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STUDY OF BI-F-HARMONIC CURVES ALONG RIEMANNIAN MAP

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Abstract. In this paper, we study bi-f-harmonic curves and helices along the Riemannian map. We find that, if a totally umbilical Riemannian map takes a horizontal bi-f-harmonic curve to bi-f-harmonic curve, then the map is totally geodesic. Then, we discuss the mean curvature vector field for horizontal bi-harmonic curves through Riemannian maps. In addition, we obtain the condition for the curvature of helix along isotropic Riemannian map.

Keywords: Bi-f-harmonic curve, bi-harmonic curve, helix, Riemannian map, totally geodesic Riemannian map.

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1. Introduction

In 1964, J. Eells and J. H. Sampson [5], introduced the concept of bi-harmonic maps by generalizing the harmonic maps. Harmonic maps are the generalization of geodesics, minimal surfaces and harmonic functions. Harmonic maps have important applications in different fields of mathematics and physics with nonlinear partial differential equations. A harmonic map $\alpha : (M, g_M) \to (N, g_N)$ between the Riemannian manifolds (M, g_M) and (N, g_N) is a critical point of the energy functional,

$$E(\alpha) = \frac{1}{2}\int_{\Gamma_M} |d\alpha|^2 v_{g_M},$$

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where Γ_M is some compact domain of M and $\tau(\alpha) = Trace_{g_M}\nabla d\alpha$ is tension field of α. The harmonic map equation is an Euler-Lagrange equation of the functional $\tau(\varphi) \equiv Trace_{g_M}\nabla d\varphi = 0$, where $\tau(\varphi) = Trace_{g_M}\nabla d\varphi$ is a tension field of φ [5]. A bi-harmonic map α between the Riemannian manifolds (M, g_M) and (N, g_N) is a critical point of the bi-energy functional, $E_2(\alpha) = \frac{1}{2}\int_{\Gamma_M} |\tau(\alpha)|^2 v_{g_M}$, where Γ_M is a compact domain of M. The bi-harmonic map equation is an Euler-Lagrange equation of the functional,

$$\tau_2(\alpha) \equiv Trace_{g_M}(\nabla^\alpha\nabla^\alpha - \nabla^\alpha_{\nabla^M})\tau(\alpha) - Trace_{g_M}R^N(d\alpha, \tau(\alpha))d\alpha = 0,$$

where $R^N = [\nabla^N_X, \nabla^N_Y]Z - \nabla^N_{[X,Y]}Z$, is a Riemann curvature tensor (N, g_N) [10]. In 1991 [4], the author introduced the bi-harmonic submanifolds of Euclidean space and stated a conjecture " any bi-harmonic submanifold of Euclidean space is harmonic, thus minimal". If the definition of bi-harmonic maps for Riemannian immersion in Euclidean space is used, then the Chen's definition of a bi-harmonic submanifold coincides with the definition given by the bi-energy functional.

Bi-f-harmonic maps are the generalization of harmonic maps, f-harmonic maps and bi-harmonic maps. There are two methods to formalize the link between bi-harmonic maps and f-harmonic maps. For the first type of formalization, the authors extended the bi-energy functional in [20, 26] to the bi-f-energy functional and obtained bi-f-harmonic maps. For the second formalization, the f-energy functional is extended to the f-bi-energy functional. In [13], the author introduced the f-bi-harmonic maps by generalizing the bi-harmonic maps. A smooth map between Riemannian manifolds is an f-bi-harmonic map if it is a critical point of the f-bi-energy function defined by the integral of f-times the square norm of the tension field, where f is a smooth function on the domain.

In 1992 [7], the author introduced the Riemannian maps between Riemannian manifolds. Isometric immersions and Riemannian submersions are particular cases of Riemannian maps. The theory of isometric immersions is one of the active research areas in differential geometry [1, 2, 3]. In [6], authors studied the characterization of submanifold by taking the hyperelastic curves along an immersion. The basic properties of Riemannian submersions were studied in [8, 15]. Let $\vartheta : (M, g_M) \to (N, g_N)$ be a smooth map between Riemannian manifolds such that $0 < rank\vartheta < min\{m, n\}$, where $dimM = m$ and $dimN = n$. Then kernel space $(Ker\vartheta_*)$ of differential map ϑ_* and g_M-orthogonal component $((Ker\vartheta_*)^\perp)$ at a point $p \in M$, are known as horizontal and vertical spaces, respectively. Thus, the tangent space T_pM of M at point p can be decomposed as $T_pM = Ker\vartheta_{*p} \oplus (Ker\vartheta_{*p})^\perp$. The range of

ϑ_* and g_2-orthogonal component at $F(p)$ on N, are denoted by $range\vartheta_*$ and $(range\vartheta_*)^\perp$, respectively. Hence, the tangent space at $F(p)$ on N, follows the decomposition

$$T_{F(p)}N = \text{Range}\vartheta_{*p} \oplus (\text{Range}\vartheta_{*p})^\perp.$$

A Riemannian map at a point $p \in M$ is a horizontal restriction

$$\vartheta^h_{*p} : \left((Ker\vartheta_{*p})^\perp, g_M(p)|_{(Ker\vartheta_{*p})^\perp}\right) \to (range\vartheta_{*p}),$$

of smooth map $\vartheta : (M, g_M) \to (N, g_N)$, such that $g_M(\vartheta_* S, \vartheta_* K) = g_N(S, K)$, where S and K are smooth sections of $\Gamma(Ker\vartheta_{*p})^\perp$ [7]. In [11, 12, 14, 18, 19, 22, 25], authors studied various types of curves such as circles, hyperelastic curves and proper curves with various maps such as immersion, embedding, Riemannian map and Clairaut Riemannian map.

We organize our paper as follows: Section 2 of this paper contains basic concepts about bi-f-harmonic curves and Riemannian maps. In section 3, we study the bi-f-harmonic curves and bi-harmonic curves through the Riemannian maps. We show that, if a totally umbilical Riemannian map takes a horizontal bi-f-harmonic curve to bi-f-harmonic curves, then the map is totally geodesic. In the same section, conditions for the mean curvature vector field are obtained by taking horizontal bi-harmonic curves through Riemannian maps. In the final section, we study helix along the Riemannian maps.

2. Preliminaries

A bi-f-harmonic map $\alpha : (M, g_M) \to (N, g_N)$ between Riemannian manifolds (M, g_M) and (N, g_N) is a critical point of bi-f-energy functional, $E_{2,f}(\alpha) = \frac{1}{2}\int_{\Gamma_M} |\tau_f(\alpha)|^2 v_{g_M}$, where Γ_M is a compact domain of M and an Euler-Lagrange equation of the functional is defined by

$$\tau^2_f(\alpha) \equiv f J^\alpha(\tau_f(\alpha)) - \nabla^\alpha_{grad\alpha}\tau_f(\alpha) = 0,$$

where $\tau_f(\alpha)$ is the $f-$tension field of α and J^α is the Jacobi operator of the map defined by $J^\alpha(X) = -(Trace_{g_M}\nabla^\alpha\nabla^\alpha X - \nabla^\alpha_{\nabla^M} X - R^N(d\alpha, X)d\alpha)$ [17, 20]. A curve $\alpha : I \to M$ on (M, g_M) is a bi-f-harmonic curve if and only if α satisfies the condition

$$\begin{aligned}(ff''' + f'f'')X_1 + (3ff'' + 2f'^2)\nabla_{X_1}X_1 + 4ff'\nabla^2_{X_1}X_1 \\ +f^2\nabla^3_{X_1}X_1 + f^2 R(\nabla_{X_1}X_1, X_1)X_1 = 0, \qquad (2.1)\end{aligned}$$

where $f : I \to (0, \infty)$ is a smooth function, ∇ is a Levi-Civita connection and R is a Riemannian curvature tensor on M.

