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PRINCIPAL NORMAL INDICATRIX (N) OF CURVES ACCORDING TO THEIR ALTERNATIVE FRAMES IN EUCLIDEAN 3-SPACE

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Abstract. In this paper, we define a new family of curves called principal normal indicatrix (briefly, PNI) of space curves with unit speed in 3-dimensional Euclidean space. During the definition, we use alternative frames and we give some conditions for space curves to be general helix, slant helix, plane curve or involute curve.

Keywords: Spherical indicatrix, Alternative frame

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1. Introduction

Curve theory is a fascinating area of differential geometry and therefore, attracts many researchers. Curve theory investigates the properties and classifications of curves. On the other hand, curves contain some special curves within themselves. Special curves are such curves that satisfy certain conditions or exhibit interesting geometric behaviors. Special curves are studied in different spaces and different frames and are closely related to many application areas such as physics, engineering, computer aided design, robotics and medicine. Helices, involute-evolute, Bertrand and Mannheim curve pairs are some examples of well-known special curves.

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In curve theory, one of the most attractive special curves are combined curves. If there exists a mathematical relationship between two or more curves, these curves are called combined curves. Spherical indicatrix of curves are also combined curves in curve theory. Izumiya and Takeuchi [1] defined a new kind of slant helix in Euclidean 3-space. They showed that γ is a slant helix iff the geodesic curvature of spherical image of principal normal indicatrix (N) of a space curve γ is $\sigma = \frac{\kappa^2}{(\kappa^2+\tau^2)^{3/2}}\left(\frac{\tau}{\kappa}\right)'$. Kula and Yaylı [2] studied the spherical indicatrix curves of slant helices. They showed that their spherical indicatrices were spherical helices. They [3] also gave the characterizations of slant helices via certain differential equations verified for each one of spherical indicatrix in Euclidean 3-space. In [4], Uzunoğlu et al. studied a curve whose spherical images (the tangent and binormal indicatrices) are spherical slant helices by using alternative frame and called it as a C−slant helix.

In this paper, we firstly define the principal normal indicatrix (PNI) of space curves with unit speed in 3-dimensional Euclidean space by using alternative frame. Then, we investigate the geometric properties of PNI of space curves and give some relationships between these space curves and special well-known curves such as general helix, slant helix, plane curves and involute curves.

2. Preliminaries

A regular curve $\gamma(s) : I \subset \mathbb{R} \to \mathbb{R}^3$ has three orthonormal vectors denoted by $\vec{T}(s)$, $\vec{N}(s)$ and $\vec{B}(s)$ which are the tangent, the principal normal and the binormal unit vectors, respectively. The set $\left\{\vec{T}(s), \vec{N}(s), \vec{B}(s)\right\}$ is called Frenet frame of γ and the unit vectors are calculated by $\vec{T}(s) = \frac{\gamma'(s)}{\|\gamma'(s)\|}$, $\vec{B}(s) = \frac{\gamma'(s)\times\gamma''(s)}{\|\gamma'(s)\times\gamma''(s)\|}$, $\vec{N}(s) = \vec{B}(s) \times \vec{T}(s)$. The orthonormal frame $\left\{\vec{T}(s), \vec{N}(s), \vec{B}(s)\right\}$ has the Frenet-Serret formulas as

$$\begin{aligned} \vec{T}'(s) &= v\kappa(s)\vec{N}(s), \\ \vec{N}'(s) &= -v\left(\kappa(s)\vec{T}(s) + \tau(s)\vec{B}(s)\right), \\ \vec{B}'(s) &= -v\tau(s)\vec{N}(s), \end{aligned} \tag{2.1}$$

where $v = \|\gamma'(s)\|$, $\kappa(s)$ is the first and $\tau(s)$ is the second curvature functions of γ. Besides, the second curvature $\tau(s)$ is also known as torsion. The curvature functions are calculated as $\kappa(s) = \frac{\|\gamma'(s)\times\gamma''(s)\|}{\|\gamma'(s)\|^3}$, $\tau(s) = \frac{\langle\vec{\gamma}'(s)\times\vec{\gamma}''(s),\gamma'''(s)\rangle}{\|\gamma'(s)\times\gamma''(s)\|^2}$. A Darboux vector is the angular velocity vector $\vec{\omega}$ of the Frenet frame of a point moving with unit speed along a curve which enables us to interpret the curvature and torsion geometrically. The Darboux vector $\vec{\omega}$ is defined by

