\frac{1}{\sqrt{2}}\tau\right)N,\\
{}^{TB}_{T}\psi(s,v)_{ss} &= \left(\frac{1}{\sqrt{2}}\kappa\tau-\left(v+\frac{1}{\sqrt{2}}\right)\kappa^2\right)T+\left(\left(v+\frac{1}{\sqrt{2}}\right)\kappa'-\frac{1}{\sqrt{2}}\tau'\right)N\\
&\quad+\left(\left(v+\frac{1}{\sqrt{2}}\right)\kappa\tau-\frac{1}{\sqrt{2}}\tau^2\right)B,\\
{}^{TB}_{T}\psi(s,v)_v &= T, \qquad {}^{TB}_{T}\psi(s,v)_{sv}=\kappa N, \qquad {}^{TB}_{T}\psi(s,v)_{vv}=0.
\end{aligned}$$

From Equation 2.3, the normal vector of ${}^{TB}_{T}\psi(s,v)$ is computed as

$$ {}^{TB}_{T}n = -\epsilon_3 B. $$

By recalling Equation 2.6, the components for the fundamental forms can be obtained. Then, by substituting those in Equation 2.4 and 2.5, the proof is completed. □

Corollary 3.8.

- *${}^{TB}_{T}\psi(s,v)$ is always developable.*

- *${}^{TB}_{T}\psi(s,v)$ is minimal when $\gamma(s)$ is a plane curve.*

Theorem 3.8. *The normal curvature ${}^{TB}_{T}\kappa_n$, the geodesic curvature ${}^{TB}_{T}\kappa_g$ and the geodesic torsion ${}^{TB}_{T}\tau_g$ of the ${}^{TB}_{T}\psi(s,v)$ surface are given as follows*

$$ {}^{TB}_{T}\kappa_n = \epsilon_3 \frac{\kappa\tau\,(h-1)}{\sqrt{2}}, \qquad {}^{TB}_{T}\kappa_g = -\epsilon_3\kappa, \qquad {}^{TB}_{T}\tau_g = -\eta\kappa\tau, \tag{3.19} $$

respectively.

Proof. Recall the Equation 3.18, since the base is the same $TB-$ Smarandache curve. The derivative of the normal vector of ${}^{TB}_{T}\psi(s,v)$ computed before as ${}^{TB}_{T}n = -\epsilon_3 B$ is given in the following

$$ {}^{TB}_{T}n' = \epsilon_3\tau N. $$

By substituting the relations given above into Equation 2.7, the proof is completed. □

Without the need for proof, from Definition 2.2 and the Equation 3.19 of Theorem 3.8, we have the following corollary:

Corollary 3.9.

- *The $TB-$ Smarandache curve of $\gamma(s)$ cannot be geodesic on ${}^{TB}_{T}\psi(s,v)$.*
- *If $\gamma(s)$ is a plane curve , then its corresponding $TB-$ Smarandache curve lies both as asymptotic and curvature line on ${}^{TB}_{T}\psi(s,v)$.*

3.2.3. *The characteristics of the ruled surface ${}^{TB}_{B}\psi(s,v)$.*

Theorem 3.9. *The 1^{st} and 2^{nd} fundamental forms, and the curvatures of Gaussian and mean for the ruled surface, ${}^{TB}_{B}\psi(s,v)$ are given as following:*

$$ \begin{aligned} {}^{TB}_{B}I =& \left(\frac{1}{\sqrt{2}}\kappa - \left(v + \frac{1}{\sqrt{2}}\right)\tau\right)^2 ds^2 + dv^2, \\ {}^{TB}_{B}II =& -\epsilon_4\left(\frac{1}{\sqrt{2}}\kappa^2 - \left(v + \frac{1}{\sqrt{2}}\right)\kappa\tau\right) ds^2, \\ {}^{TB}_{B}K =& 0, \\ {}^{TB}_{B}H =& -\epsilon_4 \frac{\kappa}{\sqrt{2}\left(\kappa - \tau\left(\sqrt{2}v+1\right)\right)}, \end{aligned} $$

respectively, where $\epsilon_4 = sign\left(\kappa - \tau\left(\sqrt{2}v+1\right)\right)$.

Proof. The partial derivatives of ${}_{B}^{TB}\psi(s,v)$ are given as follows

$$
\begin{aligned}
{}_{B}^{TB}\psi(s,v)_s &= \left(\frac{1}{\sqrt{2}}\kappa - \left(v+\frac{1}{\sqrt{2}}\right)\tau\right)N,\\
{}_{B}^{TB}\psi(s,v)_{ss} &= \left(-\frac{1}{\sqrt{2}}\kappa^2 + \left(v+\frac{1}{\sqrt{2}}\right)\kappa\tau\right)T + \left(\frac{1}{\sqrt{2}}\kappa' - \left(v+\frac{1}{\sqrt{2}}\right)\tau'\right)N\\
&\quad + \left(\frac{1}{\sqrt{2}}\kappa\tau - \left(v+\frac{1}{\sqrt{2}}\right)\tau^2\right)B,\\
{}_{B}^{TB}\psi(s,v)_v &= B, \qquad {}_{N}^{TN}\psi(s,v)_{sv} = -\tau N, \qquad {}_{N}^{TN}\psi(s,v)_{vv} = 0.
\end{aligned}
$$

From Equation 2.3, the normal vector of ${}_{B}^{TB}\psi(s,v)$ is computed as

$$
{}_{B}^{TB}n = \epsilon_4 T.
$$

By recalling Equation 2.6, the components for the fundamental forms can be obtained. Then, by substituting those in Equation 2.4 and 2.5, the proof is completed. □

Corollary 3.10.

- ${}_{B}^{TB}\psi(s,v)$ *is always developable.*
- ${}_{B}^{TB}\psi(s,v)$ *can not be minimal.*
- *If* $\gamma(s)$ *is a plane curve, then* ${}_{B}^{TB}\psi(s,v)$ *is a constant-mean-curvature (CMC) surface.*

Theorem 3.10. *The normal curvature* ${}_{B}^{TB}\kappa_n$*, the geodesic curvature* ${}_{B}^{TB}\kappa_g$ *and the geodesic torsion* ${}_{B}^{TB}\tau_g$ *of the* ${}_{B}^{TB}\psi(s,v)$ *surface are given as follows*

$$
{}_{B}^{TB}\kappa_n = \epsilon_4\frac{\kappa^2(h-1)}{\sqrt{2}}, \qquad {}_{B}^{TB}\kappa_g = \epsilon_4\tau, \qquad {}_{B}^{TB}\tau_g = \eta\kappa\tau, \tag{3.20}
$$

respectively.

Proof. By recalling again the Equation 3.18, and taking the derivative of the normal of ${}_{B}^{TB}\psi(s,v)$ computed before as ${}_{B}^{TB}n = -T$, we have

$$
{}_{B}^{TB}n' = -\kappa N.
$$

By substituting the relations given above into Equation 2.7, the proof is completed. □

Without the need for proof, from Definition 2.2 and the Equation 3.20 of Theorem 3.10, we have the following corollary:

Corollary 3.11.

- *The* $TB-$ *Smarandache curve of* $\gamma(s)$ *cannot be asymptotic on* ${}_{B}^{TB}\psi(s,v)$*.*

- *If $\gamma(s)$ is a plane curve, then its corresponding $TB-$ Smarandache curve lies both as geodesic and curvature line on ${}^{TB}_{B}\psi(s,v)$.*

Example 3.2. *By reconsidering the curve given in Example 3.1, and using the Equations 3.14, 3.15 and 3.16 the ruled surfaces ${}^{TB}_{T}\psi(s,v)$, ${}^{TB}_{N}\psi(s,v)$ and ${}^{TB}_{B}\psi(s,v)$ can be easily obtained and illustrated in Fig. 2.*

3.3. Ruled Surfaces with the Base NB− Smarandache Curve.

Definition 3.3. *Let $\gamma : s \in I \subset \Re \to E^3$ be a regular unit speed curve of C^2 class and $\{T,\ N,\ B\}$ denotes the set of its Frenet vectors. The original definition of $NB-$ Smarandache ruled surface introduced in [11] is as following:*

$$ {}^{NB}_{T}\psi(s,v) = \frac{N+B}{\sqrt{2}} + vT. \tag{3.21} $$

As before, the other two ruled surfaces with the base of $NB-$ Smarandache curve and with the ruling of other two Frenet vectors are discussed, which are parameterized in the following

$$ \begin{aligned} {}^{NB}_{N}\psi(s,v) =& \frac{N+B}{\sqrt{2}} + vN, \\ {}^{NB}_{B}\psi(s,v) =& \frac{N+B}{\sqrt{2}} + vB. \end{aligned} \tag{3.22} $$

The conditions for the base curve to be asymptotic, geodesic and curvature line on each ruled surface are examined, as well.

