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ON ROUGH $\mathcal{I}$-STATISTICAL CONVERGENCE OF COMPLEX UNCERTAIN SEQUENCES

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Abstract. In this paper, we introduce the notion of rough $\mathcal{I}$-statistical convergence of complex uncertain sequences in four aspects of uncertainty, viz., almost surely, measure, mean, distribution as an extension of rough convergence, rough statistical convergence, and rough $\mathcal{I}$-convergence of complex uncertain sequences. Also, we explore the concept of rough $\mathcal{I}$-statistical convergence in p-distance, and rough $\mathcal{I}$-statistical convergence in metric of complex uncertain sequences. Overall, this study mainly presents a diagrammatic scenario of interrelationships among all rough $\mathcal{I}$-statistical convergence concepts of complex uncertain sequences and include some observations about the above convergence concepts.

Keywords: Uncertainty space, Complex uncertain variable, Complex uncertain sequence, Almost surely, Rough $\mathcal{I}$-statistical convergence.

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1. Introduction

The idea of the convergence of a sequence of real numbers has been extended to statistical convergence independently by Fast [12] and Steinhaus [30]. Later, it was studied by Fridy [13] and many other researchers. A sequence (x_m) is said to be statistically convergent to ℓ provided that for each $\varepsilon > 0$ such that

$$\lim_{m\to\infty} \frac{1}{m} |\{k \leq m : |x_k - \ell| \geq \varepsilon\}| = 0, \ m \in \mathbb{N}.$$

The concept of $\mathcal{I}$-convergence was introduced by Kostyrko et al. [20] as a generalization

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of statistical convergence. The idea of $\mathcal{I}$-convergence was further extended to $\mathcal{I}$-statistical convergence by Savas and Das [27]. Later on, more investigation in this direction can be found in the works of [11, 15, 28].

The idea of rough convergence was first introduced by Phu [23] in finite-dimensional normed spaces. A sequence (x_m) is said to be rough convergent to ℓ provided that for each $\varepsilon > 0$ $\exists$ $m_\varepsilon \in \mathbb{N}$ such that

$$|x_m - \ell| < r + \varepsilon \text{ for all } m \geq m_\varepsilon,$$

where r is a non-negative real number and called roughness degree. After that, Dündar and Çakan [10] introduced the notion of rough $\mathcal{I}$-convergence of sequence. The concept of rough $\mathcal{I}$-statistical convergence of sequences was introduced by Savaş et al. [29] in the year 2018.

On the other hand, in 2007, Liu [21] introduced a theory named uncertainty theory, including different types of convergence of uncertain sequences and identifying the relationships among various forms of convergence, such as convergence in measure, distribution, mean, and convergence a.s. Then the concept has been extended to the c.u.v.s by Peng [22]. After that, Chen et al. [2] subsequently studied the idea of convergence of c.u.s.s using c.u.v.s. In 2017, Tripathy and Nath [31] proposed the idea of statistical convergence of c.u.s.s in the context of uncertainty theory. After that, Debnath and Das [6,7] introduced the notion of rough convergence and rough statistical convergence of c.u.s.s, and this field has also seen a lot of exciting changes; for details, see [1, 3–5, 9, 14, 16, 17, 19, 24–26]. The concept of rough $\mathcal{I}$-convergence of complex uncertain sequences was recently introduced by Debnath and Halder [8].

Inspired by the above works, in this paper we introduce the notion of rough $\mathcal{I}$-statistical convergence of c.u.s.s in four aspects of uncertainty, viz., a.s., measure, mean, and distribution. We also explore the concepts of rough $\mathcal{I}$-statistical convergence in p-distance, and rough $\mathcal{I}$-statistical convergence in metric of c.u.s.s. Finally, we try to establish the relationship among all rough $\mathcal{I}$-statistical convergence concepts of c.u.s.s with an attached diagramatic section.

2. Definitions and Preliminaries

In this section, we provide some basic ideas and results on generalized convergence concepts and the theory of uncertainty that will be used throughout the article.

Definition 2.1. *[20] Consider a non-empty set S. An ideal on S is defined as a family of subsets $\mathcal{I}$ that satisfies the following conditions:*

(i) The empty set, ϕ, belongs to $\mathcal{I}$.

(ii) For any $U, V \in \mathcal{I}$, the union of U and V, denoted as $U \cup V$, is also in $\mathcal{I}$.

(iii) For any $U \in \mathcal{I}$ and any subset $V \subset U$, V is a member of $\mathcal{I}$.

An ideal $\mathcal{I}$ is called non-trivial if $\mathcal{I} \neq \{\Phi\}$ and $S \notin \mathcal{I}$.

A non-trivial ideal $\mathcal{I}$ is called an admissible ideal in S if and only if $\{\{s\} : s \in S\} \subset \mathcal{I}$.

Example 2.1. (*i*) $\mathcal{I}_f$:= *The set of all finite subsets of $\mathbb{N}$ forms a non-trivial admissible ideal.*

(*ii*) $\mathcal{I}_d$:= *The set of all subsets of $\mathbb{N}$ whose natural density is zero forms a non-trivial admissible ideal.*

Definition 2.2. *[20] A sequence (x_m) is said to be $\mathcal{I}$-convergent to ℓ, if for every $\varepsilon > 0$, the set $\{m \in \mathbb{N} : |x_m - \ell| \geq \varepsilon\} \in \mathcal{I}$.*

The usual convergence of sequences is a special case of $\mathcal{I}$-convergence ($\mathcal{I}$=$\mathcal{I}_f$-the ideal of all finite subsets of $\mathbb{N}$). The statistical convergence of sequences is also a special case of $\mathcal{I}$-convergence. In this case, $\mathcal{I}$=$\mathcal{I}_d$ $= \left\{A \subseteq \mathbb{N} : \lim_{m\to\infty} \frac{|A \cap \{1,2,\cdots,m\}|}{m} = 0\right\}$, where $|A|$ is the cardinality of the set A.