$\vec{\omega} = \tau\vec{T} + \kappa\vec{B}$ [5] and the unit Darboux vector is given by

$$\vec{W} = \frac{\vec{\omega}}{\|\vec{\omega}\|} = \frac{1}{\sqrt{\kappa^2+\tau^2}}\left(\tau\vec{T} + \kappa\vec{B}\right). \tag{2.2}$$

Because $\vec{W} \perp \vec{N}$, a new unit vector $\vec{C}$ is obtained as $\vec{C} = \vec{W} \times \vec{N}$. In this case, a new orthonormal frame can be constructed along γ. This frame is called as alternative frame and is denoted by $\left\{\vec{N}, \vec{C}, \vec{W}\right\}$. The derivative formulas of the frame $\left\{\vec{N}, \vec{C}, \vec{W}\right\}$ is presented by

$$\begin{bmatrix} \vec{N}' \\ \vec{C}' \\ \vec{W}' \end{bmatrix} = \begin{bmatrix} 0 & f & 0 \\ -f & 0 & g \\ 0 & -g & 0 \end{bmatrix} \begin{bmatrix} \vec{N} \\ \vec{C} \\ \vec{W} \end{bmatrix}, \tag{2.3}$$

where

$$f = \sqrt{\kappa^2+\tau^2}, \;\; g = \frac{\kappa^2}{\kappa^2+\tau^2}\left(\frac{\tau}{\kappa}\right)'. \tag{2.4}$$

The curve γ is a slant helix iff g/f is constant [4]. Moreover, the relationship between these frames is as follows:

$$\begin{cases} \vec{C} = -\kappa^*\vec{T} + \tau^*\vec{B}, \;\; \vec{W} = \tau^*\vec{T} + \kappa^*\vec{B}, \\ \vec{T} = -\kappa^*\vec{C} + \tau^*\vec{W}, \;\; \vec{B} = \tau^*\vec{C} + \kappa^*\vec{W}, \end{cases} \tag{2.5}$$

where $\kappa^* = \kappa/f$ and $\tau^* = \tau/f$ [6].

3. Principal Normal Indicatrix (N) of Curves

In this section, we firstly define a new spherical image by translating the unit principal normal vector $\vec{N}$ of a space curve with unit speed to the center of the unit sphere. Then, we will give the following definitions, theorems and propositions by using similar ideas in [2, 4, 7].

Definition 3.1. *Let $\gamma : I \subset \mathbb{R}^3$ and $\gamma_N : I \subset \mathbb{R}^3 \to S_0^2$ be unit speed curves. Then γ_N is called principal normal indicatrix (PNI) of the curve γ and satisfies the equation as*

$$\vec{\gamma}_N(s_N) = \vec{N}(s)$$

where S_0^2 denotes a unit sphere and, s and s_N are arc length parameters of γ and γ_N, respectively.