3.3.1. *The characteristics of the ruled surface ${}^{NB}_{T}\psi(s,v)$.*

[11] claimed the following corollaries for the ruled surface ${}^{NB}_{T}\psi(s,v)$

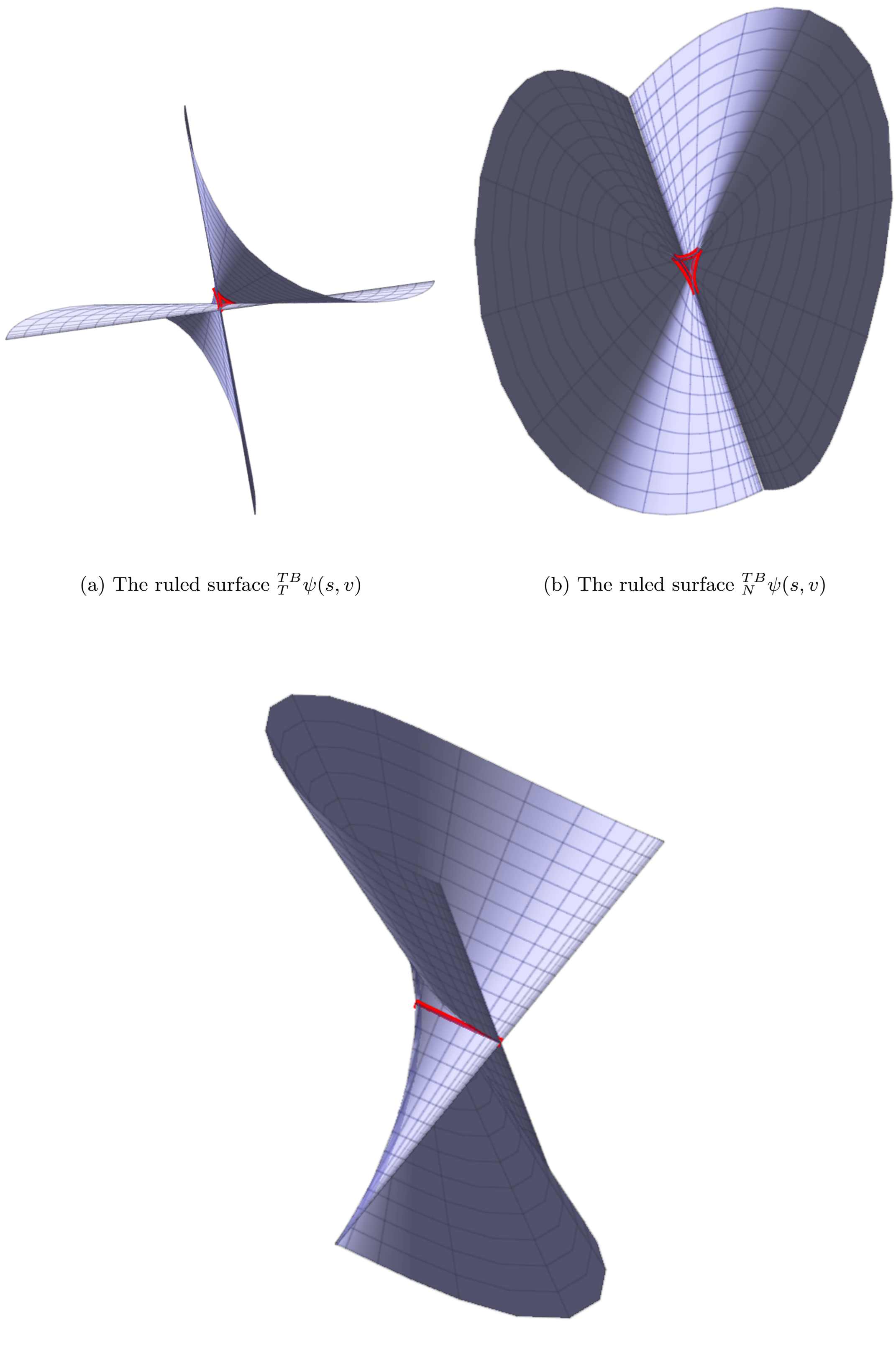
Corollary 3.12. *[11]*

- *If $\gamma(s)$ is a plane curve, then ${}^{NB}_{T}\psi(s,v)$ is developable, and if it is developable, then it is also minimal.*
- *${}^{NB}_{T}\psi(s,v)$ is minimal if the following relation holds*

$$ \frac{\tau\left(2\kappa^2\tau - 2\tau^2 - \kappa^2\right)}{\sqrt{2}} + v\left(\kappa'\tau + 2\kappa\tau^2 - \kappa^2\tau'\right) - \sqrt{2}v^2\kappa^2\tau = 0. $$

Remark 3.3. *We note that in order for the ruled surface ${}^{NB}_{T}\psi(s,v)$ be minimal, the following relation should hold*

$$ v\sqrt{2}\left(2\kappa\tau^2 + \kappa'\tau - \kappa\tau'\right) - \tau\left(2\tau^2 - \kappa^2\left(1 - 2v^2\right)\right) = 0. $$



(a) The ruled surface ${}^{TB}_{T}\psi(s,v)$ (b) The ruled surface ${}^{TB}_{N}\psi(s,v)$

(c) The ruled surface ${}^{TB}_{B}\psi(s,v)$

FIGURE 2. Ruled surfaces with base curve of TB-Smarandache curve (red) where $s \in [-2\pi, 2\pi]$ and $v \in [-2, 2]$

The base curve characteristics of the ${}_{T}^{NB}\psi(s,v)$ surface associated to the curve $\gamma(s)$ is given with the following theorem.

Theorem 3.11. *The normal curvature , the geodesic curvature and the geodesic torsion of the ${}_{T}^{NB}\psi(s,v)$ surface are given as follows*

$$\begin{aligned}
{}_{T}^{NB}\kappa_n =& \frac{\left(\sqrt{2}v-2h\right)\tau^2-\left(\tau+h'\kappa\right)\sqrt{2}v-\tau\kappa}{2\sqrt{h^2-\sqrt{2}hv+v^2}},\\
{}_{T}^{NB}\kappa_g =& \frac{\left(2h^2+1-2\sqrt{2}hv\right)\sqrt{2}\tau+2\left(\left(\sqrt{2}h-v\right)h'-v\kappa\right)}{2\left(2h^2+1\right)\sqrt{h^2-\sqrt{2}hv+v^2}},\\
{}_{T}^{NB}\tau_g =& \frac{\sqrt{2}v(h')^2+\tau\left(2h^2-\sqrt{2}hv+2v^2+1\right)h'}{\left(h^2-\sqrt{2}hv+v^2\right)\left(2h^2+1\right)^{\frac{3}{2}}}+\frac{\left(h-\sqrt{2}v\right)\left(2h^2-\sqrt{2}hv+1\right)\kappa^2}{2\left(h^2-\sqrt{2}hv+v^2\right)\sqrt{2h^2+1}},
\end{aligned} \tag{3.23}$$

respectively.