Definition 2.3. *[29] A sequence (x_m) is said to be rough $\mathcal{I}$-statistically convergent to $\ell \in \mathbb{R}$, if for every $\delta, \upsilon > 0$,*

$$\left\{m \in \mathbb{N} : \frac{1}{m} |\{k \leq m : |x_k - \ell| \geq r + \delta\}| \geq \upsilon\right\} \in \mathcal{I},$$

where r is called roughness degree. For $r = 0$, rough $\mathcal{I}$-statistical convergence coincides with $\mathcal{I}$-statistical convergence.

Definition 2.4. *[21] Let $\mathscr{P}$ be a σ-algebra on a non-empty set Υ. If the set function $\mathcal{X}$ on Υ satisfies the following axioms, it is referred to be an uncertain measure:*

- *The first axiom, which deals with normality, is $\mathcal{X}\{\Upsilon\} = 1$;*
- *The second, which deals with duality, is $\mathcal{X}\{\Xi\} + \mathcal{X}\{\Xi^c\} = 1$ for any $\Xi \in \mathscr{P}$;*
- *The third, which deals with subadditivity is for every countable sequence of $\{\Xi_m\} \in \mathscr{P}$,*

$$\mathcal{X}\{\bigcup_{m=1}^{\infty} \Xi_m\} \leq \sum_{m=1}^{\infty} \mathcal{X}\{\Xi_m\}.$$

An u.s. is denoted by the triplet $(\Upsilon, \mathscr{P}, \mathcal{X})$, and an event is denoted by each member Ξ in $\mathscr{P}$.

Definition 2.5. *[21] A c.u.v. is represented by a variable ζ in the uncertainty space $(\Upsilon, \mathscr{P}, \mathscr{X})$ if and only if both its real part ξ and imaginary part η are uncertain variables.*

Here, ξ and η correspond to the real and imaginary components of the complex variable $\zeta = \xi + i\eta$, respectively.

Definition 2.6. *[22] Let $\zeta = \xi + i\eta$ be a c.u.v., where ξ is the real part and η is the imaginary part of ζ. Then the complex uncertainty distribution of ζ is denoted by $\Psi : \mathbb{C} \to [0, 1]$ and is defined by $\Psi(z) = \mathcal{X}\{\xi \le s, \eta \le t\}$ for any complex number $z = s + it$.*

Definition 2.7. *[22] Let $\zeta = \xi + i\eta$ be a c.u.v. If the expected value of ξ and η i.e., $E[\xi]$ and $E[\eta]$ exists, then the expected value of ζ is defined by*

$$E[\zeta] = E[\xi] + iE[\eta].$$

Definition 2.8. *[25] Let ζ and ζ^* be two c.u.v.s. Then the p-distance between them is defined as*

$$d_p(\zeta, \zeta^*) = \left(E\left[\|\zeta - \zeta^*\|^p\right]\right)^{\frac{1}{p+1}}, p > 0.$$

Definition 2.9. *[26] A c.u.s. sequence (ζ_m) is considered statistically convergent in p-distance to ζ if*

$$\lim_{m\to\infty} \frac{1}{m}\left|\left\{k \le m : \left(E\left[\|\zeta_k - \zeta\|^p\right]\right)^{\frac{1}{p+1}} \ge \varepsilon\right\}\right| = 0 \text{ for every } \varepsilon > 0.$$

Definition 2.10. *[4] Let ζ and ζ^* be two c.u.v.s, then the metric between them is defined as follows*

$$D(\zeta, \zeta^*) = \inf\{t : \mathcal{X}\{\|\zeta - \zeta^*\| \le t\} = 1\}.$$

Definition 2.11. *[4] If the condition $\lim_{m\to\infty} D(\zeta_m, \zeta) = 0$ is hold for a c.u.s. (ζ_m), then (ζ_m) is called convergent in metric to ζ.*

Definition 2.12. *[8] A c.u.s. (ζ_m) is considered to be rough $\mathcal{I}$-convergent a.s. to ζ if for every small positive value δ, and for any event Ξ where $\mathcal{X}\{\Xi\} = 1$ we have the following condition satisfied for every element $\varrho \in \Xi$:*

$$\{m \in \mathbb{N} : \|\zeta_m(\varrho) - \zeta(\varrho)\| \ge r + \delta\} \in \mathcal{I},$$

where r is called roughness degree.

Definition 2.13. *[8] A c.u.s. (ζ_m) is considered to be rough $\mathcal{I}$-convergent in measure to ζ if, for every given small positive values ε and δ, there exists a set satisfying the condition*

$$\{m \in \mathbb{N} : \mathcal{X}(\|\zeta_m - \zeta\| \ge \varepsilon) \ge r + \delta\} \in \mathcal{I},$$

where r is called roughness degree.

Definition 2.14. *[8] Let $\Psi, \Psi_1, \Psi_2, \cdots$ denote the complex uncertainty distributions of c.u.v.s $\zeta, \zeta_1, \zeta_2, \cdots$, respectively. The c.u.s. (ζ_m) is called rough $\mathcal{I}$-convergent in distribution to ζ if, for every small positive values δ, there exists a set satisfying the condition:*

$$\{m \in \mathbb{N} : \|\Psi_m(z) - \Psi(z)\| \geq r + \delta\} \in \mathcal{I},$$

where r is called roughness degree and for all z at which $\Psi(z)$ is continuous.

Definition 2.15. *[8] A c.u.s. (ζ_m) is considered to be rough $\mathcal{I}$-convergent in mean to ζ if, for every given small positive values δ, there exists a set satisfying the condition*

$$\{m \in \mathbb{N} : E\left[\|\zeta_m - \zeta\|\right] \geq r + \delta\} \in \mathcal{I},$$

where r is called roughness degree.

In this article, we assume that $\mathcal{I}$ to be a non-trivial admissible ideal of $\mathbb{N}$ and r as a non-negative real number .