Now, let calculate the ratio ds/ds_N.

$$s_N = \int_0^s \|\vec{\gamma}_N'\|\, du = \int_0^s \left\|f\vec{C}\right\| du = \int_0^s |f|\, du$$

The differential of s_N yields

$$ds_N = f ds.$$

where $f > 0$. Then, we get

$$\frac{ds}{ds_N} = \frac{1}{f}.$$

Theorem 3.1. *Let γ_N be the PNI of a unit speed curve γ in Euclidean 3-space. Then the Frenet frame of γ_N is computed with regards to the alternative frame of γ as follows:*

$$\begin{aligned} \vec{T}_N &= \vec{C}, \\ \vec{N}_N &= -m\vec{N} + n\vec{W}, \\ \vec{B}_N &= n\vec{N} + m\vec{W} \end{aligned} \tag{3.6}$$

where $m = \frac{f}{\sqrt{f^2+g^2}}$ and $n = \frac{g}{\sqrt{f^2+g^2}}$, and the sets $\left\{\vec{N}, \vec{C}, \vec{W}\right\}$ and $\left\{\vec{T}_N, \vec{N}_N, \vec{B}_N\right\}$ are alternative frame of γ and Frenet frame of γ_N, respectively.

Proof. The derivative of $\vec{\gamma}_N(s_N) = \vec{N}(s)$ with respect to s is obtained as

$$\frac{d\vec{\gamma}_N}{ds} = \vec{\gamma}_N' = f\vec{C} \tag{3.7}$$

From here, we have

$$\vec{T}_N = \frac{d\vec{\gamma}_N}{ds_N} = \frac{d\vec{\gamma}_N}{ds}\frac{ds}{ds_N} = f\vec{C}\frac{1}{f} = \vec{C}$$

On the other hand, the second derivative of γ_N with respect to s yields

$$\vec{\gamma}_N'' = -f^2\vec{N} + f'\vec{C} + fg\vec{W}. \tag{3.8}$$

The cross product of (3.7) and (3.8) gives Binormal vector of γ_N as

$$\vec{B}_N = \frac{\vec{\gamma}_N' \times \vec{\gamma}_N''}{\left\|\vec{\gamma}_N' \times \vec{\gamma}_N''\right\|} = \frac{g}{\sqrt{f^2+g^2}}\vec{N} + \frac{f}{\sqrt{f^2+g^2}}\vec{W}. \tag{3.9}$$

From (3.7) and (3.9), we get

$$\vec{N}_N = \vec{B}_N \times \vec{T}_N = -\frac{f}{\sqrt{f^2+g^2}}\vec{N} + \frac{g}{\sqrt{f^2+g^2}}\vec{W}. \tag{3.10}$$

Thus, we complete the proof. □

We should note that since $f > 0$, we obtain $m > 0$. Then, the following theorem is given:

Theorem 3.2. *Let γ_N be the PNI of a unit speed curve γ. Then the following invariants are given:*

$$\kappa_N = \frac{1}{m}, \tau_N = \frac{m}{f}\left(\frac{g}{f}\right)' \tag{3.11}$$

where κ_N and τ_N are curvature and torsion of γ_N, respectively and $\kappa_N > 0$.

Proof. From the cross product of (3.7) and (3.8), we get

$$\left\|\vec{\gamma}_N' \times \vec{\gamma}_N''\right\| = f^2\sqrt{f^2+g^2}.$$

Since the curvature function of γ_N is computed by $\kappa_N = \frac{\|\vec{\gamma}_N' \times \vec{\gamma}_N''\|}{\|\vec{\gamma}_N'\|^3}$, we have

$$\kappa_N = \frac{1}{m}.$$

The torsion of γ_N is also computed by $\tau_N = \frac{\langle \vec{\gamma}_N' \times \vec{\gamma}_N'', \gamma_N''' \rangle}{\|\vec{\gamma}_N' \times \vec{\gamma}_N''\|^2}$. Then, we get

$$\tau_N = \frac{m}{f}\left(\frac{g}{f}\right)'.$$

Lastly, since $m > 0$, then $\kappa_N > 0$. □

Theorem 3.3. *The unit speed curve γ is a general helix iff the curve γ_N is a circle or a part of a circle.*