Proof. By referring the Equation (2.1), the tangent and its derivative, and the second order derivative of $NB-$ Smarandache curve are given as

$$\begin{aligned}
T_{NB} =& -\frac{T+h\left(N-B\right)}{\sqrt{2h^2+1}},\\
T'_{NB} =& \frac{h\left(2h\tau+2h'+\kappa\right)T-\left(2h^3\tau+3h\tau+h'+\kappa\right)N-\left(2h^3\tau+h\tau-h'\right)B}{\left(2h^2+1\right)^{\frac{3}{2}}},\\
\left(\frac{N+B}{\sqrt{2}}\right)'' =& \frac{\left(-\kappa'+\kappa\tau\right)T-\left(\tau^2+h\kappa'+h'\kappa+\kappa^2\right)N-\left(\tau^2-h\kappa'-h'\kappa\right)B}{\sqrt{2}}.
\end{aligned} \tag{3.24}$$

Moreover, the derivative of the normal of $NB-$ surface defined as ${}_{T}^{NB}n=\frac{\tau N+\left(\tau-\sqrt{2}\kappa v\right)B}{\sqrt{\tau^2+\left(\tau-\sqrt{2}\kappa v\right)^2}}$ in [11], but expressed by the harmonic curvature function as ${}_{T}^{NB}n=\frac{hN+\left(h-\sqrt{2}v\right)B}{\sqrt{h^2+\left(h-\sqrt{2}v\right)^2}}$ is given in the following

$$\begin{aligned}
{}_{T}^{NB}n' =& -\left(\frac{\tau\sqrt{2}}{2\sqrt{h^2-\sqrt{2}hv+v^2}}\right)T+\left(\frac{\tau\left(2v-h\sqrt{2}\right)\left(h^2-\sqrt{2}hv+v^2\right)+h'v\left(\sqrt{2}v-h\right)}{2\left(h^2-\sqrt{2}hv+v^2\right)^{\frac{3}{2}}}\right)N\\
&+\left(\frac{h\left(\left(h^2+v^2\right)\tau\sqrt{2}-v\left(2h\tau-h'\right)\right)}{2\left(h^2-\sqrt{2}hv+v^2\right)^{\frac{3}{2}}}\right)B.
\end{aligned}$$

By substituting the relations given above into Equation 2.7, the proof is completed. □

Without the need for proof, from Definition 2.2 and the Equation 3.23 of Theorem 3.11, we have the following corollary:

Corollary 3.13.

- *The $NB-$ Smarandache curve of $\gamma(s)$ cannot be geodesic on ${}_{T}^{NB}\psi(s,v)$.*
- *If $\gamma(s)$ is plane curve, then its corresponding $NB-$ Smarandache curve lies both as asymptotic and curvature line on ${}_{T}^{NB}\psi(s,v)$, while the geodesic curvature simplifies to ${}_{T}^{NB}\kappa_n = -\kappa$.*

3.3.2. *The characteristics of the ruled surface ${}_{N}^{NB}\psi(s,v)$.*

Theorem 3.12. *The 1^{st} and 2^{nd} fundamental forms, and the Gaussian and mean curvature of the ruled surface, ${}_{N}^{NB}\psi(s,v)$ are given as following:*

$$\begin{aligned}
{}_{N}^{NB}I &= \left(\left(v^2+v\sqrt{2}+1\right)(\kappa^2+\tau^2)-\frac{\kappa^2}{2}\right)ds^2-\tau\sqrt{2}dsdv+dv^2,\\
{}_{N}^{NB}II &= \frac{(\tau\kappa'-\kappa\tau')\left|\sqrt{2}v+1\right|}{\sqrt{2}\sqrt{\kappa^2+\tau^2}}ds^2,\\
{}_{N}^{NB}K &= 0,\\
{}_{N}^{NB}H &= \frac{(\kappa'\tau-\kappa\tau')}{\left|2v+\sqrt{2}\right|(\kappa^2+\tau^2)^{\frac{3}{2}}},
\end{aligned}$$

respectively.

Proof. The partial derivatives of ${}_{N}^{NB}\psi(s,v)$ are given as follows

$$\begin{aligned}
{}_{N}^{NB}\psi(s,v)_s &= -\left(v+\frac{1}{\sqrt{2}}\right)\kappa T-\frac{1}{\sqrt{2}}\tau N+\left(v+\frac{1}{\sqrt{2}}\right)\tau B,\\
{}_{N}^{NB}\psi(s,v)_{ss} &= \left(\frac{1}{\sqrt{2}}\kappa\tau-\left(v+\frac{1}{\sqrt{2}}\right)\kappa'\right)T-\left(\left(\frac{1}{\sqrt{2}}+v\right)(\kappa^2+\tau^2)+\frac{1}{\sqrt{2}}\tau'\right)N\\
&\quad -\left(\frac{1}{\sqrt{2}}\tau^2-\left(v+\frac{1}{\sqrt{2}}\right)\tau'\right)B,\\
{}_{N}^{NB}\psi(s,v)_v &= N, \qquad {}_{N}^{NB}\psi(s,v)_{sv} = -\kappa T+\tau B, \qquad {}_{N}^{NB}\psi(s,v)_{vv} = 0.
\end{aligned}$$

From Equation 2.3, the normal vector of ${}_{N}^{NB}\psi(s,v)$ is computed as

$${}_{N}^{NB}n = -\epsilon_1\frac{\tau T+\kappa B}{\sqrt{\kappa^2+\tau^2}}.$$

Let us remind that $\epsilon_1 = sign(\sqrt{2}v+1)$ was already defined in the proof of the Theorem (3.4). Then, by recalling Equation 2.6, the components for the fundamental forms can be obtained, and by substituting those in the Equations 2.4 and 2.5, the proof is completed. □

Corollary 3.14.

- ${}_{N}^{NB}\psi(s,v)$ *is always developable.*
- ${}_{N}^{NB}\psi(s,v)$ *is minimal when* $\gamma(s)$ *is a general helix.*

Theorem 3.13. *The normal curvature* ${}_{N}^{NB}\kappa_n$*, the geodesic curvature* ${}_{N}^{NB}\kappa_g$ *and the geodesic torsion* ${}_{N}^{NB}\tau_g$ *of the* ${}_{N}^{NB}\psi(s,v)$ *surface are given as follows*

$$\begin{aligned} {}_{N}^{NB}\kappa_n &= -\epsilon_1 \frac{\kappa h'}{\sqrt{2\left(h^2+1\right)}}, \\ {}_{N}^{NB}\kappa_g &= -\epsilon_1 \frac{\left(\tau h\left(2h^2+3\right)+h'+\kappa\right)}{\left(2h^2+1\right)\sqrt{h^2+1}}, \\ {}_{N}^{NB}\tau_g &= -\frac{h'\left(\tau h\left(2h^2+3\right)+h'+\kappa\right)}{\left(h^2+1\right)\left(2h^2+1\right)^{\frac{3}{2}}}, \end{aligned} \tag{3.25}$$

respectively.

Proof. Recall the Equation 3.24, since the base is the same $NB-$ Smarandache curve. The derivative of the normal vector of ${}_{N}^{NB}\psi(s,v)$ ruled surface expressed by the harmonic curvature function as ${}_{N}^{NB}n = -\epsilon_1 \dfrac{(hT+B)}{\sqrt{h^2+1}}$ is given in the following

$${}_{N}^{NB}n' = -\epsilon_1 \sigma\kappa\left(T-hB\right).$$

By substituting the relations given above into Equation 2.7, the proof is completed. ☐

Without the need for proof, from Definition 2.2 and the Equation 3.25 of Theorem 3.13, we have

Corollary 3.15.

- *The* $NB-$ *Smarandache curve of* $\gamma(s)$ *cannot be geodesic on* ${}_{N}^{NB}\psi(s,v)$.
- *If* $\gamma(s)$ *is a plane curve , then its* $NB-$ *Smarandache curve lies both as asymptotic and curvature line on* ${}_{N}^{NB}\psi(s,v)$*, while the geodesic curvature is* ${}_{N}^{NB}\kappa_g = -\kappa$.

Remark 3.4. *The two corollaries expressed for the ruled surfaces* ${}_{T}^{TN}\psi(s,v)$ *and* ${}_{N}^{TN}\psi(s,v)$ *are exactly the same for the ruled surface* ${}_{N}^{NB}\psi(s,v)$.

3.3.3. *The characteristics of the ruled surface* ${}^{NB}_{B}\psi(s,v)$.