Let $\vartheta : (M, g_M) \to (N, g_N)$ be a Riemannian map between Riemannian manifolds (M, g_M) and (N, g_N). Then a curve α on M is a horizontal curve if $\dot{\alpha}(t) \in (ker\vartheta_*)^\perp$ for every $t \in I$. If ∇^N is the Levi-Civita connection on (N, g_N) and $p_2 = \vartheta(p_1) \in N$, then the second fundamental form of ϑ is given by

$$(\nabla\vartheta_*)(X, Y) = \overset{N}{\nabla}{}^{\vartheta}_X \vartheta_*(Y) - \vartheta_*(\nabla^M_X Y), \quad \forall X, Y \in \Gamma(TM), \tag{2.2}$$

where $\overset{N}{\nabla}{}^{\vartheta}$ is the pullback connection of ∇^N [16]. The second fundamental form of a Riemannian map is symmetric and has no components in $range\vartheta_*$, that is $(\nabla\vartheta_*)(X, Y) \in (range\vartheta_*)^\perp$, $\forall\ X, Y \in \Gamma((ker\vartheta_*)^\perp)$ [23]. The scalar product of the second fundamental form is

$$g_N((\nabla\vartheta_*)(X, Y), \vartheta_*(Z)) = 0, \tag{2.3}$$

for all $X, Y, Z \in \Gamma((ker\vartheta_*)^\perp)$. Now, if $X, Y \in \Gamma((kerV_*)^\perp)$ and $V \in \Gamma((range\vartheta_*)^\perp)$, then

$$\nabla^N_{\vartheta_*(X)} V = -S_V \vartheta_*(X) + \nabla^{F^\perp}_X V, \tag{2.4}$$

where $S_V\vartheta_*(X)$ is the tangential component of $\nabla^N_{\vartheta_*(X)} V$. Since $(\nabla\vartheta_*)$ is symmetric and S_V is a symmetric linear transformation of $range\vartheta_*$, therefore

$$g_N(S_V\vartheta_*(X), \vartheta_*(Y)) = g_N(V, (\nabla\vartheta_*)(X, Y)). \tag{2.5}$$

From equations (2.2) and (2.4), we get

$$\begin{aligned} R^N(\vartheta_*(X), \vartheta_*(Y))\vartheta_*(Z) &= -S_{(\nabla\vartheta_*)(Y,Z)}\vartheta_* X + S_{(\nabla\vartheta_*)(X,Z)}\vartheta_* Y \\ &+ \vartheta_*(R^M(X, Y)Z) + (\tilde{\nabla}_X(\nabla\vartheta_*))(Y, Z) - (\tilde{\nabla}_Y(\nabla\vartheta_*))(X, Z), \end{aligned} \tag{2.6}$$

where $\tilde{\nabla}$ is the covariant derivative of the second fundamental form. The covariant derivative of $\nabla\vartheta_*$ and S are, respectively

$$(\tilde{\nabla}_X(\nabla\vartheta_*))(Y, Z) = \nabla^{\vartheta\perp}_X(\nabla\vartheta_*)(Y, Z) - (\nabla\vartheta_*)(\nabla^M_X Y, Z) - (\nabla\vartheta_*)(Y, \nabla^M_X Z), \tag{2.7}$$

and

$$(\tilde{\nabla}_X S)_V\vartheta_*(Y) = \vartheta_*(\nabla^M_X {}^*\vartheta_*(S_V\vartheta_*(Y))) - S_{\nabla^{\vartheta\perp}_X V}\vartheta_*(Y) - S_V Q \overset{N}{\nabla}{}^{\vartheta}_X \vartheta_*(Y), \tag{2.8}$$

where Q is a projection morphism on $range\vartheta_*$ and ${}^*\vartheta_*$ is an adjoint map of ϑ_*. From (2.7) and (2.8), we obtain

$$g_N((\tilde{\nabla}_X(\nabla\vartheta_*))(Y, Z), V) = g_N((\tilde{\nabla}_X S)_V\vartheta_*(Y), \vartheta_*(Z)). \tag{2.9}$$

Let $\vartheta : (M, g_M) \to (N, g_N)$ be a Riemannian map between Riemannian manifolds (M, g_M) and (N, g_N). Then ϑ is an umbilical Riemannian map if and only if

$$(\nabla\vartheta_*)(X, Y) = g_M(X, Y)H_2, \tag{2.10}$$

where $X, Y \in \Gamma((ker\vartheta_*)^\perp)$ and H_2 is non zero vector field on $(range\vartheta_*)^\perp$ [21]. The Riemannian map $\vartheta : (M, g_M) \to (N, g_N)$ is h-isotropic at $p \in M$ if

$$\mu(X) = \frac{||(\nabla\vartheta_*)(X, X)||}{||\vartheta_* X||^2}. \tag{2.11}$$

If the map is h-isotropic at every point, then the map is called h-isotropic. The map ϑ is h-isotropic at $p \in M$ if and only if $\nabla\vartheta_*$ satisfies the condition

$$g_N((\nabla\vartheta_*)(X, X), (\nabla\vartheta_*)(X, Y)) = 0, \tag{2.12}$$

for all orthogonal pair $X, Y \in \Gamma((ker\vartheta_*)^\perp)$.

3. Characterization of bi-f-harmonic curves

Let $\alpha : I \to M$ be a curve in an m-dimensional Riemannian manifold M with an orthonormal frame $\{W_0, W_1, ...W_{m_1-1}\}$ in ΓTM_1, where $W_0 = T$, $W_1 = N$ and $W_2 = U$ are the unit tangent vector, the unit normal vector and the unit binormal vector of α, respectively. Then the Frenet equations are given by

$$\nabla_T W_j = -\kappa_j W_{j-1} + \kappa_{j+1} W_{j+1}, \qquad 0 \leq j \leq m-1, \tag{3.13}$$

where $\kappa_0 = \kappa_m = 0$, $\kappa_1 = \kappa = ||\nabla_T T||$ is curvature and $\tau = \kappa_2 = -\langle\nabla_T W_1, W_2\rangle$ is torsion of α on M, respectively.

Next, we introduce the concept horizontal bi-f-harmonic curve

Definition 3.1. *Let $\vartheta : (M, g_M) \to (N, g_N)$ be a Riemannian map between Riemannian manifolds (M, g_M) and (N, g_N). Then a horizontal curve on M with (2.1) is said to be a horizontal bi-f-harmonic curve on M.*

Lemma 3.1. *: Let $\vartheta : (M, g_M) \to (N, g_N)$ be a Riemannian map between Riemannian manifolds (M, g_M) and (N, g_N). If $\bar{\alpha} = \vartheta \circ \alpha$ is a curve on N, where α is a horizontal curve* *on M, then*

$$(i)\qquad \bar{\nabla}^3_{\vartheta_*(X_1)}\vartheta_*(X_1) = -(\nabla\vartheta_*)(X_1, {}^*\vartheta_*(S_{(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(X_1))) + \vartheta_*(\nabla^3_{X_1}X_1)$$
$$-\vartheta_*(\nabla_{X_1}{}^*\vartheta_*(S_{(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(X_1))) - S_{\nabla^{\vartheta\perp}_{X_1}(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(X_1)$$
$$-S_{(\nabla\vartheta_*)(X_1,\nabla_{X_1}X_1)}\vartheta_*(X_1) + \nabla^{\vartheta\perp}_{X_1}(\nabla\vartheta_*)(X_1,\nabla_{X_1}X_1)$$
$$+ (\nabla^{\vartheta\perp}_{X_1})^2(\nabla\vartheta_*)(X_1,X_1) + (\nabla\vartheta_*)(X_1,\nabla^2_{X_1}X_1), \tag{3.14}$$

$$(ii)\qquad \bar{R}(\vartheta_*(\nabla_{X_1}X_1),\vartheta_*(X_1))\vartheta_*(X_1) = -S_{(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(\nabla_{X_1}X_1)$$
$$+S_{(\nabla\vartheta_*)(\nabla_{X_1}X_1,X_1)}\vartheta_*(X_1) + \vartheta_*(R(\nabla_{X_1}X_1,X_1)X_1)$$
$$+(\tilde{\nabla}_{\nabla_{X_1}X_1}(\nabla\vartheta_*))(X_1,X_1) - (\tilde{\nabla}_{X_1}(\nabla\vartheta_*))(\nabla_{X_1}X_1,X_1), \tag{3.15}$$

where ∇ and $\bar{\nabla}$ are the Levi-Civita connections of M and N.