Proof. Let γ be a general helix. Then from (2.4), we have that τ/κ is constant, i.e., $g = 0$. Thus, we have $\kappa_N = 1$ and $\tau_N = 0$ which means γ_N is a circle or a part of a circle. Conversely, assume that γ_N is a circle or a part of a circle. Then, κ_N is a non-zero constant and $\tau_N = 0$. Since $f > 0$ and $m > 0$, we get $g = 0$. Therefore, γ is a general helix. □

Theorem 3.4. *The unit speed curve γ is a slant helix iff the curve γ_N is planar.*

Proof. Let γ be a slant helix. We know that the curve γ is a slant helix iff g/f is constant [4]. Then, from (3.11), we obtain $\tau_N = 0$ which means γ_N is planar. Conversely, assume that γ_N is planar. Thus, we get $\tau_N = 0$ and it implies that g/f is constant. □

Now we will take $\mu = \frac{1}{f(f^2+g^2)^{3/2}}\left(\frac{g}{f}\right)'$ to simplify the equations further. Then, we can give the following theorem:

Theorem 3.5. *Let γ_N be the PNI of a unit speed curve γ. Then the Darboux-like vector of Frenet frame $\left\{\vec{T}_N, \vec{N}_N, \vec{B}_N\right\}$ is given by*

$$\vec{\omega}_N = \frac{n}{m}\vec{N} + \frac{f\mu}{m^2}\vec{C} + \vec{W}.$$

Proof. We know that the Darboux vector of the Frenet frame $\left\{\vec{T}, \vec{N}, \vec{B}\right\}$ is given by $\vec{\omega} = \tau\vec{T} + \kappa\vec{B}$. Similarly, the Darboux vector of the Frenet frame $\left\{\vec{T}_N, \vec{N}_N, \vec{B}_N\right\}$ is given by $\vec{\omega}_N = \tau_N\vec{T}_N + \kappa_N\vec{B}_N$. Substituting the equalities in (3.6) and (3.11) into $\vec{\omega}_N$, we get $\vec{\omega}_N = \frac{g}{f}\vec{N} + \frac{1}{\sqrt{f^2+g^2}}\left(\frac{g}{f}\right)'\vec{C} + \vec{W}$. Since $m = \frac{f}{\sqrt{f^2+g^2}}$ and $n = \frac{g}{\sqrt{f^2+g^2}}$, we see that $\frac{g}{f} = \frac{n}{m}$ and

$m^2+n^2=1$. Taking $\frac{1}{f(f^2+g^2)^{3/2}}\left(\frac{g}{f}\right)'=\mu$, we write $\frac{1}{\sqrt{f^2+g^2}}\left(\frac{g}{f}\right)'=\frac{f\mu}{m^2}$. Then, we complete the proof. □

From (3.6), we see that there exists an orthonormal frame such as $\left\{\vec{N}_N,\vec{C}_N,\vec{W}_N\right\}$. In this case, we give the following corollary:

Corollary 3.1. *Let γ_N be the PNI of a unit speed curve γ in Euclidean 3-space. Then the alternative frame of γ_N is computed with regards to the alternative frame of γ as follows:*

$$\begin{aligned}
\vec{N}_N &= -m\vec{N}+n\vec{W},\\
\vec{C}_N &= \tfrac{\mu nf}{\sqrt{m^2+\mu^2f^2}}\vec{N}-\tfrac{m}{\sqrt{m^2+\mu^2f^2}}\vec{C}+\tfrac{\mu mf}{\sqrt{m^2+\mu^2f^2}}\vec{W},\\
\vec{W}_N &= \tfrac{mn}{\sqrt{m^2+\mu^2f^2}}\vec{N}+\tfrac{\mu f}{\sqrt{m^2+\mu^2f^2}}\vec{C}+\tfrac{m^2}{\sqrt{m^2+\mu^2f^2}}\vec{W},
\end{aligned}\tag{3.12}$$

where the sets $\left\{\vec{N},\vec{C},\vec{W}\right\}$ and $\left\{\vec{N}_N,\vec{C}_N,\vec{W}_N\right\}$ are alternative frames of γ and γ_N, respectively.