3. Main Results

Definition 3.1. *A c.u.s. (ζ_m) is considered to be rough $\mathcal{I}$-statistically convergent a.s. to ζ if, for every small positive value δ and υ, and for any event Ξ where $\mathcal{X}\{\Xi\} = 1$ we have the following condition satisfied for every element $\varrho \in \Xi$:*

$$\left\{m \in \mathbb{N} : \frac{1}{m} \left|\{k \leq m : \|\zeta_k(\varrho) - \zeta(\varrho)\| \geq r + \delta\}\right| \geq \upsilon\right\} \in \mathcal{I},$$

where r is called roughness degree. If we take $r = 0$ we obtain the notion of $\mathcal{I}$-statistical convergence a.s. of c.u.s. which was introduced by Halder and Debnath [14].

Definition 3.2. *A c.u.s. (ζ_m) is considered to be rough $\mathcal{I}$-statistically convergent in measure to ζ if, for every given small positive values ε, δ and υ, there exists a set satisfying the condition*

$$\left\{m \in \mathbb{N} : \frac{1}{m} \left|\{k \leq m : \mathcal{X}\left(\|\zeta_k - \zeta\| \geq \varepsilon\right) \geq r + \delta\}\right| \geq \upsilon\right\} \in \mathcal{I},$$

where r is called roughness degree.

Definition 3.3. *A c.u.s. (ζ_m) is considered to be rough $\mathcal{I}$-statistically convergent in mean to ζ if, for every given small positive values δ, and υ, there exists a set satisfying the condition*

$$\left\{m \in \mathbb{N} : \frac{1}{m} \left|\{k \leq m : E\left[\|\zeta_k - \zeta\|\right] \geq r + \delta\}\right| \geq \upsilon\right\} \in \mathcal{I},$$

where r is called roughness degree.

Definition 3.4. *Let $\Psi, \Psi_1, \Psi_2, \cdots$ denote the complex uncertainty distributions of c.u.v.s $\zeta, \zeta_1, \zeta_2, \cdots$, respectively. The c.u.s. (ζ_m) is called rough $\mathcal{I}$-statistically convergent in distribution to ζ if, for every small positive values δ and υ, there exists a set satisfying the condition:*

$$\left\{m \in \mathbb{N} : \frac{1}{m} \left|\{k \leq m : \|\Psi_k(z) - \Psi(z)\| \geq r + \delta\}\right| \geq \upsilon\right\} \in \mathcal{I},$$

where r is called roughness degree and for all z at which $\Psi(z)$ is continuous.

Theorem 3.1. *The c.u.s. (ζ_m) where $\zeta_m = \xi_m + i\eta_m$ is rough $\mathcal{I}$-statistically convergent in measure to $\zeta = \xi + i\eta$ if and only if the uncertain sequence (ξ_m) and (η_m) are rough $\mathcal{I}$-statistically convergent in measure to ξ and η, respectively.*

Proof. Omitted, since it can be established using standard technique. □

Theorem 3.2. *If a c.u.s. (ζ_m) is rough $\mathcal{I}$-statistically convergent in mean to ζ, then it is rough $\mathcal{I}$-statistically convergent in measure to ζ.*

Proof. The proof follows from the following Markov inequality. □

Remark 3.1. *However, the reverse of the above theorem does not hold in general.*

Example 3.1. *Consider the u.s. $(\Upsilon, \mathscr{P}, \mathcal{X})$ to be $\{\varrho_1, \varrho_2, \cdots\}$ with power set and $\mathcal{X}\{\Upsilon\} = 1$, $\mathcal{X}\{\Phi\} = 0$ and*

$$\mathcal{X}\{\Xi\} = \begin{cases} \sup\limits_{\varrho_m \in \Xi} \frac{m}{(2m+1)}, & if \ \sup\limits_{\varrho_m \in \Xi} \frac{m}{(2m+1)} < \frac{1}{2} \\ 1 - \sup\limits_{\varrho_m \in \Xi^c} \frac{m}{(2m+1)}, & if \ \sup\limits_{\varrho_m \in \Xi^c} \frac{m}{(2m+1)} < \frac{1}{2} \\ \frac{1}{2}, & otherwise \end{cases} \quad for \ m = 1, 2, 3, \cdots$$

Also, $\zeta_m(\varrho)$ (the c.u.v.s) are defined by

$$\zeta_m(\varrho) = \begin{cases} im, \ if \ \varrho = \varrho_m \\ 0, \quad otherwise \end{cases} \quad for \ m = 1, 2, 3, \cdots$$

and $\zeta \equiv 0$. Take $\mathcal{I}$=$\mathcal{I}_d$.

For every $\varepsilon, \delta, \upsilon > 0$ and $r \geq 0$ we have,

$$\begin{aligned} &\left\{m \in \mathbb{N} : \tfrac{1}{m} \left|\{k \leq m : \mathcal{X}\left(\|\zeta_k - \zeta\| \geq \varepsilon\right) \geq r + \delta\}\right| \geq \upsilon\right\} \\ &= \left\{m \in \mathbb{N} : \tfrac{1}{m} \left|\{k \leq m : \mathcal{X}\left(\varrho : \|\zeta_k(\varrho) - \zeta(\varrho)\| \geq \varepsilon\right) \geq r + \delta\}\right| \geq \upsilon\right\} \\ &= \left\{m \in \mathbb{N} : \tfrac{1}{m} \left|\{k \leq m : \mathcal{X}\{\varrho_k\} \geq r + \delta\}\right| \geq \upsilon\right\} \\ &= \left\{m \in \mathbb{N} : \tfrac{1}{m} \left|\left\{k \leq m : \tfrac{k}{2k+1} \geq r + \delta\right\}\right| \geq \upsilon\right\} \in \mathcal{I}. \end{aligned}$$

Thus the sequence (ζ_m) is rough $\mathcal{I}$-statistically convergent in measure to ζ for $r = \frac{1}{2}$.

However, for each m, we have the complex uncertainty distributions of uncertain variable

$\|\zeta_m - \zeta\|$ *is*

$$\Psi_m(t) = \begin{cases} 0, & \text{if } t < 0 \\ 1 - \frac{m}{2m+1}, & \text{if } 0 \le t < m \\ 1, & \text{if } t \ge m \end{cases} \quad \text{for } m = 1, 2, 3, \cdots .$$

Now $E\left[\|\zeta_m - \zeta\|\right] = \int_0^{+\infty} (1 - \Psi_m(t))\, dt = \int_0^m \frac{m}{2m+1} dt = \frac{m^2}{2m+1}$.