Proof. Let α be a horizontal curve with curvature κ on Riemannian manifold M and $\bar{\alpha} = \vartheta\circ\alpha$ is a curve with curvature $\bar{\kappa}$ on N. Then a vector field $\vartheta_*(X_1)$ along $\bar{\alpha}$ is defined by

$$\vartheta_*(X_1) = \vartheta_{*\alpha}X_1, \tag{3.16}$$

for all vector field $X_1(s) = X_1$ along $\alpha(s) = \alpha$.

(i) From (2.2) and (2.4), we have

$$\bar{\nabla}^2_{\vartheta_*(X_1)}\vartheta_*(X_1) = \bar{\nabla}_{\vartheta_*(X_1)}((\nabla\vartheta_*)(X_1,X_1) + \vartheta_*(\nabla_{X_1}X_1)) \tag{3.17}$$
$$= -S_{(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(X_1) + \nabla^{\vartheta\perp}_{X_1}(\nabla\vartheta_*)(X_1,X_1)$$
$$+(\nabla\vartheta_*)(X_1,\nabla_{X_1}X_1) + \vartheta_*(\nabla^2_{X_1}X_1). \tag{3.18}$$

Taking covariant derivative of (3.18) and using (2.2) and (2.4), we get the required condition. (ii) From (2.6) and (2.2), we get the required equation. □

Lemma 3.2. *Let $\vartheta : (M, g_M) \to (N, g_N)$ be a Riemannian map between Riemannian manifolds (M, g_M) and (N, g_N). If $\bar{\alpha} = \vartheta\circ\alpha$ is a bi-f-harmonic curve on N, where α is a horizontal curve on M, then $(\nabla\vartheta_*)(X_1, U_1) = 0$ and*

$$ff''' + f'f'' - 3\kappa\kappa'f^2 - 4ff'\kappa^2 = 4ff'||(\nabla\vartheta_*)(X_1,X_1)||^2 + \frac{3}{2}f^2\nabla^{\vartheta\perp}_{X_1}||(\nabla\vartheta_*)(X_1,X_1)||^2.$$

Proof. Let $\vartheta : (M, g_M) \to (N, g_N)$ be a Riemannian map such that α is a horizontal curve on M and $\bar{\alpha}$ is a bi-f-harmonic curve on N, then we have

$$(ff''' + f'f'')\vartheta_*(X_1) + (3ff'' + 2f'^2)\bar{\nabla}_{\vartheta_*(X_1)}\vartheta_*(X_1) + 4ff'\bar{\nabla}^2_{\vartheta_*(X_1)}\vartheta_*(X_1)$$
$$+f^2\bar{\nabla}^3_{\vartheta_*(X_1)}\vartheta_*(X_1) + f^2\bar{R}(\vartheta_*(\nabla_{X_1}X_1), \vartheta_*(X_1))\vartheta_*(X_1) = 0. \quad (3.19)$$

From Lemma 3.1 and (3.19), we have

$$(ff''' + f'f'')\vartheta_*(X_1) + (3ff'' + 2f'^2)(\nabla\vartheta_*)(X_1, X_1) + (3ff'' + 2f'^2)\vartheta_*(\nabla_{X_1}X_1)$$
$$+f^2(\nabla^{\vartheta\perp}_{X_1})^2(\nabla\vartheta_*)(X_1, X_1) - f^2 S_{(\nabla\vartheta_*)(X_1,\nabla_{X_1}X_1)}\vartheta_*(X_1) + f^2\nabla^{\vartheta\perp}_{X_1}(\nabla\vartheta_*)(X_1, \nabla_{X_1}X_1)$$
$$-4ff'S_{(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(X_1) + 4ff'\nabla^{F\perp}_{X_1}(\nabla\vartheta_*)(X_1, X_1) + 4ff'(\nabla\vartheta_*)(X_1, \nabla_{X_1}X_1)$$
$$+4ff'\vartheta_*(\nabla^2_{X_1}X_1) - f^2(\nabla\vartheta_*)(X_1, {}^*\vartheta_*(S_{(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(X_1)))$$
$$-f^2\vartheta_*(\nabla_{X_1}{}^*\vartheta_*(S_{(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(X_1))) - f^2 S_{\nabla^{\vartheta\perp}_{X_1}(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(X_1)$$
$$+f^2(\nabla\vartheta_*)(X_1, \nabla^2_{X_1}X_1) + f^2\vartheta_*(\nabla^3_{X_1}X_1) - f^2 S_{(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(\nabla_{X_1}X_1)$$
$$+f^2 S_{(\nabla\vartheta_*)(\nabla_{X_1}X_1,X_1)}\vartheta_*(X_1) + f^2\vartheta_*(R(\nabla_{X_1}X_1, X_1)X_1)$$
$$+f^2(\bar{\nabla}_{\nabla_{X_1}X_1}(\nabla\vartheta_*))(X_1, X_1) - f^2(\bar{\nabla}_{X_1}(\nabla\vartheta_*))(\nabla_{X_1}X_1, X_1) = 0. \quad (3.20)$$

The rangeϑ_*, component of (3.20) is

$$f^2\vartheta_*(\nabla^3_{X_1}X_1) - f^2\vartheta_*(\nabla_{X_1}{}^*\vartheta_*(S_{(\nabla\vartheta_*)(X_1,X_1)}F_*(X_1))) - f^2 S_{\nabla^{\vartheta\perp}_{X_1}(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(X_1)$$
$$+f^2\vartheta_*(R(\nabla_{X_1}X_1, X_1)X_1) + (3ff'' + 2f'^2)\vartheta_*(\nabla_{X_1}X_1) + (ff''' + f'f'')\vartheta_*(X_1)$$
$$-4ff'S_{(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(X_1) + 4ff'\vartheta_*(\nabla^2_{X_1}X_1) - f^2 S_{(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(\nabla_{X_1}X_1) = 0. \quad (3.21)$$

From (2.8) and (2.7), we get

$$\vartheta_*(\nabla_{X_1}{}^*\vartheta_*(S_{(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(X_1))) = (\tilde{\nabla}_{X_1}S)_{(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(X_1)$$
$$+S_{\nabla^{\vartheta\perp}_{X_1}(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(X_1) + S_{(\nabla\vartheta_*)(X_1,X_1)}Q\bar{\nabla}^F_{X_1}\vartheta_*(X_1), \quad (3.22)$$

and

$$(\tilde{\nabla}_{X_1}(\nabla\vartheta_*))(X_1, X_1) = \nabla^{\vartheta\perp}_{X_1}(\nabla\vartheta_*)(X_1, X_1) - 2(\nabla\vartheta_*)(X_1, \nabla_{X_1}X_1). \quad (3.23)$$