Proof. The norm of $\vec{\omega}_N$ in Theorem 3.5 is obtained as $\|\vec{\omega}_N\|=\frac{\sqrt{m^2+\mu^2f^2}}{m^2}$. So, we get the Darboux unit vector $\vec{W}_N=\frac{\vec{\omega}_N}{\|\vec{\omega}_N\|}$. Since $\vec{C}_N=\vec{W}_N\times\vec{N}_N$, we satisfy the desired result. □

Theorem 3.6. *Let γ_N be the PNI of a unit speed curve γ. Then the following invariants are given:*

$$f_N=\sqrt{\frac{1}{m^2}+\frac{\mu m^4}{f^3}},\; g_N=\frac{f^3}{f^3+\mu m^2}\left(\frac{\mu m^5}{f^3}\right)'$$

where f_N and g_N are curvatures of γ_N, and $f_N>0$.

Proof. From (2.4), we can write $f_N=\sqrt{\kappa_N^2+\tau_N^2}$ and $g_N=\frac{\kappa_N^2}{\kappa_N^2+\tau_N^2}\left(\frac{\tau_N}{\kappa_N}\right)'$. Substituting $\kappa_N=\frac{1}{m}$, $\tau_N=\frac{m^2}{f}\left(\frac{g}{f}\right)'$ in (3.11) into f_N and g_N, we get (3.13). □

Theorem 3.7. *Let that $\bar{\gamma}$ be an involute of the unit speed curve γ and the set $\left\{\vec{\bar{T}},\vec{\bar{N}},\vec{\bar{B}}\right\}$ denotes the Frenet frame of $\bar{\gamma}$. Then the relationship between the Frenet frames $\left\{\vec{T}_N,\vec{N}_N,\vec{B}_N\right\}$ and $\left\{\vec{\bar{T}},\vec{\bar{N}},\vec{\bar{B}}\right\}$ of γ_N and $\bar{\gamma}$, respectively is computed as*

$$\begin{cases}
\vec{T}_N=\vec{\bar{N}},\\
\vec{N}_N=-m\vec{\bar{T}}+n\vec{\bar{B}},\\
\vec{B}_N=n\vec{\bar{T}}+m\vec{\bar{B}}.
\end{cases}\tag{3.13}$$

Proof. Since $\bar{\gamma}$ is an involute of the unit speed curve γ, we know that

$$\begin{aligned}
\vec{\bar{T}} &= \vec{N},\\
\vec{\bar{N}} &= -\tfrac{\kappa}{f}\vec{T}+\tfrac{\tau}{f}\vec{B},\\
\vec{\bar{B}} &= \tfrac{\tau}{f}\vec{T}+\tfrac{\kappa}{f}\vec{B}
\end{aligned}\tag{3.14}$$

where $f = \sqrt{\kappa^2 + \tau^2}$ [2, 5, 6]. From (2.4) and (2.5), we can rewrite (3.14) as follows:

$$\vec{\bar{T}} = \vec{N}, \quad \vec{\bar{N}} = \vec{C}, \quad \vec{\bar{B}} = \vec{W}. \tag{3.15}$$

Substituting the last three equalities into (3.6), we complete the proof. □

Remark 3.1. *The alternative frame* $\left\{\vec{N}, \vec{C}, \vec{W}\right\}$ *of the unit speed curve* γ *and the Frenet frame* $\left\{\vec{\bar{T}}, \vec{\bar{N}}, \vec{\bar{B}}\right\}$ *of its involute* $\bar{\gamma}$ *coincide with each other at any moment.*

Proposition 3.1. *Let* $\bar{\kappa}$ *and* $\bar{\tau}$ *be the curvature and torsion functions of involute* $\bar{\gamma}$ *of the unit speed curve* γ*, respectively. Then, PNI* γ_N *of the curve* γ *is an involute of* $\bar{\gamma}$ *iff* $f = \bar{\kappa}$ *and* $g = \bar{\tau}$*.*