Theorem 3.14. *The* 1^{st} *and* 2^{nd} *fundamental forms, and the curvatures of Gaussian and mean for the ruled surface,* ${}^{NB}_{B}\psi(s,v)$ *are given as following:*

$$
\begin{aligned}
{}^{NB}_{B}I =& \left(\frac{\kappa^2}{2}+\left(v^2+v\sqrt{2}+1\right)\tau^2\right)ds^2+\tau\sqrt{2}dsdv+dv^2,\\
{}^{NB}_{B}II =& -\kappa\left(\frac{\left(\tau^2\left(\sqrt{2}v+1\right)-\left(\tau\kappa'+\kappa\tau'\right)\right)\left(\sqrt{2}v+1\right)-\left(\tau^2+\kappa^2\right)}{\sqrt{2}\sqrt{\kappa^2+\left(\tau\left(\sqrt{2}v+1\right)\right)^2}}\right)ds^2\\
&-\frac{2\kappa\tau}{\sqrt{\kappa^2+\left(\tau\left(\sqrt{2}v+1\right)\right)^2}}dsdv,\\
{}^{NB}_{B}K =& -2\left(\frac{\kappa\tau}{\kappa^2+\left(\tau\left(\sqrt{2}v+1\right)\right)^2}\right)^2,\\
{}^{NB}_{B}H =& -\frac{\left(\sqrt{2}v+1\right)\left(\kappa\tau'+\tau\kappa'\right)+\left(2\tau^2v\left(\sqrt{2}+v\right)+\kappa^2\right)\kappa}{\sqrt{2}\left(\kappa^2+\left(\tau\left(\sqrt{2}v+1\right)\right)^2\right)^{\frac{3}{2}}},
\end{aligned}
$$

respectively.

Proof. The partial derivatives of ${}^{NB}_{B}\psi(s,v)$ are given as follows

$$
\begin{aligned}
{}^{NB}_{B}\psi(s,v)_s =& -\frac{\kappa}{\sqrt{2}}T-\tau\left(v+\frac{1}{\sqrt{2}}\right)N+\frac{\tau}{\sqrt{2}}B,\\
{}^{NB}_{B}\psi(s,v)_{ss} =& \left(\left(v+\frac{1}{\sqrt{2}}\right)\kappa\tau-\frac{\kappa'}{\sqrt{2}}\right)T-\left(\frac{\kappa^2+\tau^2}{\sqrt{2}}+\tau'\left(v+\frac{1}{\sqrt{2}}\right)\right)N\\
&-\left(\tau^2\left(v+\frac{1}{\sqrt{2}}\right)-\frac{\tau'}{\sqrt{2}}\right)B,
\end{aligned}
$$

$$
{}^{NB}_{B}\psi(s,v)_v = B, \qquad {}^{NB}_{B}\psi(s,v)_{sv} = -\tau N, \qquad {}^{NB}_{B}\psi(s,v)_{vv} = 0.
$$

From 2.3, the normal vector of ${}^{NB}_{B}\psi(s,v)$ is computed as

$$
{}^{NB}_{B}n = -\frac{\tau\left(\sqrt{2}v+1\right)T-\kappa N}{\sqrt{\kappa^2+\left(\tau\left(\sqrt{2}v+1\right)\right)^2}}.
$$

By using the Equation (2.6), the components for the fundamental forms can be obtained. Then, by substituting those in the Equation (2.4) and (2.5), the proof is completed. □

Corollary 3.16.

- ${}_{B}^{NB}\psi(s,v)$ *is developable when* $\gamma(s)$ *is a plane curve.*
- ${}_{B}^{NB}\psi(s,v)$ *is minimal if the following relation holds:*

$$\left(\sqrt{2}v+1\right)\left(\kappa\tau'+\tau\kappa'\right)+\kappa\left(2\tau^2 v\left(\sqrt{2}+v\right)+\kappa^2\right)=0.$$

Theorem 3.15. *The normal curvature* ${}_{B}^{NB}\kappa_n$*, the geodesic curvature* ${}_{B}^{NB}\kappa_g$ *and the geodesic torsion* ${}_{B}^{NB}\tau_g$ *of the* ${}_{B}^{NB}\psi(s,v)$ *surface are given as follows*

$$\begin{aligned}
{}_{B}^{NB}\kappa_n =& \frac{\sqrt{2}v\kappa' h-\tau^2\left(\sqrt{2}v+2\right)-h'\kappa-\kappa^2}{\sqrt{2}\sqrt{h^2\left(\sqrt{2}v+1\right)^2+1}},\\
{}_{B}^{NB}\kappa_g =& \frac{h'-h\tau\left(2h^2+1\right)\left(\sqrt{2}v+1\right)}{\left(2h^2+1\right)\sqrt{h^2\left(\sqrt{2}v+1\right)^2+1}},\\
{}_{B}^{NB}\tau_g =& \frac{\left(\left(2v^2+3\right)\sqrt{2}h^2 v+\left(6v^2+1\right)h^2+\sqrt{2}v+1\right)(h')^2}{\left(h^2\left(\sqrt{2}v+1\right)^2+1\right)\left(2h^2+1\right)^{\frac{3}{2}}}\\
&-\frac{\kappa\left(2h^4\left(\sqrt{2}v+1\right)-h^2\left(2v^2+1\right)-1\right)h'}{\sqrt{h^2\left(\sqrt{2}v+1\right)^2+1}\left(2h^2+1\right)^{\frac{3}{2}}}\\
&-\frac{\left(2h^2+\sqrt{2}h^2 v+1\right)\left(\sqrt{2}v+1\right)\tau^2}{\left(h^2\left(\sqrt{2}v+1\right)^2+1\right)\sqrt{2h^2+1}},
\end{aligned} \tag{3.26}$$

respectively.

Proof. By recalling again the Equation 3.24, and taking the derivative of the normal of ${}_{B}^{NB}\psi(s,v)$ expressed by the harmonic curvature function as ${}_{B}^{NB}n=-\dfrac{h\left(\sqrt{2}v+1\right)T-N}{\sqrt{h^2\left(\sqrt{2}v+1\right)^2+1}}$, we have

$$
\begin{aligned}
{}^{NB}_{B}n' = &-\left(\frac{\left(\sqrt{2}v+1\right)h' + \kappa\left(h^2\left(\sqrt{2}v+1\right)^2+1\right)}{\left(h^2\left(\sqrt{2}v+1\right)^2+1\right)^{\frac{3}{2}}}\right)T \\
&-\left(\frac{hh'\left(\sqrt{2}v+1\right)^2 + \tau\left(h^2\left(\sqrt{2}v+1\right)^2+1\right)\left(\sqrt{2}v+1\right)}{\left(h^2\left(\sqrt{2}v+1\right)^2+1\right)^{\frac{3}{2}}}\right)N \\
&+\left(\frac{\tau}{\sqrt{h^2\left(\sqrt{2}v+1\right)^2+1}}\right)B.
\end{aligned}
$$

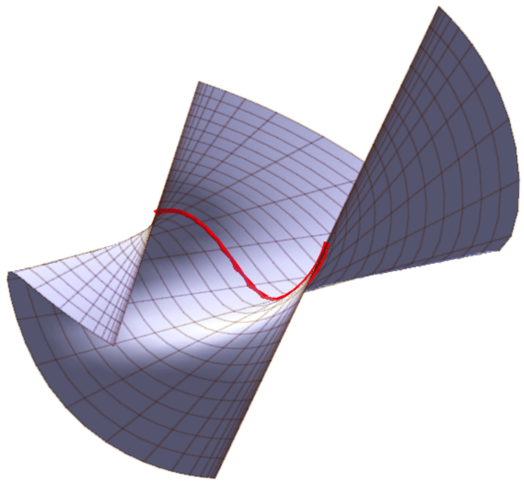
By substituting the relations given above into Equation 2.7, the proof is completed. □

Without the need for proof, from Definition 2.2 and the Equation 3.26 of Theorem 3.15, we have the following corollary:

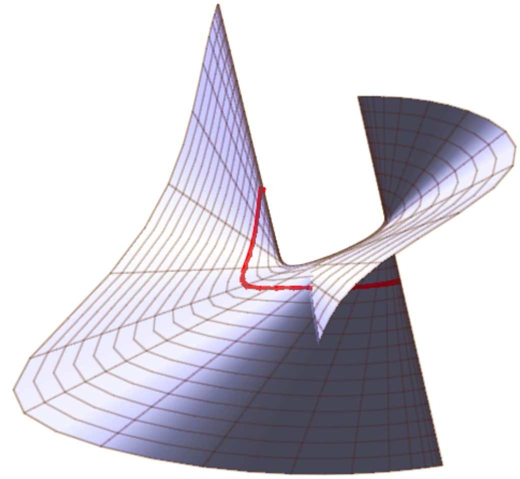
Corollary 3.17.