Substituting (3.22) and (3.23) in (3.21), we have

$$f^2\vartheta_*(\nabla^3_{X_1}X_1) - f^2(\tilde{\nabla}_{X_1}S)_{(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(X_1) - 2f^2S_{\nabla^{\vartheta\perp}_{X_1}(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(X_1)$$
$$-f^2S_{(\nabla\vartheta_*)(X_1,X_1)}Q\bar{\nabla}^F_{X_1}\vartheta_*(X_1) + f^2\vartheta_*(R(\nabla_{X_1}X_1,X_1)X_1) + (3ff'' + 2f'^2)\vartheta_*(\nabla_{X_1}X_1)$$
$$+(ff''' + f'f'')\vartheta_*(X_1) - 4ff'S_{(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(X_1) + 4ff'\vartheta_*(\nabla^2_{X_1}X_1)$$
$$-f^2S_{(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(\nabla_{X_1}X_1) = 0. \tag{3.24}$$

Using (2.7) in (3.24), we obtain

$$f^2\vartheta_*(\nabla^3_{X_1}X_1) - f^2(\tilde{\nabla}_{X_1}S)_{(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(X_1) - 2f^2S_{(\tilde{\nabla}_{X_1}(\nabla\vartheta_*))(X_1,X_1)}\vartheta_*(X_1)$$
$$-4f^2S_{(\nabla\vartheta_*)(X_1,\nabla_{X_1}X_1)}\vartheta_*(X_1) - f^2S_{(\nabla\vartheta_*)(X_1,X_1)}Q\bar{\nabla}^{\vartheta}_{X_1}\vartheta_*(X_1) + f^2\vartheta_*(R(\nabla_{X_1}X_1,X_1)X_1)$$
$$+(3ff'' + 2f'^2)\vartheta_*(\nabla_{X_1}X_1) + (ff''' + f'f'')\vartheta_*(X_1) - 4ff'S_{(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(X_1)$$
$$+4ff'\vartheta_*(\nabla^2_{X_1}X_1) - f^2S_{(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(\nabla_{X_1}X_1) = 0. \tag{3.25}$$

Using Serret-Frenet equations of α in (3.25), we have

$$(ff''' + f'f'' - 3\kappa\kappa'f^2 - 4ff'\kappa^2)\vartheta_*(X_1) + (\kappa''f^2 - \kappa^3f^2 - \kappa\tau^2f^2 + 3ff''\kappa$$
$$+2f'^2\kappa + 4ff'\kappa')\vartheta_*(W_1) + f^2\vartheta_*(R(\kappa W_1,X_1)X_1) - 4ff'S_{(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(X_1)$$
$$-f^2S_{(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(\kappa W_1) + (2\kappa'\tau f^2 + \kappa\tau'f^2 + 4ff'\kappa\tau)\vartheta_*(U_1) + \kappa\tau f^2\kappa_3\vartheta_*(W_3)$$
$$-f^2(\tilde{\nabla}_{X_1}S)_{(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(X_1) - 2f^2S_{(\tilde{\nabla}_{X_1}(\nabla\vartheta_*))(X_1,X_1)}\vartheta_*(X_1)$$
$$-4f^2S_{(\nabla\vartheta_*)(X_1,\kappa W_1)}\vartheta_*(X_1) - f^2S_{(\nabla\vartheta_*)(X_1,X_1)}Q\bar{\nabla}^{\vartheta}_{X_1}\vartheta_*(X_1) = 0. \tag{3.26}$$

Taking inner product of (3.26) with $\vartheta_*(X_1)$, we get

$$ff''' + f'f'' - 3\kappa\kappa'f^2 - 4ff'\kappa^2 - 6\kappa f^2g_N((\nabla\vartheta_*)(X_1,W_1),(\nabla\vartheta_*)(X_1,X_1))$$
$$-4ff'g_N((\nabla\vartheta_*)(X_1,X_1),(\nabla\vartheta_*)(X_1,X_1)) = 2f^2g_N(S_{\nabla^{\vartheta\perp}_{X_1}(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(X_1),\vartheta_*(X_1))$$
$$-4f^2\kappa g_N(S_{(\nabla\vartheta_*)(X_1,W_1)}\vartheta_*(X_1),\vartheta_*(X_1)) + f^2g_N((\tilde{\nabla}_{X_1}S)_{(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(X_1),\vartheta_*(X_1)). \tag{3.27}$$

From equations (2.5), (2.7) and (2.9), we have

$$ff''' + f'f'' - 3\kappa\kappa'f^2 - 4ff'\kappa^2 = 4ff'g_N((\nabla\vartheta_*)(X_1,X_1),(\nabla\vartheta_*)(X_1,X_1))$$
$$+3f^2g_N(\nabla^{F\perp}_{X_1}(\nabla\vartheta_*)(X_1,X_1),(\nabla\vartheta_*)(X_1,X_1))$$
$$= 4ff'||(\nabla\vartheta_*)(X_1,X_1)||^2 + \tfrac{3}{2}f^2\nabla^{\vartheta\perp}_{X_1}||(\nabla\vartheta_*)(X_1,X_1)||^2. \tag{3.28}$$

The $(range\vartheta_*)^\perp$, component of (3.20) is

$$
\begin{aligned}
&(3ff''+2f'^2)(\nabla\vartheta_*)(X_1,X_1)+4ff'\nabla^{\vartheta\perp}_{X_1}(\nabla\vartheta_*)(X_1,X_1)+4ff'(\nabla\vartheta_*)(X_1,\nabla_{X_1}X_1)\\
&+f^2\nabla^{\vartheta\perp}_{X_1}(\nabla\vartheta_*)(X_1,\nabla_{X_1}X_1)+f^2(\nabla\vartheta_*)(X_1,\nabla^2_{X_1}X_1)+f^2(\tilde{\nabla}_{\nabla_{X_1}X_1}(\nabla\vartheta_*))(X_1,X_1)\\
&-f^2(\nabla\vartheta_*)(X_1,{}^*\vartheta_*(S_{(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(X_1)))+f^2(\nabla^{\vartheta\perp}_{X_1})^2(\nabla\vartheta_*)(X_*,X_*)\\
&-f^2(\tilde{\nabla}_{X_1}(\nabla\vartheta_*))(\nabla_{X_1}X_1,X_1)=0. \qquad (3.29)
\end{aligned}
$$

Also from (2.7), we get

$$
\begin{aligned}
(\nabla^{\vartheta\perp}_{X_1})^2(\nabla\vartheta_*)(X_1,X_1)&=(\tilde{\nabla}^2_{X_1}(\nabla\vartheta_*))(X_1,X_1)+4\nabla^{\vartheta\perp}_{X_1}(\nabla\vartheta_*)(X_1,\nabla_{X_1}X_1)\\
&-2(\nabla\vartheta_*)(\nabla_{X_1}X_1,\nabla_{X_1}X_1)-2(\nabla\vartheta_*)(X_1,\nabla^2_{X_1}X_1)). \qquad (3.30)
\end{aligned}
$$

Substituting (3.30) in (3.29) and using (2.7), we obtain

$$
\begin{aligned}
&(3ff''+2f'^2)(\nabla\vartheta_*)(X_1,X_1)+4ff'(\tilde{\nabla}_{X_1}(\nabla\vartheta_*))(X_1,X_1)+12ff'(\nabla\vartheta_*)(X_1,\nabla_{X_1}X_1)\\
&+4f^2(\tilde{\nabla}_{X_1}(\nabla\vartheta_*))(X_1,\nabla_{X_1}X_1)+3f^2(\nabla\vartheta_*)(\nabla_{X_1}X_1,\nabla_{X_1}X_1)+4f^2(\nabla\vartheta_*)(X_1,\nabla^2_{X_1}X_1)\\
&-f^2(\nabla\vartheta_*)(X_1,{}^*\vartheta_*(S_{(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(X_1)))+f^2(\tilde{\nabla}^2_{X_1}(\nabla\vartheta_*))(X_1,X_1)\\
&+f^2(\tilde{\nabla}_{\nabla_{X_1}X_1}(\nabla\vartheta_*))(X_1,X_1)=0. \qquad (3.31)
\end{aligned}
$$