Proof. If γ_N is an involute of the curve $\bar{\gamma}$, we get

$$\begin{aligned} \vec{T}_N &= \vec{\bar{N}}, \\ \vec{N}_N &= -\tfrac{\bar{\kappa}}{\bar{f}}\vec{\bar{T}} + \tfrac{\bar{\tau}}{\bar{f}}\vec{\bar{B}}, \\ \vec{B}_N &= \tfrac{\bar{\tau}}{\bar{f}}\vec{\bar{T}} + \tfrac{\bar{\kappa}}{\bar{f}}\vec{\bar{B}} \end{aligned} \tag{3.16}$$

where $\bar{f} = \sqrt{\bar{\kappa}^2 + \bar{\tau}^2}$ [2, 5, 6]. When we compare (3.16) with (3.13), we complete the proof. □

3.1. **Relationships Between Principal Normal Indicatrix (N) and Tangent Indicatrix (T) of Curves.**

Definition 3.2. *Let* $\gamma : I \subset \mathbb{R}^3$ *and* $\gamma_T : I \subset \mathbb{R}^3 \to S_0^2$ *be unit speed curves in Euclidean 3-space. Then* γ_T *is called tangent indicatrix of the curve* γ *and satisfies the equation as*

$$\vec{\gamma}_T(s_T) = \vec{T}(s)$$

where S_0^2 *denotes a unit sphere and,* s *and* s_T *are arc length parameters of* γ *and* γ_T*, respectively.*

Now, let calculate the ratio ds/ds_T.

$$s_T = \int_0^s \left\|\vec{\gamma}_T'\right\| du = \int_0^s \left\|\kappa\vec{N}\right\| du = \int_0^s |\kappa|\, du$$

The differential of s_T gives

$$ds_T = \kappa ds.$$

where $\kappa > 0$. Then, we get

$$\frac{ds}{ds_T} = \frac{1}{\kappa}.$$

Theorem 3.8. *Let γ_T be the tangent indicatrix of a unit speed curve γ in Euclidean 3-space. Then the Frenet frame $\left\{\vec{T}_T, \vec{N}_T, \vec{B}_T\right\}$ of γ_T is computed with regards to the alternative frame $\left\{\vec{N}, \vec{C}, \vec{W}\right\}$ of γ as follows:*

$$\begin{aligned} \vec{T}_T &= \vec{N}, \\ \vec{N}_T &= -\tfrac{\kappa}{f}\vec{T} + \tfrac{\tau}{f}\vec{B}, \\ \vec{B}_T &= \tfrac{\tau}{f}\vec{T} + \tfrac{\kappa}{f}\vec{B} \end{aligned} \tag{3.17}$$

where $f = \sqrt{\kappa^2 + \tau^2}$.

Proof. The derivative of $\vec{\gamma}_T(s_T) = \vec{T}(s)$ is obtained as

$$\frac{d\vec{\gamma}_T}{ds} = \vec{\gamma}_T' = \kappa\vec{N}.$$

From here, we have

$$\vec{T}_T = \frac{d\vec{\gamma}_T}{ds_T} = \frac{d\vec{\gamma}_T}{ds}\frac{ds}{ds_T} = \kappa\vec{N}\frac{1}{\kappa} = \vec{N}$$

The second derivative of γ_T also gives

$$\vec{\gamma}_T'' = -\kappa^2\vec{T} + \kappa'\vec{N} + \kappa\tau\vec{B}.$$

The cross product of $\vec{\gamma}_T'$ and $\vec{\gamma}_T''$ gives binormal vector of γ_T as

$$\vec{B}_T = \frac{\vec{\gamma}_T' \times \vec{\gamma}_T''}{\left\|\vec{\gamma}_T' \times \vec{\gamma}_T''\right\|} = \frac{\tau}{\sqrt{\kappa^2+\tau^2}}\vec{T} + \frac{\kappa}{\sqrt{\kappa^2+\tau^2}}\vec{B}.$$