Consequently, for any given δ *and* υ *both greater than zero, and* $r = \frac{1}{2}$,

$$\left\{m \in \mathbb{N} : \frac{1}{m} \left|\{k \le m : E\left[\|\zeta_k - \zeta\|\right] \ge r + \delta\}\right| \ge \upsilon\right\}$$
$$= \left\{m \in \mathbb{N} : \frac{1}{m} \left|\left\{k \le m : \frac{k^2}{2k+1} \ge r + \delta\right\}\right| \ge \upsilon\right\} \notin \mathcal{I}.$$

Hence the sequence (ζ_m) *is not rough* $\mathcal{I}$*-statistically convergent in mean to* ζ *for* $r = \frac{1}{2}$.

Theorem 3.3. *Let* (ξ_m) *and* (η_m) *be the real and imaginary part of a c.u.s.* (ζ_m) *are considered to be rough* $\mathcal{I}$*-statistical convergence in measure to* ξ *and* η *respectively. then* (ζ_m) *is rough* $\mathcal{I}$*-statistically convergent in distribution to* $\zeta = \xi + i\eta$.

Proof. Let $z = s + it$ be a continuous point of the complex uncertainty distribution Ψ. For any $\alpha > s$ and $\beta > t$, we can express

$$\{\xi_m \le s, \eta_m \le t\} = \{\xi_m \le s, \eta_m \le t, \xi \le \alpha, \eta \le \beta\} \cup \{\xi_m \le s, \eta_m \le t, \xi > \alpha, \eta > \beta\}$$
$$\cup \{\xi_m \le s, \eta_m \le t, \xi \le \alpha, \eta > \beta\} \cup \{\xi_m \le s, \eta_m \le t, \xi > \alpha, \eta \le \beta\}$$
$$\subset \{\xi \le \alpha, \eta \le \beta\} \cup \{|\xi_m - \xi| \ge \alpha - s\} \cup \{|\eta_m - \eta| \ge \beta - t\}.$$

By the subadditivity axiom, we can conclude that:

$$\Psi_m(z) = \Psi_m(s + it) \le \Psi(\alpha + i\beta) + \mathcal{X}\{|\xi_m - \xi| \ge \alpha - s\} + \mathcal{X}\{|\eta_m - \eta| \ge \beta - t\}.$$

Since (ξ_m) and (η_m) are rough $\mathcal{I}$-statistically convergent in measure to ξ and η respectively, then it follows that for any given δ, υ and $r \ge 0$, we can conclude that:

$$\Psi_m(z) = \Psi_m(s + it) \le \Psi(\alpha + i\beta) + \mathcal{X}\{|\xi_m - \xi| \ge \alpha - s\} + \mathcal{X}\{|\eta_m - \eta| \ge \beta - t\}.$$

Since (ξ_m) and (η_m) are rough $\mathcal{I}$-statistically convergent in measure to ξ and η respectively, then it follows that for any given δ, υ and $r \ge 0$, we can conclude that:

$$\left\{m \in \mathbb{N} : \frac{1}{m} \left|\{k \le m : \mathcal{X}\left(|\xi_k - \xi| \ge \alpha - s\right) \ge r + \delta\}\right| \ge \upsilon\right\} \in \mathcal{I}$$
$$\text{and } \left\{m \in \mathbb{N} : \frac{1}{m} \left|\{k \le m : \mathcal{X}\left(|\eta_k - \eta| \ge \beta - t\right) \ge r + \delta\}\right| \ge \upsilon\right\} \in \mathcal{I}.$$

Then for any $\alpha > s, \beta > t$ and letting $\alpha + i\beta \to s + it$, we have

$$\|\Psi_m(z) - \Psi(z)\| \le \mathcal{X}\left\{|\xi_m - \xi| \ge \alpha - s\right\} + \mathcal{X}\left\{|\eta_m - \eta| \ge \beta - t\right\}.$$

Then for every $\delta > 0$ and $r \ge 0$,

$$\{k \le m : \|\Psi_k(z) - \Psi(z)\| \ge r + \delta\}$$
$$\subseteq \{k \le m : \mathcal{X}\left\{|\xi_k - \xi| \ge \alpha - s\right\} \ge r + \delta\}$$
$$\cup \{k \le m : \mathcal{X}\left\{|\eta_k - \eta| \ge \beta - t\right\} \ge r + \delta\}.$$

$$\Rightarrow \frac{1}{m}\left|\{k \le m : \|\Psi_k(z) - \Psi(z)\| \ge r + \delta\}\right|$$
$$\le \frac{1}{m}\left|\{k \le m : \mathcal{X}\{|\xi_k - \xi| \ge \alpha - s\} \ge r + \delta\}\right|$$
$$+ \{k \le m : \mathcal{X}\{|\eta_k - \eta| \ge \beta - t\} \ge r + \delta\}.$$

For every $\upsilon > 0$,

$$\left\{m \in \mathbb{N} : \frac{1}{m}\left|\{k \le m : \|\Psi_k(z) - \Psi(z)\| \ge r + \delta\}\right| \ge \upsilon\right\}$$
$$\subseteq \left\{m \in \mathbb{N} : \frac{1}{m}\left|\{k \le m : \mathcal{X}\{|\xi_k - \xi| \ge \alpha - s\} \ge r + \delta\}\right| \ge \upsilon\right\}$$
$$\cup \{m \in \mathbb{N} : \{k \le m : \mathcal{X}\{|\eta_k - \eta| \ge \beta - t\} \ge r + \delta\} \ge \upsilon\} \in \mathcal{I}.$$

Hence the c.u.s. (ζ_m) is rough $\mathcal{I}$-statistically convergent in distribution to ζ. □

Remark 3.2. *However, the reverse of the above theorem does not hold in general.*

Example 3.2. *Consider the u.s.* $(\Upsilon, \mathscr{P}, \mathcal{X})$ *to be* $\{\varrho_1, \varrho_2\}$ *with* $\mathcal{X}(\varrho_1) = \mathcal{X}(\varrho_2) = \frac{1}{2}$. *We define a c.u.v. as*

$$\zeta(\varrho) = \begin{cases} i, & if\ \varrho = \varrho_1, \\ -i, & if\ \varrho = \varrho_2. \end{cases}$$

We also define $\zeta_m = -\zeta$ *for* $m = 1, 2, \cdots$ *and take* $\mathcal{I}$=$\mathcal{I}_d$.