Using Frenet equations in (3.31), we get

$$
\begin{aligned}
&(12\kappa ff'+4\kappa'f^2)(\nabla\vartheta_*)(X_1,W_1)+4\kappa\tau f^2(\nabla\vartheta_*)(X_1,U_1)+4\kappa f^2(\tilde{\nabla}_{X_1}(\nabla\vartheta_*))(X_1,W_1)\\
&+3\kappa^2f^2(\nabla\vartheta_*)(W_1,W_1)+\kappa f^2(\tilde{\nabla}_{W_1}(\nabla\vartheta_*))(X_1,X_1)+4ff'(\tilde{\nabla}_{X_1}(\nabla\vartheta_*))(X_1,X_1)\\
&=f^2(\nabla\vartheta_*)(X_1,{}^*\vartheta_*(S_{(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(X_1)))-f^2(\tilde{\nabla}^2{}_{X_1}(\nabla\vartheta_*))(X_1,X_1)\\
&+(4\kappa^2f^2-3ff''-2f'^2)(\nabla\vartheta_*)(X_1,X_1). \qquad (3.32)
\end{aligned}
$$

Replacing U_1 with $-U_1$ in equation (3.32), we have

$$
\begin{aligned}
&(12\kappa ff'+4\kappa'f^2)(\nabla\vartheta_*)(X_1,W_1)-4\kappa\tau f^2(\nabla\vartheta_*)(X_1,U_1)+4\kappa f^2(\tilde{\nabla}_{X_1}(\nabla\vartheta_*))(X_1,W_1)\\
&+3\kappa^2f^2(\nabla\vartheta_*)(W_1,W_1)+\kappa f^2(\tilde{\nabla}_{W_1}(\nabla\vartheta_*))(X_1,X_1)+4ff'(\tilde{\nabla}_{X_1}(\nabla\vartheta_*))(X_1,X_1)\\
&=f^2(\nabla\vartheta_*)(X_1,{}^*\vartheta_*(S_{(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(X_1)))-f^2(\tilde{\nabla}^2_{X_1}(\nabla\vartheta_*))(X_1,X_1)\\
&+(4\kappa^2f^2-3ff''-2f'^2)(\nabla\vartheta_*)(X_1,X_1). \qquad (3.33)
\end{aligned}
$$

Subtracting equation (3.33) from equation (3.32), we have

$$
(\nabla\vartheta_*)(X_1,U_1)=0. \qquad (3.34)
$$

□

Theorem 3.1. *Let $\vartheta : (M, g_M) \to (N, g_N)$ be a totally geodesic Riemannian map between Riemannian manifolds (M, g_M) and (N, g_N). If $\bar{\alpha} = \vartheta \circ \alpha$ is a bi-f-harmonic curve on N and α is a horizontal curve with curvature κ on M, then*

$$ff''' + f'f'' - 3\kappa\kappa' f^2 - 4ff'\kappa^2 = 0. \tag{3.35}$$

Proof. Using the fact that ϑ is a totally geodesic Riemannian map in equation (3.28), we get the required condition. □

Corollary 3.1. *Let $\vartheta : (M, g_M) \to (N, g_N)$ be an isotropic Riemannian map between Riemannian manifolds (M, g_M) and (N, g_N). If $\bar{\alpha} = \vartheta \circ \alpha$ is a bi-f-harmonic curve on N, where α is horizontal curve with curvature κ and constant f on M, then α is a curve of constant curvature on M.*

Proof. Since ϑ is an isotropic Riemannian map, therefore from (3.28), we have

$$ff''' + f'f'' - 3\kappa\kappa' f^2 - 4ff'\kappa^2 = 4ff'||(\nabla\vartheta_*)(X_1, X_1)||^2 \tag{3.36}$$

Also, f is a constant, therefore from (3.36), we get $\kappa = C(constant)$. □

Theorem 3.2. *Let $\vartheta : (M, g_M) \to (N, g_N)$ be a Riemannian map between Riemannian manifolds (M, g_M) and (N, g_N). If ϑ is a totally umbilical Riemannian map taking a horizontal bi-f-harmonic curve α on M to a bi-f-harmonic curve $\bar{\alpha} = \vartheta \circ \alpha$ on N, then ϑ is a totally geodesic Riemannian map.*
Conversely, a totally geodesic Riemannian map takes a horizontal bi-f-harmonic curve α on M to a bi-f-harmonic curve $\bar{\alpha} = \vartheta \circ \alpha$ on N.

Proof. Let $\vartheta : (M, g_M) \to (N, g_N)$ be a totally umbilical Riemannian map taking a horizontal bi-f-harmonic curve α on M to a bi-f-harmonic curve $\bar{\alpha} = \vartheta \circ \alpha$ on N, then from (3.31), we have

$$\begin{aligned} f^2(\nabla\vartheta_*)(X_1, {}^*\vartheta_*(S_{(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(X_1))) &= (3ff'' + 2f'^2)H_2 + 4ff'\nabla^{\vartheta\perp}_{X_1}H_2 \\ & f^2(\nabla^{\vartheta\perp}_{X_1})^2 H_2 - \kappa^2 f^2 H_2 + f^2(\tilde{\nabla}_{\nabla_{X_1}X_1}(\nabla\vartheta_*))(X_1, X_1). \end{aligned} \tag{3.37}$$

Substituting (3.37) in (3.20), we have

$$\begin{aligned} f^2\vartheta_*(\nabla_{X_1}{}^*\vartheta_*(S_{(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(X_1))) &= -4ff'S_{(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(X_1) \\ & -f^2 S_{\nabla^{\vartheta\perp}_{X_1}H_2}\vartheta_*(X_1) - \kappa f^2||H_2||^2\vartheta_*(W_1). \end{aligned} \tag{3.38}$$

Since $S_{(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(\nabla_{X_1}X_1) = \kappa||H_2||^2\vartheta_*(W_1)$, therefore from (3.22) and (3.38), we have

$$f^2(\tilde{\nabla}_{X_1}S)_{(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(X_1) + 2f^2S_{\nabla^{\vartheta\perp}_{X_1}(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(X_1) + f^2S_{(\nabla\vartheta_*)(X_1,X_1)}Q\bar{\nabla}^{\vartheta}_{X_1}\vartheta_*(X_1)$$
$$+4ff'S_{(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(X_1) + \kappa f^2||H_2||^2\vartheta_*(W_1) = 0, \quad (3.39)$$

where

$$(\tilde{\nabla}_{X_1}S)_{(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(X_1) = \frac{1}{2}\nabla^{\vartheta\perp}_{X_1}||H_2||^2\vartheta_*(X_1),$$

$$S_{\nabla^{\vartheta\perp}_{X_1}H_2}\vartheta_*(X_1) = \frac{1}{2}\nabla^{\vartheta\perp}_{X_1}||H_2||^2\vartheta_*(X_1),$$

and

$$S_{(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(X_1) = ||H_2||^2\vartheta_*(X_1).$$

Thus from (3.39), we have

$$\frac{3}{2}f^2(\nabla^{\vartheta\perp}_{X_1}||H_2||^2)\vartheta_*(X_1) + 2f^2||H_2||^2\vartheta_*(\nabla_{X_1}X_1) + 4ff'||H_2||^2\vartheta_*(X_1) = 0. \quad (3.40)$$

Taking the inner product of (3.40) with $\vartheta_*(\nabla_{X_1}X_1)$, we have

$$||H_2|| = 0 \implies H_2 = 0. \quad (3.41)$$

Hence ϑ is a totally geodesic Riemannian map.