Thus, we have

$$\vec{N}_T = \vec{B}_T \times \vec{T}_T = -\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}\vec{T} + \frac{\tau}{\sqrt{\kappa^2+\tau^2}}\vec{B}.$$

Thus, we complete the proof. □

From (2.5), we can rewrite the equations in (3.17) as follows:

$$\vec{T}_T = \vec{N}, \quad \vec{N}_T = \vec{C}, \quad \vec{B}_T = \vec{W}. \tag{3.18}$$

Therefore, from (3.15), the following corollary is given:

Corollary 3.2. *The Frenet frame $\left\{\vec{\bar{T}}, \vec{\bar{N}}, \vec{\bar{B}}\right\}$ of involute curve $\bar{\gamma}$ of the unit speed curve γ coincide with the Frenet frame $\left\{\vec{T}_T, \vec{N}_T, \vec{B}_T\right\}$ of tangent indicatrix γ_T of the curve γ.*

Corollary 3.3. *The tangent indicatrix curve γ_T is an involute of the unit speed curve γ.*

Proof. From (3.17) the proof is satisfied clearly. □

Now, from (3.6) and (3.17), we can give the following proposition:

Proposition 3.2. *Let that γ_N and γ_T be principal normal and tangent indicatrices of the unit speed curve γ in Euclidean 3-space with regards to the Frenet frames $\left\{\vec{T}_N, \vec{N}_N, \vec{B}_N\right\}$ and $\left\{\vec{T}_T, \vec{N}_T, \vec{B}_T\right\}$, respectively. Then, we have*

$$\begin{aligned}
\vec{T}_N &= \vec{N}_T, \\
\vec{N}_N &= -m\vec{T}_T + n\vec{B}_T, \\
\vec{B}_N &= n\vec{T}_T + m\vec{B}_T
\end{aligned} \tag{3.19}$$

where $m = \frac{f}{\sqrt{f^2+g^2}}$ and $n = \frac{g}{\sqrt{f^2+g^2}}$.

Proposition 3.3. *PNI γ_N of the unit speed curve γ is an involute of the tangent indicatrix γ_T of the curve γ iff $f = \kappa$ and $g = \tau$.*

3.2. **Relationships Between Principal Normal Indicatrix (N) and Binormal Indicatrix (B) of Curves.**

Definition 3.3. *Let $\gamma : I \subset \mathbb{R}^3$ and $\gamma_B : I \subset \mathbb{R}^3 \to S_0^2$ be unit speed curves in Euclidean 3-space. Then γ_B is called binormal indicatrix of the curve γ and satisfies the equation as*

$$\vec{\gamma}_B(s_B) = \vec{B}(s)$$

where S_0^2 denotes a unit sphere and, s and s_B are arc length parameters of γ and γ_B, respectively.

Now, let calculate the ratio ds/ds_B.

$$s_B = \int_0^s \left\|\vec{\gamma}_B'\right\| du = \int_0^s \left\|-\tau\vec{N}\right\| du = \int_0^s |\tau|\, du$$

The differential of s_B gives

$$ds_B = \tau ds.$$

where $\tau > 0$. Then, we get

$$\frac{ds}{ds_B} = \frac{1}{\tau}.$$

Theorem 3.9. *Let γ_B be the binormal indicatrix of a unit speed curve γ in Euclidean 3-space. Then the Frenet frame $\left\{\vec{T}_B, \vec{N}_B, \vec{B}_B\right\}$ of γ_T is computed with regards to the alternative frame $\left\{\vec{N}, \vec{C}, \vec{W}\right\}$ of γ as follows:*

$$\begin{aligned}
\vec{T}_B &= -\vec{N}, \\
\vec{N}_B &= \tfrac{\kappa}{f}\vec{T} - \tfrac{\tau}{f}\vec{B}, \\
\vec{B}_B &= \tfrac{\tau}{f}\vec{T} + \tfrac{\kappa}{f}\vec{B}
\end{aligned} \tag{3.20}$$

where $f = \sqrt{\kappa^2 + \tau^2}$.