- *If the curve $\gamma(s)$ is plane curve, then its corresponding $NB-$ Smarandache curve lies both as geodesic and curvature line on the ruled surface ${}^{NB}_{B}\psi(s,v)$, while the normal curvature simplifies to the relation ${}^{NB}_{B}\kappa_n = -\frac{\kappa^2}{\sqrt{2}}$.*
- *The $NB-$ Smarandache curve of $\gamma(s)$ cannot be asymptotic on the ruled surface ${}^{NB}_{B}\psi(s,v)$.*

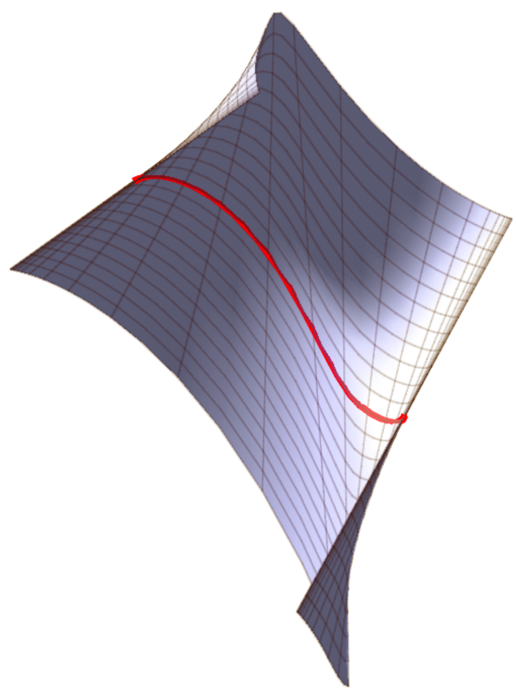
Example 3.3. *By utilizing the same curve as of previous examples, and by applying the Equations 3.14, 3.21 and 3.22 the ruled surfaces ${}^{NB}_{T}\psi(s,v)$, ${}^{NB}_{N}\psi(s,v)$ and ${}^{NB}_{B}\psi(s,v)$ can be easily obtained and illustrated in Fig. 3*



(a) The ruled surface ${}^{NB}_{T}\psi(s,v)$



(b) The ruled surface ${}^{NB}_{N}\psi(s,v)$



(c) The ruled surface ${}^{NB}_{B}\psi(s,v)$

FIGURE 3. Ruled surfaces with base curve of NB-Smarandache curve (red) where $s \in [-2\pi, 2\pi]$ and $v \in [-2, 2]$

3.4. **Ruled Surfaces with the Base TNB− Smarandache Curve.**

Definition 3.4. *Let $\gamma : s \in I \subset \Re \to E^3$ be a regular unit speed curve of C^2 class and $\{T,\ N,\ B\}$ denotes the set of its Frenet vectors. Then the ruled surfaces with the base of $TNB-$ Smarandache curve and the generator lines as each one of them are defined as following:*

$$\begin{aligned} {}_{T}^{TNB}\psi(s,v) &= \left(\frac{T+N+B}{\sqrt{3}}\right) + vT, \\ {}_{N}^{TNB}\psi(s,v) &= \left(\frac{T+N+B}{\sqrt{3}}\right) + vN, \\ {}_{B}^{TNB}\psi(s,v) &= \left(\frac{T+N+B}{\sqrt{3}}\right) + vB. \end{aligned} \tag{3.27}$$

3.4.1. *The characteristics of the ruled surface ${}_{T}^{TNB}\psi(s,v)$.*

Theorem 3.16. *The 1^{st} and 2^{nd} fundamental forms, and the curvatures of Gaussian and mean for the ruled surface, ${}_{T}^{TNB}\psi(s,v)$ are given as following:*

$$\begin{aligned} {}_{T}^{TNB}I &= \left(\frac{\kappa^2+\tau^2}{3} + \left(\frac{\kappa-\tau}{\sqrt{3}} + v\kappa\right)^2\right) ds^2 - \frac{2\kappa}{\sqrt{3}} dsdv + dv^2, \\ {}_{T}^{TNB}II &= \frac{\left(\left(\sqrt{3}v+1\right)(\tau\kappa'-\kappa\tau') - \tau\left(\tau^2+\kappa^2+\left(\kappa\left(\sqrt{3}v+1\right)-\tau\right)^2\right)\right) ds^2 + 2\sqrt{3}\kappa\tau\, dsdv}{\sqrt{3}\sqrt{\tau^2+\left(\kappa\left(\sqrt{3}v+1\right)-\tau\right)^2}}, \end{aligned}$$

$$\begin{aligned} {}_{T}^{TNB}K &= -3\left(\frac{\kappa\tau}{\tau^2+\left(\kappa\left(\sqrt{3}v+1\right)-\tau\right)^2}\right)^2, \\ {}_{T}^{TNB}H &= -\frac{\sqrt{3}}{2}\frac{\tau\left(\tau^2+\left(\sqrt{3}v\kappa-\tau\right)\left(2\kappa+\sqrt{3}v\kappa-\tau\right)\right)+(\kappa\tau'-\kappa'\tau)\left(\sqrt{3}v+1\right)}{\left(\tau^2+\left(\kappa\left(\sqrt{3}v+1\right)-\tau\right)^2\right)^{\frac{3}{2}}}, \end{aligned}$$

respectively.

Proof. The partial derivatives of ${}_{T}^{TNB}\psi(s,v)$ are given as follows

$$\begin{aligned} {}_{T}^{TNB}\psi(s,v)_s &= -\frac{\kappa}{\sqrt{3}}T + \left(\left(\frac{1}{\sqrt{3}}+v\right)\kappa - \frac{\tau}{\sqrt{3}}\right)N + \frac{\tau}{\sqrt{3}}B, \\ {}_{T}^{TNB}\psi(s,v)_{ss} &= \left(-\left(\frac{1}{\sqrt{3}}+v\right)\kappa^2 + \frac{\kappa\tau-\kappa'}{\sqrt{3}}\right)T - \left(\frac{\kappa^2+\tau^2+\tau'}{\sqrt{3}} - \left(\frac{1}{\sqrt{3}}+v\right)\kappa'\right)N \\ &\quad + \left(\left(\frac{1}{\sqrt{3}}+v\right)\tau\kappa + \frac{\tau'-\tau^2}{\sqrt{3}}\right)B, \\ {}_{T}^{TNB}\psi(s,v)_v &= T, \qquad {}_{T}^{TNB}\psi(s,v)_{sv} = \kappa N, \qquad {}_{T}^{TNB}\psi(s,v)_{vv} = 0. \end{aligned}$$

From Equation 2.3, the normal vector of ${}^{TNB}_{T}\psi(s,v)$ is computed as

$$ {}^{TNB}_{T}n = \frac{\tau N - \left(\kappa\left(\sqrt{3}v+1\right)-\tau\right)B}{\sqrt{\tau^2+\left(\kappa\left(\sqrt{3}v+1\right)-\tau\right)^2}}. $$

By recalling Equation 2.6, the components for the fundamental forms can be obtained. Then, by substituting those in Equation 2.4 and 2.5, the proof is completed. □

Corollary 3.18.

- *If the curve $\gamma(s)$ is a plane curve, then the ruled surface ${}^{TNB}_{T}\psi(s,v)$ is both developable and minimal.*
- *If the curve $\gamma(s)$ is a regular unit speed space curve, then ${}^{TNB}_{T}\psi(s,v)$ is minimal if the following relation holds*

$$ \tau\left(\tau^2+\left(\sqrt{3}v\kappa-\tau\right)\left(2\kappa+\sqrt{3}v\kappa-\tau\right)\right)-\left(\kappa\tau'-\kappa'\tau\right)\left(\sqrt{3}v+1\right)=0. $$

The base curve characteristics of the ${}^{TNB}_{T}\psi(s,v)$ surface associated to the curve $\gamma(s)$ is given with the following theorem.