Conversely, suppose that ϑ is a totally geodesic Riemannian map, then we have

$$(ff''' + f'f'')\vartheta_*(X_1) + (3ff'' + 2f'^2)\bar{\nabla}_{\vartheta_*(X_1)}\vartheta_*(X_1) + 4ff'\bar{\nabla}^2_{\vartheta_*(X_1)}\vartheta_*(X_1)$$
$$+f^2\bar{\nabla}^3_{\vartheta_*(X_1)}\vartheta_*(X_1) + f^2\bar{R}(\vartheta_*(\nabla_{X_1}X_1), \vartheta_*(X_1))\vartheta_*(X_1) = (ff''' + f'f'')\vartheta_*(X_1)$$
$$+(3ff'' + 2f'^2)\nabla_{\vartheta_*(X_1)}\vartheta_*(X_1) + 4ff'\nabla^2_{\vartheta_*(X_1)}\vartheta_*(X_1) + f^2\nabla^3_{\vartheta_*(X_1)}\vartheta_*(X_1)$$
$$+f^2R(\vartheta_*(\nabla_{X_1}X_1), \vartheta_*(X_1))\vartheta_*(X_1). \quad (3.42)$$

Since α is a horizontal bi-f-harmonic curve on M, therefore

$$(ff''' + f'f'')\vartheta_*(X_1) + (3ff'' + 2f'^2)\bar{\nabla}_{\vartheta_*(X_1)}\vartheta_*(X_1) + 4ff'\bar{\nabla}^2_{\vartheta_*(X_1)}\vartheta_*(X_1)$$
$$+f^2\bar{\nabla}^3_{\vartheta_*(X_1)}\vartheta_*(X_1) + f^2\bar{R}(\vartheta_*(\nabla_{X_1}X_1), \vartheta_*(X_1))\vartheta_*(X_1) = 0. \quad (3.43)$$

Hence $\bar{\alpha} = \vartheta \circ \alpha$ is a bi-f-harmonic curve on N. □

3.1. **Characterization of bi-harmonic curves.** A bi-harmonic curve (bi-1-harmonic curve) is a special case of bi-f-harmonic curve for $f = 1$. Let $\vartheta : (M, g_M) \to (N, g_N)$ be a Riemannian map between Riemannian manifolds (M, g_M) and (N, g_N) such that $\bar{\alpha}$ is a bi-harmonic curve on N, then

$$\bar{\nabla}^3_{\vartheta_*(X_1)}\vartheta_*(X_1) + \bar{R}(\vartheta_*(\nabla_{X_1}X_1), \vartheta_*(X_1))\vartheta_*(X_1) = 0.$$

Taking $f = 1$ in (3.26) and (3.32), we have

$$\begin{aligned}
&-3\kappa\kappa'\vartheta_*(X_1) + \big(\kappa'' - \kappa^3 - \kappa\tau^2\big)\vartheta_*(W_1) + \vartheta_*(R(\kappa W_1, X_1)X_1)\\
&-S_{(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(\kappa W_1) + (2\kappa'\tau + \kappa\tau')\vartheta_*(U_1) + \kappa\tau\kappa_3\vartheta_*(W_3)\\
&-(\tilde{\nabla}_{X_1}S)_{(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(X_1) - 2S_{(\tilde{\nabla}_{X_1}(\nabla\vartheta_*))(X_1,X_1)}\vartheta_*(X_1)\\
&-4S_{(\nabla\vartheta_*)(X_1,\kappa W_1)}\vartheta_*(X_1) - S_{(\nabla\vartheta_*)(X_1,X_1)}Q\bar{\nabla}^{\vartheta}_{X_1}\vartheta_*(X_1) = 0,
\end{aligned} \tag{3.44}$$

and

$$\begin{aligned}
&4\kappa'(\nabla\vartheta_*)(X_1, W_1) + 4\kappa\tau(\nabla\vartheta_*)(X_1, U_1) + 4\kappa(\tilde{\nabla}_{X_1}(\nabla\vartheta_*))(X_1, W_1)\\
&+3\kappa^2(\nabla\vartheta_*)(W_1, W_1) + \kappa(\tilde{\nabla}_{W_1}(\nabla\vartheta_*))(X_1, X_1) = 4\kappa^2(\nabla\vartheta_*)(X_1, X_1)\\
&-(\tilde{\nabla}^2_{X_1}(\nabla\vartheta_*))(X_1, X_1) + (\nabla\vartheta_*)(X_1, {}^*\vartheta_*(S_{(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(X_1))),
\end{aligned} \tag{3.45}$$

respectively.

Theorem 3.3. *Let $\vartheta : (M, g_M) \to (N, g_N)$ be a totally umbilical Riemannian map between Riemannian manifolds (M, g_M) and (N, g_N). If $\bar{\alpha} = \vartheta \circ \alpha$ is a bi-harmonic curve on N and α is a horizontal bi-harmonic curve on M, then the mean curvature vector field satisfies the relations*

$$(\nabla^{\vartheta\perp}_{X_1})^2 H_2 = ||H_2||^2 H_2 + \kappa^2 H_2, \tag{3.46}$$

and

$$2\kappa||H_2||^2 = \kappa'' - \kappa^3 - \kappa\tau^2 + \kappa g_M(R(W_1, X_1)X_1, W_1). \tag{3.47}$$

Conversely, let $\vartheta : (M, g_M) \to (N, g_N)$ be a totally umbilical Riemannian map and mean curvature vector field satisfies the following conditions

$$(\nabla^{\vartheta\perp}_{X_1})^2 H_2 = ||H_2||^2 H_2 + \kappa^2 H_2, \nabla^{F\perp}_{W_1} H = 2||H_2||^2\vartheta_* W_1, \tag{3.48}$$

and $||H_2||^2 = constant$. Then ϑ maps a horizontal bi-harmonic curve α on M to a bi-harmonic curve $\bar{\alpha} = \vartheta \circ \alpha$ on N.

Proof. Let $\vartheta : (M, g_M) \to (N, g_N)$ be a totally umbilical bi-harmonic Riemannian map between M and N, then from (3.45), we have

$$4\kappa(\tilde{\nabla}_{X_1}(\nabla\vartheta_*))(X_1, W_1) - \kappa^2 H_2 + \kappa(\tilde{\nabla}_{W_1}(\nabla\vartheta_*))(X_1, X_1)$$
$$= -(\tilde{\nabla}^2_{X_1}(\nabla\vartheta_*))(X_1, X_1) + (\nabla\vartheta_*)(X_1, {}^*\vartheta_*(S_{(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(X_1))). \tag{3.49}$$

Replacing W_1 with $-W_1$ in equation (3.49), we get

$$-4\kappa(\tilde{\nabla}_{X_1}(\nabla\vartheta_*))(X_1, W_1) - \kappa^2 H_2 - \kappa(\tilde{\nabla}_{W_1}(\nabla\vartheta_*))(X_1, X_1)$$
$$= -(\tilde{\nabla}^2_{X_1}(\nabla\vartheta_*))(X_1, X_1) + (\nabla\vartheta_*)(X_1, {}^*\vartheta_*(S_{(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(X_1))). \tag{3.50}$$

Subtracting (3.49) from (3.50), we obtain

$$4\kappa(\tilde{\nabla}_{X_1}(\nabla\vartheta_*))(X_1, W_1) + \kappa(\tilde{\nabla}_{W_1}(\nabla\vartheta_*))(X_1, X_1) = 0. \tag{3.51}$$

From equations (3.49) and (3.51), we get

$$(\tilde{\nabla}^2_{X_1}(\nabla\vartheta_*))(X_1, X_1) - (\nabla\vartheta_*)(X_1, {}^*\vartheta_*(S_{(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(X_1))) - \kappa^2 H_2 = 0. \tag{3.52}$$

From (2.5) and (2.10), we have

$$(\tilde{\nabla}^2_{X_1}(\nabla\vartheta_*))(X_1, X_1) = (\nabla^{\vartheta\perp}_{X_1})^2 H_2, \tag{3.53}$$

and

$$(\nabla\vartheta_*)(X_1, {}^*\vartheta_*(S_{(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(X_1))) = ||H_2||^2 H_2. \tag{3.54}$$

Equations (3.52), (3.53) and (3.54), gives the first condition.