Proof. The derivative of $\vec{\gamma}_B(s_B) = \vec{B}(s)$ is obtained as

$$\frac{d\vec{\gamma}_B}{ds} = \vec{\gamma}_B' = -\tau\vec{N}.$$

From here, we have

$$\vec{T}_B = \frac{d\vec{\gamma}_B}{ds_B} = \frac{d\vec{\gamma}_B}{ds}\frac{ds}{ds_B} = -\tau\vec{N}\frac{1}{\tau} = -\vec{N}$$

The second derivative of γ_B also gives

$$\vec{\gamma}_B'' = \kappa\tau\vec{T} - \tau'\vec{N} - \tau^2\vec{B}.$$

The cross product of $\vec{\gamma}_B'$ and $\vec{\gamma}_B''$ gives binormal vector of γ_B as

$$\vec{B}_B = \frac{\vec{\gamma}_B' \times \vec{\gamma}_B''}{\left\|\vec{\gamma}_B' \times \vec{\gamma}_B''\right\|} = \frac{\tau}{\sqrt{\kappa^2+\tau^2}}\vec{T} + \frac{\kappa}{\sqrt{\kappa^2+\tau^2}}\vec{B}.$$

Thus, we have

$$\vec{N}_B = \vec{B}_B \times \vec{T}_B = \frac{\kappa}{\sqrt{\kappa^2+\tau^2}}\vec{T} - \frac{\tau}{\sqrt{\kappa^2+\tau^2}}\vec{B}.$$

Thus, we complete the proof. □

From (2.5), we can rewrite the equations in (3.20) as follows:

$$\vec{T}_B = -\vec{N}, \quad \vec{N}_B = -\vec{C}, \quad \vec{B}_B = \vec{W}. \tag{3.21}$$

Therefore, from (3.15), the following corollary is given:

Corollary 3.4. *The Frenet frame* $\left\{\vec{\bar{T}}, \vec{\bar{N}}, \vec{\bar{B}}\right\}$ *of involute curve* $\bar{\gamma}$ *of the unit speed curve* γ *coincide with the Frenet frame* $\left\{\vec{T}_B, \vec{N}_B, \vec{B}_B\right\}$ *of binormal indicatrix* γ_B *of the curve* γ*, i.e.,* $\vec{\bar{T}} = -\vec{T}_B, \quad \vec{\bar{N}} = -\vec{N}_B, \quad \vec{\bar{B}} = \vec{B}_B$.

Corollary 3.5. *The binormal indicatrix curve* γ_B *is an involute of the unit speed curve* γ*.*

Proof. From (3.20) the proof is satisfied clearly. □

Now, from (3.6) and (3.20), we can give the following proposition:

Proposition 3.4. *Let that* γ_N *and* γ_B *be principal normal and binormal indicatrices of the unit speed curve* γ *in Euclidean 3-space with regards to the Frenet frames* $\left\{\vec{T}_N, \vec{N}_N, \vec{B}_N\right\}$ *and* $\left\{\vec{T}_B, \vec{N}_B, \vec{B}_B\right\}$*, respectively. Then, we have*

$$\begin{aligned}\vec{T}_N &= -\vec{N}_B,\\ \vec{N}_N &= m\vec{T}_B + n\vec{B}_B,\\ \vec{B}_N &= -n\vec{T}_B + m\vec{B}_B\end{aligned} \tag{3.22}$$

where $m = \frac{f}{\sqrt{f^2+g^2}}$ *and* $n = \frac{g}{\sqrt{f^2+g^2}}$.

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