Then the sequence (ζ_m) *and* ζ *have the same distribution as:*

$$\Psi_m(z) = \Psi_m(s + it) = \begin{cases} 0, & if\ s < 0, -\infty < t < +\infty, \\ 0, & if\ s \ge 0, t < -1, \\ \frac{1}{2}, & if\ s \ge 0, -1 \le t < 1, \\ 1, & if\ s \ge 0, t \ge 1. \end{cases}$$

So the sequence (ζ_m) *is rough* $\mathcal{I}$*-statistically convergent in distribution to* ζ.

However, for a given $\varepsilon, \delta, \upsilon > 0$ *and* $r \ge 0$, *we have*

$$\left\{m \in \mathbb{N} : \frac{1}{m}\left|\{k \le m : \mathcal{X}(\|\zeta_k - \zeta\| \ge \varepsilon) \ge r + \delta\}\right| \ge \upsilon\right\} \notin \mathcal{I}.$$

Thus the sequence (ζ_m) *is not rough* $\mathcal{I}$*-statistically convergent in measure to* ζ *for* $r = 0.1$.

Definition 3.5. *A c.u.s.* (ζ_m) *is said to be rough* $\mathcal{I}$*-statistically convergent in p-distance to* ζ *if for every* $\delta, \upsilon > 0$ *such that*

$$\left\{m \in \mathbb{N} : \frac{1}{m}\left|\left\{k \le m : (E[\|\zeta_k - \zeta\|^p])^{\frac{1}{p+1}} \ge r + \delta\right\}\right| \ge \upsilon\right\} \in \mathcal{I},$$

where r *is called roughness degree.*

Theorem 3.4. *Let* $\zeta, \zeta_1, \zeta_2, \cdots$ *be c.u.v.s defined on u.s.* $(\Upsilon, \mathscr{P}, \mathcal{X})$. *Then* (ζ_m) *is considered to be rough* $\mathcal{I}$*-statistically convergent in measure to* ζ *if it is rough* $\mathcal{I}$*-statistically convergent in p-distance to* ζ.

Proof. Let the c.u.s. (ζ_m) be rough $\mathcal{I}$-statistically convergent in p-distance to ζ, then for every choice of δ and υ greater than zero, we obtain

$$\left\{m \in \mathbb{N} : \frac{1}{m}\left|\left\{k \leq m : \left(E\left[\|\zeta_k - \zeta\|^p\right]\right)^{\frac{1}{p+1}} \geq r + \delta\right\}\right| \geq \upsilon\right\} \in \mathcal{I}.$$

Then for any given $\varepsilon, p > 0$, we have

$\mathcal{X}\left(\|\zeta_m - \zeta\| \geq \varepsilon\right) \leq \dfrac{E\left[\|\zeta_m - \zeta\|^p\right]}{\varepsilon^p}$ (Using Markov Inequality).

So for every $\delta > 0$ and $r \geq 0$,

$$\{k \leq m : \mathcal{X}\left(\|\zeta_k - \zeta\| \geq \varepsilon\} \geq r + \delta\right\}$$
$$\subseteq \left\{k \leq m : \left(E\left[\|\zeta_k - \zeta\|^p\right]\right)^{\frac{1}{p+1}} \geq r' + \delta'\right\}, \text{ where } r' + \delta' = \left[(r+\delta)\cdot\varepsilon^p\right]^{\frac{1}{p+1}}.$$

For every $\upsilon > 0$,

$$\left\{m \in \mathbb{N} : \frac{1}{m}\left|\{k \leq m : \mathcal{X}\left(\|\zeta_k - \zeta\| \geq \varepsilon\right) \geq r + \delta\}\right| \geq \upsilon\right\}$$
$$\subseteq \left\{m \in \mathbb{N} : \frac{1}{m}\left|\left\{k \leq m : \left(E\left[\|\zeta_k - \zeta\|^p\right]\right)^{\frac{1}{p+1}} \geq r' + \delta'\right\}\right| \geq \upsilon\right\} \in \mathcal{I}.$$

Hence the sequence (ζ_m) is rough $\mathcal{I}$-statistically convergent in measure to ζ. □

Remark 3.3. *However, the reverse of the above theorem does not hold in general.*

Example 3.3. *Let* $\mathbb{N} = \bigcup_{j=1}^{\infty} D_j$, *where* $D_j = \{2^{j-1}j^* : 2$ *does not divide* $j^*, j^* \in \mathbb{N}\}$ *be the decomposition of* $\mathbb{N}$ *such that each* D_j *is infinite and* $D_j \cap D_{j^*} = \Phi$, *for* $j \neq j^*$. *Let* $\mathcal{I}$ *be the class of all subsets of* $\mathbb{N}$ *that can intersect only finite number of* $D_j{}'s$. *Then* $\mathcal{I}$ *is a non-trivial admissible ideal of* $\mathbb{N}$ *(see for details in [20]).*

Now we consider the u.s. $(\Upsilon, \mathscr{P}, \mathcal{X})$ *to be* $\{\varrho_1, \varrho_2, \cdots\}$ *with power set and* $\mathcal{X}\{\Upsilon\} = 1$, $\mathcal{X}\{\Phi\} = 0$ *and*

$$\mathcal{X}\{\Xi\} = \begin{cases} \sup\limits_{\varrho_m \in \Xi} \beta_m, & if \ \sup\limits_{\varrho_m \in \Xi} \beta_m < \frac{1}{2} \\ 1 - \sup\limits_{\varrho_m \in \Xi^c} \beta_m, & if \ \sup\limits_{\varrho_m \in \Xi^c} \beta_m < \frac{1}{2} \\ \frac{1}{2}, & otherwise \end{cases}$$

where $\beta_m = \frac{1}{j+1}$, *if* $m \in D_j$ *for* $m = 1, 2, 3, \cdots$.