Now, taking the inner product of (3.44) with $\vartheta_*(W_1)$, we have

$$g_N((\tilde{\nabla}_{X_1}S)_{(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(X_1), \vartheta_*(W_1)) + g_N(4S_{(\nabla\vartheta_*)(X_1,\kappa W_1)}\vartheta_*(X_1), \vartheta_*(W_1))$$
$$-g_N(\vartheta_*(R(\kappa W_1, X_1)X_1), \vartheta_*(W_1)) + g_N(S_{(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(\kappa W_1), \vartheta_*(W_1))$$
$$-\kappa'' + \kappa^3 + \kappa\tau^2 + g_N(2S_{(\tilde{\nabla}_{X_1}(\nabla\vartheta_*))(X_1,X_1)}\vartheta_*(X_1), \vartheta_*(W_1))$$
$$+g_N(S_{(\nabla\vartheta_*)(X_1,X_1)}Q\bar{\nabla}^{\vartheta}_{X_1}\vartheta_*(X_1), \vartheta_*(W_1)) = 0. \tag{3.55}$$

Since ϑ is a totally umbilical Riemannian map, therefore

$$\begin{cases} g_N((\tilde{\nabla}_{X_1}S)_{(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(X_1), \vartheta_*(W_1)) = 0, \\ g_N(S_{(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(\kappa W_1), \vartheta_*(W_1)) = \kappa||H_2||^2, \\ g_N(S_{(\nabla\vartheta_*)(X_1,X_1)}Q\bar{\nabla}^{\vartheta}_{X_1}\vartheta_*(X_1), \vartheta_*(W_1)) = \kappa||H_2||^2. \end{cases} \tag{3.56}$$

From equation (3.55) and (3.56), we get the required condition.

Conversely, suppose that ϑ is a totally umbilical Riemannian map, then for a curve $\bar{\alpha} = \vartheta \circ \alpha$ on N, where α is a curve on M, we have

$$\begin{aligned}\bar{\nabla}^3_{\vartheta_*(X_1)}\vartheta_*(X_1) + \bar{R}(\vartheta_*(\nabla_{X_1}X_1), \vartheta_*(X_1))\vartheta_*(X_1) = -||H_2||^2 H_2 \\ -\vartheta * (\nabla_{X_1}{}^*\vartheta_*(||H_2||^2\vartheta_*(X_1))) - \tfrac{1}{2}(\nabla^{\vartheta\perp}_{X_1}||H_2||^2)\vartheta_*(X_1) \\ +(\nabla^{\vartheta\perp}_{X_1})^2 H_2 - \kappa^2 H_2 - \kappa||H_2||^2\vartheta_*(W_1) + \kappa\nabla^{F\perp}_{W_1}H_2 \\ +\vartheta_*(\nabla^3_{X_1}X_1) + \vartheta_*(R(\nabla_{X_1}X_1, X_1)X_1).\end{aligned} \tag{3.57}$$

Taking $||H||^2 = constant$ in equation (3.57), we have

$$\begin{aligned}\bar{\nabla}^3_{\vartheta_*(X_1)}\vartheta_*(X_1) + \bar{R}(\vartheta_*(\nabla_{X_1}X_1), \vartheta_*(X_1))\vartheta_*(X_1) = -||H_2||^2 H_2 \\ +(\nabla^{\vartheta\perp}_{X_1})^2 H_2 - \kappa^2 H_2 - 2\kappa||H_2||^2\vartheta_*(W_1) + \kappa\nabla^{F\perp}_{W_1}H_2 \\ +\vartheta_*(\nabla^3_{X_1}X_1) + \vartheta_*(R(\nabla_{X_1}X_1, X_1)X_1).\end{aligned} \tag{3.58}$$

Using equation (3.48) in (3.58), we get

$$\begin{aligned}\bar{\nabla}^3_{\vartheta_*(X_1)}\vartheta_*(X_1) + \bar{R}(\vartheta_*(\nabla_{X_1}X_1), \vartheta_*(X_1))\vartheta_*(X_1) \\ = \vartheta_*(\nabla^3_{X_1}X_1) + \vartheta_*(R(\nabla_{X_1}X_1, X_1)X_1).\end{aligned} \tag{3.59}$$

Hence, from equation (3.59), we can say that the curve $\bar{\alpha} = \vartheta \circ \alpha$ on N is bi-harmonic curve on N iff α is a horizontal bi-harmonic curve on M. □

4. Helices along the Riemannian map

A regular curve $\alpha = \alpha(s)$ parametrized by arc length s is an ordinary helix if their exist unit vector fields W_1 and U_1 along α and constants κ and τ ($\kappa, \tau \geq 0$) such that

$$\begin{cases} \nabla_{X_1}X_1 = \kappa W_1, \\ \nabla_{X_1}W_1 = -\kappa X_1 + \tau U_1, \\ \nabla_{X_1}U_1 = -\tau W_1, \end{cases} \tag{4.60}$$

where κ is known as the curvature of the helix and τ is known as the torsion of the helix [9]. If $\tau = 0$, then α reduces to the circle and if both $\kappa = 0$ and $\tau = 0$, then α reduces to the geodesic. Hence for a proper ordinary helix κ and τ both are positive constants.

Theorem 4.1. *Let $\vartheta : (M, g_M) \to (N, g_N)$ be an Riemannian map between Riemannian manifolds (M, g_M) and (N, g_N). If $\bar{\alpha} = \vartheta \circ \alpha$ is a helix on N, where α is a horizontal curve on M, then $(\nabla\vartheta_*)(X_1, U_1) = 0$ and $\nabla^{\vartheta\perp}_{X_1}||(\nabla\vartheta_*)(X_1, X_1)||^2 + 2\kappa\kappa' = 0$.*

Proof. Let $\vartheta : (M, g_M) \to (N, g_N)$ be a Riemannian map such that α is a horizontal curve on M and $\bar{\alpha} = \vartheta \circ \alpha$ is a helix on N, then

$$\bar{\nabla}^3_{\vartheta_*(X_1)}\vartheta_*(X_1) + \lambda^2\bar{\nabla}_{\vartheta_*(X_1)}\vartheta_*(X_1) = 0. \tag{4.61}$$

From Lemma 3.1 and (4.61), we have

$$\begin{aligned}
&-(\nabla\vartheta_*)(X_1, {}^*\vartheta_*(S_{(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(X_1))) - \vartheta_*(\nabla_{X_1}{}^*\vartheta_*(S_{(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(X_1)))\\
&-S_{\nabla^{\vartheta\perp}_{X_1}(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(X_1) + (\nabla^{\vartheta\perp}_{X_1})^2(\nabla\vartheta_*)(X_1, X_1) - S_{(\nabla\vartheta_*)(X_1,\nabla_{X_1}X_1)}\vartheta_*(X_1)\\
&+\nabla^{\vartheta\perp}_{X_1}(\nabla\vartheta_*)(X_1, \nabla_{X_1}X_1) + (\nabla\vartheta_*)(X_1, \nabla^2_{X_1}X_1) + \lambda^2(\nabla\vartheta_*)(X_1, X_1)\\
&+\vartheta_*(\nabla^3_{X_1}X_1) + \lambda^2\vartheta_*(\nabla_{X_1}X_1) = 0.
\end{aligned} \tag{4.62}$$