Theorem 3.17. *The normal curvature ${}^{TNB}_{T}\kappa_n$, the geodesic curvature ${}^{TNB}_{T}\kappa_g$ and the geodesic torsion ${}^{TNB}_{T}\tau_g$ of the ${}^{TNB}_{T}\psi(s,v)$ surface are given as follows*

$$ \begin{aligned} {}^{TNB}_{T}\kappa_n &= -\frac{\left(\sqrt{3}v+1\right)\left(h'\kappa+h\kappa'\right)-h\left(\kappa'+\kappa^2\left(\left(\sqrt{3}v-2h\right)(h-1)-2\right)\right)}{\sqrt{3}\sqrt{\left(\sqrt{3}v-h+1\right)^2+h^2}}, \\ {}^{TNB}_{T}\kappa_g &= -\frac{\left(\sqrt{3}v-2h+1\right)h'+2\kappa\left(\sqrt{3}v-h+1\right)\left(h^2-h+1\right)}{2\left(h^2-h+1\right)\sqrt{\left(\sqrt{3}v-h+1\right)^2+h^2}}, \\ {}^{TNB}_{T}\tau_g &= \frac{\begin{array}{l}(h')^2\left(\sqrt{3}v+1\right)(2h-1) \\ -\kappa h'\left(3hv^2(1-2h)+v\sqrt{3}\left(2h^2-3h+2\right)(h+1)-2\left(h^2-h+1\right)\left(2h^2-1\right)\right)\end{array}}{2\sqrt{2}\left(\left(\sqrt{3}v-h+1\right)^2+h^2\right)\left(h^2-h+1\right)^{\frac{3}{2}}} \\ &\quad -\epsilon_5\frac{\kappa\tau\left(\sqrt{3}v-h+1\right)\left((h-1)\left(\sqrt{3}v-2h\right)-2\right)}{\sqrt{2}\left(\left(\sqrt{3}v-h+1\right)^2+h^2\right)\sqrt{h^2-h+1}}, \end{aligned} \tag{3.28} $$

respectively, where $\epsilon_5 = sign(h^2-h+1)$.

Proof. By referring the Equation (2.1), the tangent and its derivative, and the second order derivative of $TNB-$ Smarandache curve are given as

$$T_{TNB} = \frac{-T-(h-1)N+hB}{\sqrt{2}\sqrt{h^2-h+1}},$$

$$T'_{TNB} = \frac{\begin{array}{l}\left(h'(2h-1)+2\kappa(h-1)\left(h^2-h+1\right)\right)T \\ \quad -\left(h'(h+1)+2\kappa\left(h^2+1\right)\left(h^2-h+1\right)\right)N \\ \quad\quad -\left(h'(h-2)+2\tau(h-1)\left(h^2-h+1\right)\right)B\end{array}}{2\sqrt{2}(h^2-h+1)^{\frac{3}{2}}}, \tag{3.29}$$

$$\left(\frac{T+N+B}{\sqrt{3}}\right)'' = \frac{\begin{array}{l}-\left(\kappa'-\kappa^2(h-1)\right)T-\left(\kappa'(h-1)+\kappa\left(h'+\kappa\left(h^2+1\right)\right)\right)N \\ \quad +\left(h\kappa'+\kappa\left(h'-\tau(h-1)\right)\right)B\end{array}}{\sqrt{3}}.$$

Moreover, the derivative of the normal of ${}_{T}^{TNB}\psi(s,v)$ ruled surface which is expressed by the harmonic curvature function as ${}_{T}^{TNB}n = \dfrac{hN-\left(\sqrt{3}v-h+1\right)B}{\sqrt{\left(\sqrt{3}v-h+1\right)^2+h^2}}$ is given in the following

$$\begin{aligned}{}_{T}^{TNB}n' = &-\frac{\tau}{\sqrt{\left(\sqrt{3}v-h+1\right)^2+h^2}}T \\ &-\frac{\left(\tau\left(\left(\sqrt{3}v-h+1\right)^2+h^2\right)+h'\left(\sqrt{3}v+1\right)\right)\left(\sqrt{3}v-h+1\right)}{\left(\left(\sqrt{3}v-h+1\right)^2+h^2\right)^{\frac{3}{2}}}N \\ &-\frac{h\left(\tau\left(\left(\sqrt{3}v-h+1\right)^2+h^2\right)+h'\left(\sqrt{3}v+1\right)\right)}{\left(\left(\sqrt{3}v-h+1\right)^2+h^2\right)^{\frac{3}{2}}}B.\end{aligned}$$

By substituting the relations given above into Equation 2.7, the proof is completed. □

Without the need for proof, from Definition 2.2 and the Equation 3.28 of Theorem 3.17, we have the following corollary:

Corollary 3.19.

- *The $TNB-$ Smarandache curve of $\gamma(s)$ cannot be geodesic on the ruled surface ${}_{T}^{TNB}\psi(s,v)$.*
- *If the curve $\gamma(s)$ is plane curve, then its corresponding $TNB-$ Smarandache curve lies both as asymptotic and curvature line on the ruled surface ${}_{T}^{TNB}\psi(s,v)$, while the geodesic curvature simplifies to ${}_{T}^{TNB}\kappa_n = -\kappa$.*

3.4.2. *The characteristics of the ruled surface* ${}_{N}^{TNB}\psi(s,v)$.

Theorem 3.18. *The* 1^{st} *and* 2^{nd} *fundamental forms, and the curvatures of Gaussian and mean for the ruled surface,* ${}_{N}^{TNB}\psi(s,v)$ *are given as following:*

$$
\begin{aligned}
{}_{N}^{TNB}I &= \left(\left(\frac{1}{\sqrt{3}}+v\right)^2\left(\kappa^2+\tau^2\right)+\frac{(\kappa-\tau)^2}{3}\right)ds^2+\frac{2}{\sqrt{3}}(\kappa-\tau)dsdv+dv^2,\\
{}_{N}^{TNB}II &= \left(\frac{\left|\sqrt{3}v+1\right|(\kappa'\tau-\kappa\tau')}{\sqrt{3}\sqrt{(\tau^2+\kappa^2)}}\right)ds^2,\\
{}_{N}^{TNB}K &= 0,\\
{}_{N}^{TNB}H &= \frac{\sqrt{3}}{2}\frac{\tau\kappa'-\kappa\tau'}{\left|\sqrt{3}v+1\right|(\kappa^2+\tau^2)^{\frac{3}{2}}},
\end{aligned}
$$

respectively.

Proof. The partial derivatives of ${}_{N}^{TNB}\psi(s,v)$ are given as follows

$$
\begin{aligned}
{}_{N}^{TNB}\psi(s,v)_s &= -\left(\frac{1}{\sqrt{3}}+v\right)\kappa T+\frac{(\kappa-\tau)}{\sqrt{3}}N+\left(\frac{1}{\sqrt{3}}+v\right)\tau B,\\
{}_{N}^{TNB}\psi(s,v)_{ss} &= -\left(\frac{(\kappa-\tau)\kappa}{\sqrt{3}}+\kappa'\left(\frac{1}{\sqrt{3}}+v\right)\right)T-\left(\left(\frac{1}{\sqrt{3}}+v\right)\left(\tau^2+\kappa^2\right)-\frac{(\kappa'-\tau')}{\sqrt{3}}\right)N\\
&\quad+\left(\left(\frac{1}{\sqrt{3}}+v\right)\tau'+\frac{(\kappa-\tau)\tau}{\sqrt{3}}\right)B,\\
{}_{N}^{TNB}\psi(s,v)_v &= N, \qquad {}_{N}^{TNB}\psi(s,v)_{sv}=-\kappa T+\tau B, \qquad {}_{N}^{TNB}\psi(s,v)_{vv}=0.
\end{aligned}
$$

From Equation 2.3, the normal vector of ${}_{N}^{TNB}\psi(s,v)$ is computed as

$$
{}_{N}^{TNB}n=-\epsilon_6\frac{(\tau T+\kappa B)}{\sqrt{(\tau^2+\kappa^2)}},
$$

where $\epsilon_6=sign(\sqrt{3}v+1)$. By recalling Equation 2.6, the components for the fundamental forms can be obtained. Then, by substituting those in Equation 2.4 and 2.5, the proof is completed. □

Corollary 3.20.