Also, the c.u.v.s are defined by

$$\zeta_m(\varrho) = \begin{cases} i(m+1), \ if \ \varrho = \varrho_m \\ 0, \quad otherwise \end{cases} \qquad for \ m = 1, 2, 3, \cdots$$

and $\zeta \equiv 0$.

It can be shown that, the sequence (ζ_m) *is rough* $\mathcal{I}$*-statistically convergent in measure to* $\zeta \equiv 0$ *but it is not rough* $\mathcal{I}$*-statistically convergent in* p*-distance to* $\zeta \equiv 0$.

Theorem 3.5. *Let $\zeta, \zeta_1, \zeta_2, \cdots$ be c.u.v.s defined on u.s. $(\Upsilon, \mathscr{P}, \mathcal{X})$. Then (ζ_m) is considered to be rough $\mathcal{I}$-statistically convergent in distribution to ζ if it is rough $\mathcal{I}$-statistically convergent in p-distance to ζ.*

Proof. If the sequence (ζ_k) exhibits rough $\mathcal{I}$-statistically convergent in p-distance to ζ, then, according to theorems 3.4 and 3.3, it also demonstrates rough $\mathcal{I}$-statistically convergent in distribution to the same limit ζ. □

Remark 3.4. *However, the reverse of the above theorem does not hold in general.*

Example 3.4. *In example 3.3, the complex uncertainty distributions of (ζ_m) are*

$$\Psi_m(z) = \Psi_m(s+it) = \begin{cases} 0, & \text{if } s<0, t<\infty \\ 0, & \text{if } s \geq 0, t<0 \\ 1-\beta_m, & \text{if } s \geq 0, 0 \leq t < (m+1) \\ 1, & \text{if } s \geq 0, t \geq (m+1) \end{cases} \quad \text{for } m = 1,2,3,\cdots$$

and the complex uncertainty distributions of ζ is

$$\Psi(z) = \Psi(s+it) = \begin{cases} 0, & \text{if } s<0, t<\infty \\ 0, & \text{if } s \geq 0, t<0 \\ 1, & \text{if } s \geq 0, t \geq 0. \end{cases}$$

It can be shown that the c.u.s. (ζ_m) is rough $\mathcal{I}$-statistically convergent in distribution to $\zeta \equiv 0$ but it is not rough $\mathcal{I}$-statistically convergent in p-distance to $\zeta \equiv 0$.

Definition 3.6. *A c.u.s. (ζ_m) is said to be rough $\mathcal{I}$-statistically convergent in metric to ζ if for every $\delta, \upsilon > 0$ such that*

$$\left\{m \in \mathbb{N} : \frac{1}{m} \left|\left\{k \leq m : D\left(\zeta_k, \zeta\right) \geq r+\delta\right\}\right| \geq \upsilon\right\} \in \mathcal{I},$$

where r is called roughness degree.

Theorem 3.6. *Let $\zeta, \zeta_1, \zeta_2, \cdots$ be c.u.v.s defined on u.s. $(\Upsilon, \mathscr{P}, \mathcal{X})$. Then (ζ_m) is considered to be rough $\mathcal{I}$-statistically convergent in mean to ζ if it is rough $\mathcal{I}$-statistically convergent in metric to ζ.*

Proof. Let the c.u.s. (ζ_m) be rough $\mathcal{I}$-statistically convergent in metric to ζ, then for every $\delta, \upsilon > 0$ and $r \geq 0$ we have,

$$\left\{m \in \mathbb{N} : \frac{1}{m} \left|\left\{k \leq m : D\left(\zeta_k, \zeta\right) \geq r+\delta\right\}\right| \geq \upsilon\right\} \in \mathcal{I},$$

where $D(\zeta_m, \zeta) = \inf\{t : \mathcal{X}\{\|\zeta_m - \zeta\| \le t\} = 1\}$.

Let $D(\zeta_m, \zeta) = q$ and $\Psi_m(t)$ represent the complex uncertainty distributions of the uncertain variable $\|\zeta_m - \zeta\|$. Then, we have $D(\zeta_m, \zeta) = \inf\{t : \Psi_m(t) = 1\}$.

Now for any positive number ℓ,

$$E[\|\zeta_m - \zeta\|] = \int_0^{+\infty} (1 - \Psi_m(t))\,dt = \int_0^{q+\ell} (1 - \Psi_m(t))\,dt + \int_{q+\ell}^{+\infty} (1 - \Psi_m(t))\,dt$$
$$= \int_0^{q+\ell} (1 - \Psi_m(t))\,dt < 1 \cdot (q+\ell) = q + \ell$$

$$\Rightarrow E[\|\zeta_m - \zeta\|] \le q \Rightarrow E[\|\zeta_m - \zeta\|] \le D(\zeta_m, \zeta).$$

So for every $\delta > 0$ and $r \ge 0$,

$$\{k \le m : E[\|\zeta_k - \zeta\|] \ge r + \delta\} \subseteq \{k \le m : D(\zeta_k, \zeta) \ge r + \delta\}$$
$$\Rightarrow \frac{1}{m}\left|\{k \le m : E[\|\zeta_k - \zeta\|] \ge r + \delta\}\right| \le \frac{1}{m}\left|\{k \le m : D(\zeta_k, \zeta) \ge r + \delta\}\right|.$$

Then for every $\upsilon > 0$,

$$\left\{m \in \mathbb{N} : \frac{1}{m}\left|\{k \le m : E[\|\zeta_k - \zeta\|] \ge r + \delta\}\right| \ge \upsilon\right\}$$
$$\subseteq \left\{m \in \mathbb{N} : \frac{1}{m}\left|\{k \le m : D(\zeta_k, \zeta) \ge r + \delta\}\right| \ge \upsilon\right\} \in \mathcal{I}.$$

Hence the sequence (ζ_m) is rough $\mathcal{I}$-statistically convergent in mean to ζ. □

Remark 3.5. *However, the reverse of the above theorem does not hold in general.*

Example 3.5. *Consider the u.s.* $(\Upsilon, \mathscr{P}, \mathcal{X})$ *to be* $\{\varrho_1, \varrho_2, \cdots\}$ *with power set and* $\mathcal{X}\{\Upsilon\} = 1$, $\mathcal{X}\{\Phi\} = 0$ *and*

$$\mathcal{X}\{\Xi\} = \begin{cases} \sup\limits_{\varrho_m \in \Xi} \frac{m\beta_m}{2m+1}, & if \ \sup\limits_{\varrho_m \in \Xi} \frac{m\beta_m}{2m+1} < \frac{1}{2} \\ 1 - \sup\limits_{\varrho_m \in \Xi^c} \frac{m\beta_m}{2m+1}, & if \ \sup\limits_{\varrho_m \in \Xi^c} \frac{m\beta_m}{2m+1} < \frac{1}{2} \\ \frac{1}{2}, & otherwise \end{cases}$$

where $\beta_m = \begin{cases} 1, \ if \ m = k^2, k \in \mathbb{N} \\ 0, \quad otherwise \end{cases} \quad for \ m = 1, 2, 3, \cdots.$

Also, the c.u.v.s are defined by

$$\zeta_m(\varrho) = \begin{cases} i(m+1), \ if \ \varrho = \varrho_m \\ 0, \quad otherwise \end{cases} \quad for \ m = 1, 2, 3, \cdots$$

and $\zeta \equiv 0$. *Take* $\mathcal{I} = \mathcal{I}_d$.

The complex uncertainty distributions associated with the uncertain variable $\|\zeta_m - \zeta\|$ *is*

$$\Psi_m(t) = \begin{cases} 0, & if\ t < 0 \\ 1 - \frac{m\beta_m}{2m+1}, & if\ 0 \le t < (m+1) \\ 1, & if\ t \ge (m+1) \end{cases} \quad for \ m = 1, 2, 3, \cdots.$$

Now $E\left[\|\zeta_m-\zeta\|\right]=\int_0^{+\infty}\left(1-\Psi_m(t)\right)dt=\int_0^{(m+1)}\frac{m\beta_m}{2m+1}dt=\frac{m(m+1)\beta_m}{2m+1}$.

Then for every $\delta,\upsilon>0$ *and* $r\geq 0$, *we have*

$$\left\{m\in\mathbb{N}:\frac{1}{m}\left|\left\{k\leq m:E\left[\|\zeta_k-\zeta\|\right]\geq r+\delta\right\}\right|\geq\upsilon\right\}$$
$$=\left\{m\in\mathbb{N}:\frac{1}{m}\left|\left\{k\leq m:\frac{k(k+1)\beta_k}{2k+1}\geq r+\delta\right\}\right|\geq\upsilon\right\}\in\mathcal{I}.$$

Again the metric between complex uncertain veriables ζ_m *and* ζ *is given by*

$$D\left(\zeta_m,\zeta\right)=\inf\left\{t:\mathcal{X}\left\{\|\zeta_m-\zeta\|\leq t\right\}=1\right\}=\inf\left\{t:\Psi_m(t)=1\right\}=m+1.$$

Thus for every $\delta,\upsilon>0$ *and* $r\geq 0$,

$$\left\{m\in\mathbb{N}:\frac{1}{m}\left|\left\{k\leq m:D\left(\zeta_k,\zeta\right)\geq r+\delta\right\}\right|\geq\upsilon\right\}$$
$$=\left\{m\in\mathbb{N}:\frac{1}{m}\left|\left\{k\leq m:(k+1)\geq r+\delta\right\}\right|\geq\upsilon\right\}\notin\mathcal{I}.$$

Hence the c.u.s. (ζ_m) *is rough* $\mathcal{I}$*-statistically convergent in mean to* $\zeta\equiv 0$ *but it is not rough* $\mathcal{I}$*-statistically convergent in metric to* $\zeta\equiv 0$.

Theorem 3.7. *Let* $\zeta,\zeta_1,\zeta_2,\cdots$ *be c.u.v.s defined on u.s.* $(\Upsilon,\mathscr{P},\mathcal{X})$. *If* (ζ_m) *is rough* $\mathcal{I}$*-statistically convergent in metric to* ζ, *then it is rough* $\mathcal{I}$*-statistically convergent in measure to* ζ.

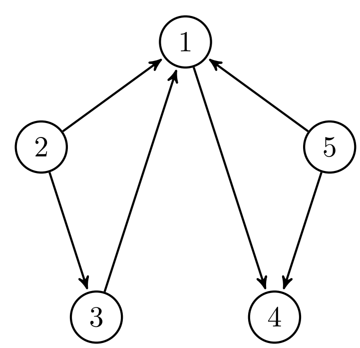
Proof. Let (ζ_m) be rough $\mathcal{I}$-statistically convergent in metric to ζ, then it is rough $\mathcal{I}$-statistically convergent in measure to ζ by theorem 3.6 and 3.2. □

Remark 3.6. *However, the reverse of the above theorem does not hold in general.*

Example 3.6. *From example 3.5, it can be shown that the c.u.s.* (ζ_m) *is rough* $\mathcal{I}$*-statistically convergent in measure to* $\zeta\equiv 0$ *but it is not rough* $\mathcal{I}$*-statistically convergent in metric to* $\zeta\equiv 0$.

4. Diagramatic representation among all convergence concepts

1. rough $\mathcal{I}$-statistically convergence in measure
2. rough $\mathcal{I}$-statistically convergence in metric
3. rough $\mathcal{I}$-statistically convergence in mean
4. rough $\mathcal{I}$-statistically convergence in distribution
5. rough $\mathcal{I}$-statistically convergence in p-distance



5. Conclusion

This paper has mainly discussed some rough $\mathcal{I}$-statistical convergence concepts of c.u.s.s, such as rough $\mathcal{I}$-statistical convergence in measure, mean, distribution, a.s., and established the relationships among them. Also, we initiate the notion of rough $\mathcal{I}$-statistical convergence in p-distance, and rough $\mathcal{I}$-statistical convergence in metric of c.u.s.s and include some interesting examples related to the notion. Furthermore, this paper is a more generalized form of rough $\mathcal{I}$-convergence of c.u.s.s, which was introduced by Debnath and Halder [8], which is a very recent and a new approach in complex uncertainty theory. In this paper, we try to establish relationships among all rough $\mathcal{I}$-statistical convergence concepts of c.u.s.s. However, we observe that certain concepts are unrelated to each other. It may attract future researchers in this direction.