- ${}_{N}^{TNB}\psi(s,v)$ *is always developable.*
- ${}_{N}^{TNB}\psi(s,v)$ *is minimal when* $\gamma(s)$ *is a general helix.*

Theorem 3.19. *The normal curvature ${}_{N}^{TNB}\kappa_n$, the geodesic curvature ${}_{N}^{TNB}\kappa_g$ and the geodesic torsion ${}_{N}^{TNB}\tau_g$ of the ${}_{N}^{TNB}\psi(s,v)$ surface are given as follows*

$$\begin{aligned}
{}_{N}^{TNB}\kappa_n &= -\epsilon_6 \frac{h'\kappa}{\sqrt{3}\sqrt{h^2+1}}, \\
{}_{N}^{TNB}\kappa_g &= -\epsilon_6 \frac{(h+1)\,h' + 2\kappa\left(h^2+1\right)\left(h^2-h+1\right)}{2\sqrt{h^2+1}\,(h^2-h+1)}, \\
{}_{N}^{TNB}\tau_g &= -\frac{(h+1)\,(h')^2 + 2\kappa h'\left(h^2+1\right)\left(h^2-h+1\right)}{2\sqrt{2}\,(h^2+1)\,(h^2-h+1)^{\frac{3}{2}}},
\end{aligned} \tag{3.30}$$

respectively.

Proof. By using the Equation 3.29, and taking the derivative of the normal of ${}_{N}^{TNB}\psi(s,v)$ ruled surface expressed by the harmonic curvature function as ${}_{N}^{TNB}n = -\epsilon_6 \frac{hT+B}{\sqrt{h^2+1}}$ is given in the following

$${}_{N}^{TNB}n' = -\epsilon_6 \sigma\kappa\,(T - hB)\,.$$

By substituting the relations given above into Equation 2.7, the proof is completed. □

Without the need for proof, from Definition 2.2 and the Equation 3.30 of Theorem 3.19, we have the following corollary:

Corollary 3.21.

- *The $TNB-$ Smarandache curve of $\gamma(s)$ cannot be geodesic on the ruled surface ${}_{N}^{TNB}\psi(s,v)$.*
- *If $\gamma(s)$ is a general helix such that h is constant, then its corresponding $TNB-$ Smarandache curve lies both as asymptotic and curvature line on ${}_{N}^{TNB}\psi(s,v)$, while the geodesic curvature simplifies to ${}_{N}^{TNB}\kappa_g = -\kappa\sqrt{h^2+1}$.*

3.4.3. *The characteristics of the ruled surface* ${}_{B}^{TNB}\psi(s,v)$.

Theorem 3.20. *The* 1^{st} *and* 2^{nd} *fundamental forms, and the curvatures of Gaussian and mean for the ruled surface,* ${}_{B}^{TNB}\psi(s,v)$ *are given as following:*

$$
{}_{B}^{TNB}I=\left(\frac{\kappa^2+\tau^2}{3}+\left(\frac{1}{\sqrt{3}}\left(\kappa-\tau\right)-v\tau\right)^2\right)ds^2+\frac{2\tau}{\sqrt{3}}dsdv+dv^2,
$$

$$
{}_{B}^{TNB}II=-\frac{\left(\kappa\left(\kappa^2+\tau^2+\left(\kappa-\tau\left(\sqrt{3}v+1\right)\right)^2\right)+\left(\sqrt{3}v+1\right)\left(\kappa\tau'-\tau\kappa'\right)\right)ds^2-2\sqrt{3}\kappa\tau dsdv}{\sqrt{3}\sqrt{\kappa^2+\left(\kappa-\tau\left(\sqrt{3}v+1\right)\right)^2}},
$$

$$
{}_{B}^{TNB}K=-3\left(\frac{\kappa\tau}{\kappa^2+\left(\kappa-\tau\left(\sqrt{3}v+1\right)\right)^2}\right)^2,
$$

$$
{}_{B}^{TNB}H=-\frac{\sqrt{3}}{2}\frac{\left(\kappa\tau'-\kappa'\tau\right)\left(\sqrt{3}v+1\right)-\kappa\left(\tau^2-\kappa^2-\left(\kappa-\tau\left(\sqrt{3}v+1\right)\right)^2\right)}{\left(\left(\kappa-\tau\left(v\sqrt{3}+1\right)\right)^2+\kappa^2\right)^{\frac{3}{2}}},
$$

respectively.

Proof. The partial derivatives of ${}_{B}^{TNB}\psi(s,v)$ are given as follows

$$
\begin{aligned}
{}_{B}^{TNB}\psi(s,v)_s &= -\frac{1}{\sqrt{3}}\left(\kappa T-\tau B\right)+\left(\frac{1}{\sqrt{3}}\left(\kappa-\tau\right)-v\tau\right)N,\\
{}_{B}^{TNB}\psi(s,v)_{ss} &= -\left(\frac{1}{\sqrt{3}}\kappa'+\left(\frac{(\kappa-\tau)}{\sqrt{3}}-v\tau\right)\kappa\right)T\\
&\quad -\left(\frac{1}{\sqrt{3}}\left(\kappa^2+\tau^2-\kappa'\right)+\left(\frac{1}{\sqrt{3}}+v\right)\tau'\right)N\\
&\quad +\left(\frac{1}{\sqrt{3}}\tau'+\left(\frac{1}{\sqrt{3}}\left(\kappa-\tau\right)-v\tau\right)\tau\right)B,\\
{}_{B}^{TNB}\psi(s,v)_v &= B, \qquad {}_{B}^{TNB}\psi(s,v)_{sv}=-\tau N, \qquad {}_{B}^{TNB}\psi(s,v)_{vv}=0.
\end{aligned}
$$

From Equation 2.3, the normal vector of ${}_{B}^{TNB}\psi(s,v)$ is computed as

$$
{}_{B}^{TNB}n=\frac{\left(\kappa-\tau\left(\sqrt{3}v+1\right)\right)T+\kappa N}{\sqrt{\kappa^2+\left(\kappa-\tau\left(\sqrt{3}v+1\right)\right)^2}}.
$$

By recalling Equation 2.6, the components for the fundamental forms can be obtained. Then, by substituting those in Equation 2.4 and 2.5, the proof is completed. ☐

Corollary 3.22.

- ${}_{B}^{TNB}\psi(s,v)$ *is developable, when* $\gamma(s)$ *is a plane curve.*
- ${}_{B}^{TNB}\psi(s,v)$ *can not be minimal.*
- *If* $\gamma(s)$ *is a plane curve, then* ${}_{B}^{TNB}\psi(s,v)$ *is a constant-mean-curvature (CMC) surface.*

Theorem 3.21. *The normal curvature* ${}_{B}^{TNB}\kappa_n$*, the geodesic curvature* ${}_{B}^{TNB}\kappa_g$ *and the geodesic torsion* ${}_{B}^{TNB}\tau_g$ *of the* ${}_{B}^{TNB}\psi(s,v)$ *surface are given as follows*

$$\begin{aligned}
{}_{B}^{TNB}\kappa_n &= -\frac{h'\kappa-\sqrt{3}h\kappa' v+\kappa^2\left(h\left(h-1\right)\left(\sqrt{3}v+2\right)+2\right)}{\sqrt{3}\sqrt{\left(h\left(\sqrt{3}v+1\right)-1\right)^2+1}},\\
{}_{B}^{TNB}\kappa_g &= \frac{h'-\left(h'+2\tau\left(h^2-h+1\right)\right)\left(h\left(\sqrt{3}v+1\right)-1\right)}{2\left(h^2-h+1\right)\sqrt{\left(h\left(\sqrt{3}v+1\right)-1\right)^2+1}},\\
{}_{B}^{TNB}\tau_g &= \frac{\begin{array}{c}(2-h)\left((h'')^2\left(\sqrt{3}v+1\right)+\kappa h' 3h^2v^2\right)\\ -\kappa h'\left(\sqrt{3}hv\left(h+1\right)\left(2h^2-3h+2\right)+2\left(h^2-h+1\right)\left(h^2-2\right)\right)\end{array}}{2\sqrt{2}(h^2-h+1)^{\frac{3}{2}}\left(\left(h\left(\sqrt{3}v+1\right)-1\right)^2+1\right)}\\
&\quad-\epsilon_5\frac{\kappa\tau\left(2h^2+\left(\sqrt{3}hv-2\right)\left(h-1\right)\right)\left(h\left(\sqrt{3}v+1\right)-1\right)}{\sqrt{2}\sqrt{h^2-h+1}\left(\left(h\left(\sqrt{3}v+1\right)-1\right)^2+1\right)},
\end{aligned} \tag{3.31}$$

respectively, where $\epsilon_5 = sign(h^2-h+1)$ *as already defined in the Theorem 3.17.*