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References

[1] Baliarsingh, P. (2021). On statistical deferred A-convergence of uncertain sequences. Internat. J. Uncertain. Fuzziness Knowledge-Based Systems, 29(4), 499-515.

[2] Chen, X., Ning, Y., & Wang, X. (2016). Convergence of complex uncertain sequences. J. Intell. Fuzzy Syst., 30(6), 3357-3366.

[3] Choudhury, C., & Debnath, S. (2022). On rough $\mathcal{I}$-statistical convergence of sequences in gradual normed linear spaces. Mat. Vesnik, 74(3), 218-228.

[4] Das, B. (2022). Convergence of complex uncertain triple sequence via metric operator, p-distance and complete convergence. Facta Univ. Ser. Math. Inform., 37(2), 377-396.

[5] Das, B., & Tripathy, B. C. (2023). On λ^2-statistical convergence of complex uncertain sequences. Asian-Eur. J. Math., 16(5), Article number-2350083.

[6] Debnath, S., & Das, B. (2023). On rough statistical convergence of complex uncertain sequences. New Math. Nat. Comput., 19(1), 1-17.

[7] Debnath, S., & Das, B. (2023). On rough convergence of complex uncertain sequences. J. Uncertain Syst., 14(4), Article number-2150021.

[8] Debnath, S., & Halder, A. (2024). On rough $\mathcal{I}$-convergence of complex uncertain sequences. Boletim da Sociedade Paranaense de Matemática. (Accepted).

[9] Dowari, P. J., & Tripathy, B. C. (2023). Lacunary statistical convergence of sequences of complex uncertain variables. Bol. Soc. Paran. Mat., 41, 1-10.

[10] Dündar, E., & Çakan, C. (2019). Rough $\mathcal{I}$-convergence. Gulf J. Math., 2(1), 45-51.

[11] Dündar, E., & Ulusu, U. (2023). On rough $\mathcal{I}$-convergence and $\mathcal{I}$-Cauchy sequence for functions defined on amenable semigroups. Universal J. Math. Appl., 6(2), 86-90.

[12] Fast, H. (1951). Sur la convergence statistique. Colloq. Math., 2(3-4), 241-244.

[13] Fridy, J. A. (1985). On statistical convergence. Analysis, 5, 301-313.

[14] Halder, A., & Debnath, S. (2023). On $\mathcal{I}$-statistical convergence almost surely of complex uncertain sequences. Adv. Math. Sci. Appl., 32(2), 431-445.

[15] Hazarika, B. (2014). Ideal convergence in locally solid Riesz spaces. Filomat, 28(4), 797–809.

[16] Khan, V.A., Hazarika, B., Khan, I.A., & Rahman, Z. (2022). A study on $\mathcal{I}$-deferred strongly Cesàro summable and μ-deferred $\mathcal{I}$-statistically convergence for complex uncertain sequences. Filomat, 36(20), 7001–7020.

[17] Khan, V.A., Khan, I.A., & Hazarika, B. (2022). On μ-deferred $\mathcal{I}_2$-statistical convergence of double sequence of complex uncertain variables. Rev. Real Acad. Cienc. Exactas Fis. Nat. Ser. A-Mat, 116, Article: 121.

[18] Kişi, Ö., & Gürdal, M. (2022). Rough statistical convergence of complex uncertain triple sequence. Acta Math. Univ. Comen., 91(4), 365–376.

[19] Kişi, Ö., & Gürdal, M. (2023). On I_2 and I_2^*-convergence in almost surely of complex uncertain double sequences. Probl. Anal. Issues Anal., 12(30) (2), 51-67.

[20] Kostyrko, P., Mačaj, M., & Sleziak, M. (2000/2001). $\mathcal{I}$-convergence. Real Anal. Exchange, 26, 669-686.

[21] Liu, B. (2015). Uncertainty Theory. (4^{th} edition), Springer-Verlag, Berlin.

[22] Peng, Z. (2012). Complex uncertain variable. Doctoral Dissertation, Tsinghua University.

[23] Phu, H. X. (2001). Rough convergence in normed linear spaces. Numer. Funct. Anal. Optim., 22(1-2), 199-222.

[24] Raj, K., Sharma, S., & Mursaleen, M. (2022). Almost λ-statistical convergence of complex uncertain sequences. Internat. J. Uncertain. Fuzziness Knowledge-Based Systems, 30(5), 795-811.

[25] Roy, S., Tripathy, B. C., & Saha, S. (2016). Some results on $p-$distance and sequence of complex uncertain variables. Commun. Korean. Math. Soc., 35(3), 907-916.

[26] Saha, S., Tripathy, B. C. & Roy, S. (2021). Relationships between statistical convergence concepts of complex uncertain sequences. Appl. Sci., 23, 137-144.

[27] Savaş, E., & Das, P. 2011, A generalized statistical convergence via ideals. A Math. Lett., 24, 826-830.

[28] Savaş, E., & Das, P. (2014). On $\mathcal{I}$-statistically pre-Cauchy sequences. Taiwanese J. Math., 18(1), 115-126.

[29] Savaş, E., Debnath, S., & Rakshit, D. (2018). On $\mathcal{I}$-statistically rough convergence. Publ. Inst. Math., 105(119), 145-150.

[30] Steinhaus, H. (1951). Sur la convergence ordinaire et la convergence asymptotique. Colloq. Math., 2(1), 73-74.

[31] Tripathy, B. C., & Nath, P. K. (2017). Statistical convergence of complex uncertain sequences. New Math. Nat. Comput., 13(3), 359-374.

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