Proof. By using the Equation 3.29, and taking the derivative of the normal of ${}_{B}^{TNB}\psi(s,v)$ ruled surface expressed by the harmonic curvature function as ${}_{B}^{TNB}n = -\frac{\left(h\left(\sqrt{3}v+1\right)-1\right)T-B}{\sqrt{\left(h\left(\sqrt{3}v+1\right)-1\right)^2+1}}$ is given in the following

$$\begin{aligned}
{}_{B}^{TNB}n' = &-\frac{2\kappa+\left(h'+\tau\left(h\left(\sqrt{3}v+1\right)-2\right)\right)\left(\sqrt{3}v+1\right)}{\left(\left(h\left(\sqrt{3}v+1\right)-1\right)^2+1\right)^{\frac{3}{2}}}T\\
&-\frac{\sqrt{3}\left(2\kappa+\left(h'+\tau\left(h\left(\sqrt{3}v+1\right)-2\right)\right)\left(\sqrt{3}v+1\right)\right)\left(h\left(\sqrt{3}v+1\right)-1\right)}{\left(\left(h\left(\sqrt{3}v+1\right)-1\right)^2+1\right)^{\frac{3}{2}}}N\\
&+\frac{\tau}{\sqrt{\left(h\left(\sqrt{3}v+1\right)-1\right)^2+1}}B.
\end{aligned}$$

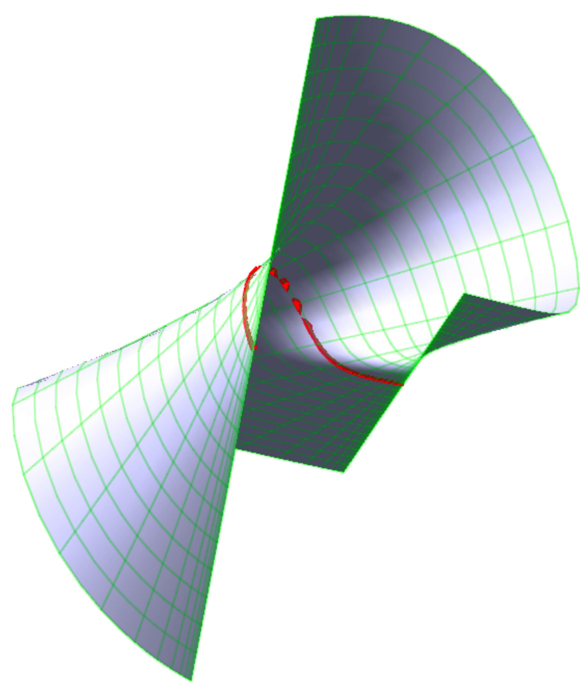
By substituting the relations given above into Equation 2.7, the proof is completed. □

Without the need for proof, from Definition 2.2 and the Equation 3.31 of Theorem 3.21, we have the following corollary:

Corollary 3.23.

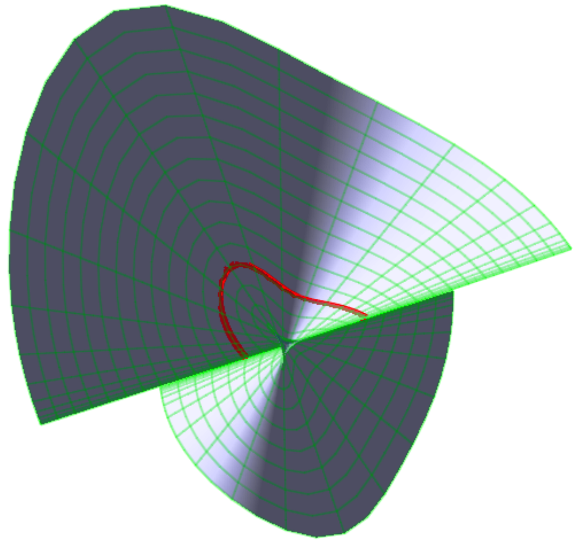
- *The $TNB-$ Smarandache curve of $\gamma(s)$ cannot be asymptotic on ${}^{TNB}_{B}\psi(s,v)$.*
- *If $\gamma(s)$ is plane curve, then its corresponding $TNB-$ Smarandache curve lies both as geodesic and curvature line on the ruled surface ${}^{TNB}_{B}\psi(s,v)$, while the asymptotic curvature simplifies to ${}^{TNB}_{B}\kappa_n = -\frac{\kappa}{\sqrt{3}}$.*

Example 3.4. *By utilizing the same curve as of previous examples, and by applying the Equations 3.14, and 3.27 the ruled surfaces ${}^{TNB}_{T}\psi(s,v)$, ${}^{TNB}_{N}\psi(s,v)$ and ${}^{TNB}_{B}\psi(s,v)$ can be easily obtained and illustrated in Fig. 4.*

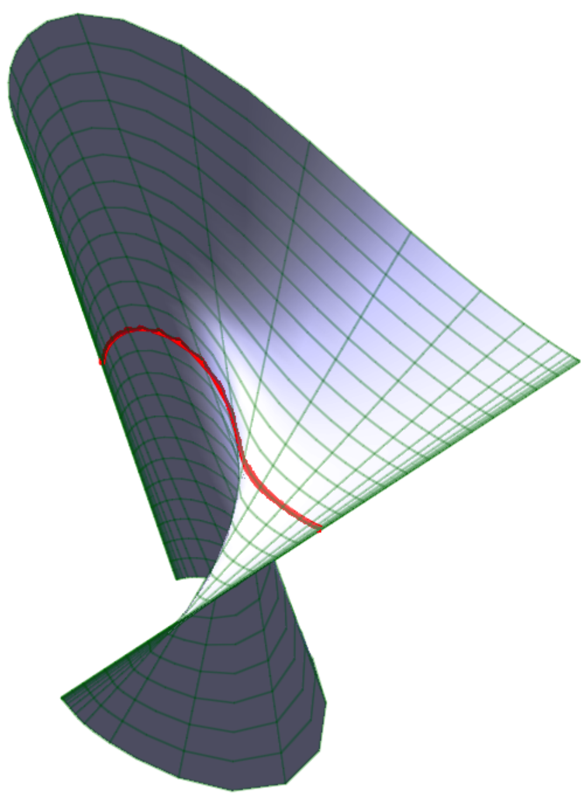


(a) The ruled surface ${}^{TNB}_{T}\psi(s,v)$

FIGURE 4. Ruled surfaces with base curve of TNB-Smarandache curve (red) where $s \in [-2\pi, 2\pi]$ and $v \in [-2, 2]$



(b) The ruled surface ${}_{N}^{TNB}\psi(s,v)$



(c) The ruled surface ${}_{B}^{TNB}\psi(s,v)$

FIGURE 4. Ruled surfaces with base curve of TNB-Smarandache curve (red) where $s \in [-2\pi, 2\pi]$ and $v \in [-2, 2]$

4. Conclusion

The theory of ruled surfaces plays an important role in the field of geometric modeling, since they are the most preferred ones for computational designs. This study introduces a series of new ruled surfaces and provides some of their metric properties. Such properties as developability and minimality are discussed in terms of the fundamental forms and principal curvatures. Hence, the required conditions are provided for each ruled surface to meet these characteristics. Moreover, asymptotic, geodesic and curvature line characteristics of the each Smarandache curve as a base curve are discussed. This way of generating and characterizing new ruled surfaces as in this study can be extended by referring other orthonormal frames and by using different space forms. Finally, researchers can be interested to examine the dual expressions for these surfaces.

Conflict of interest. The authors declare no conflict of interest.

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