The $range\vartheta_*$ and $(range\vartheta_*)^\perp$, components of (4.62) are

$$\begin{aligned}
&-\vartheta_*(\nabla_{X_1}{}^*\vartheta_*(S_{(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(X_1))) - S_{\nabla^{\vartheta\perp}_{X_1}(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(X_1)\\
&-S_{(\nabla\vartheta_*)(X_1,\nabla_{X_1}X_1)}\vartheta_*(X_1) + \vartheta_*(\nabla^3_{X_1}X_1) + \lambda^2\vartheta_*(\nabla_{X_1}X_1) = 0,
\end{aligned} \tag{4.63}$$

and

$$\begin{aligned}
&-(\nabla\vartheta_*)(X_1, {}^*\vartheta_*(S_{(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(X_1))) + (\nabla^{\vartheta\perp}_{X_1})^2(\nabla\vartheta_*)(X_1, X_1)\\
&+\nabla^{\vartheta\perp}_{X_1}(\nabla\vartheta_*)(X_1, \nabla_{X_1}X_1) + (\nabla\vartheta_*)(X_1, \nabla^2_{X_1}X_1) + \lambda^2(\nabla\vartheta_*)(X_1, X_1) = 0,
\end{aligned} \tag{4.64}$$

respectively. From (2.8) and (2.7), we get

$$\begin{aligned}
\vartheta_*(\nabla_{X_1}{}^*\vartheta_*(S_{(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(X_1))) &= (\tilde{\nabla}_{X_1}S)_{(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(X_1)\\
&+S_{\nabla^{\vartheta\perp}_{X_1}(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(X_1) + S_{(\nabla\vartheta_*)(X_1,X_1)}Q\bar{\nabla}^F_{X_1}\vartheta_*(X_1),
\end{aligned} \tag{4.65}$$

and

$$\begin{aligned}
(\nabla^{\vartheta\perp}_{X_1})^2(\nabla\vartheta_*)(X_1, X_1) &= (\tilde{\nabla}^2_{X_1}(\nabla\vartheta_*))(X_1, X_1) + 4\nabla^{\vartheta\perp}_{X_1}(\nabla\vartheta_*)(X_1, \nabla_{X_1}X_1)\\
&-2(\nabla\vartheta_*)(\nabla_{X_1}X_1, \nabla_{X_1}X_1) - 2(\nabla\vartheta_*)(X_1, \nabla^2_{X_1}X_1).
\end{aligned} \tag{4.66}$$

Substituting (4.66) in (4.64), we get

$$-(\nabla\vartheta_*)(X_1, {}^*\vartheta_*(S_{(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(X_1))) + (\tilde{\nabla}^2_{X_1}(\nabla\vartheta_*))(X_1, X_1)$$
$$+5\nabla^{\vartheta\perp}_{X_1}(\nabla\vartheta_*)(X_1, \nabla_{X_1}X_1) - 2(\nabla\vartheta_*)(\nabla_{X_1}X_1, \nabla_{X_1}X_1)$$
$$-(\nabla\vartheta_*)(X_1, \nabla^2_{X_1}X_1) + \lambda^2(\nabla\vartheta_*)(X_1, X_1) = 0. \tag{4.67}$$

Using (2.7) in (4.67), we obtain

$$-(\nabla\vartheta_*)(X_1, {}^*\vartheta_*(S_{(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(X_1))) + (\tilde{\nabla}^2_{X_1}(\nabla\vartheta_*))(X_1, X_1)$$
$$+5(\tilde{\nabla}_{X_1}(\nabla\vartheta_*))(X_1, \nabla_{X_1}X_1) + 3(\nabla\vartheta_*)(\nabla_{X_1}X_1, \nabla_{X_1}X_1)$$
$$+4(\nabla\vartheta_*)(X_1, \nabla^2_{X_1}X_1) + \lambda^2(\nabla\vartheta_*)(X_1, X_1) = 0. \tag{4.68}$$

Using Serret-Frenet equations in (4.68), we get

$$-(\nabla\vartheta_*)(X_1, {}^*\vartheta_*(S_{(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(X_1))) + (\tilde{\nabla}^2_{X_1}(\nabla\vartheta_*))(X_1, X_1)$$
$$+5\kappa(\tilde{\nabla}_{X_1}(\nabla\vartheta_*))(X_1, W_1) + 3\kappa^2(\nabla\vartheta_*)(W_1, W_1)$$
$$+4\kappa'(\nabla\vartheta_*)(X_1, W_1) - 4\kappa^2(\nabla\vartheta_*)(X_1, X_1)$$
$$+4\kappa\tau(\nabla\vartheta_*)(X_1, U_1) + \lambda^2(\nabla\vartheta_*)(X_1, X_1) = 0. \tag{4.69}$$

From (4.69), we have

$$-\tfrac{1}{4\kappa\tau}\Big\{-(\nabla\vartheta_*)(X_1, {}^*\vartheta_*(S_{(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(X_1))) + (\tilde{\nabla}^2_{X_1}(\nabla\vartheta_*))(X_1, X_1)$$
$$+5\kappa(\tilde{\nabla}_{X_1}(\nabla\vartheta_*))(X_1, W_1) + 3\kappa^2(\nabla\vartheta_*)(W_1, W_1)$$
$$+4\kappa'(\nabla\vartheta_*)(X_1, W_1) - 4\kappa^2(\nabla\vartheta_*)(X_1, X_1)$$
$$+\lambda^2(\nabla\vartheta_*)(X_1, X_1)\Big\} = (\nabla\vartheta_*)(X_1, U_1). \tag{4.70}$$

Changing U_1 into $-U_1$ in (4.69), we get

$$-(\nabla\vartheta_*)(X_1, {}^*\vartheta_*(S_{(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(X_1))) + (\tilde{\nabla}^2_{X_1}(\nabla\vartheta_*))(X_1, X_1)$$
$$+5\kappa(\tilde{\nabla}_{X_1}(\nabla\vartheta_*))(X_1, W_1) + 3\kappa^2(\nabla\vartheta_*)(W_1, W_1)$$
$$+4\kappa'(\nabla\vartheta_*)(X_1, W_1) - 4\kappa^2(\nabla\vartheta_*)(X_1, X_1)$$
$$+4\kappa\tau(\nabla\vartheta_*)(X_1, U_1) + \lambda^2(\nabla\vartheta_*)(X_1, X_1) = 0, \tag{4.71}$$

and then subtracting from (4.69), we have

$$(\nabla\vartheta_*)(X_1,U_1)=0. \tag{4.72}$$

Now, for second condition substituting (4.65) and (4.66) in (4.63), we have

$$\begin{aligned}-(\tilde{\nabla}_{X_1}S)_{(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(X_1)-2S_{\left(\tilde{\nabla}_{X_1}(\nabla\vartheta_*)\right)(X_1,X_1)}\vartheta_*(X_1),\\ -5S_{(\nabla\vartheta_*)(X_1,\nabla_{X_1}X_1)}\vartheta_*(X_1)-S_{(\nabla\vartheta_*)(X_1,X_1)}Q\bar{\nabla}^{\vartheta}_{X_1}\vartheta_*(X_1),\\ \vartheta_*(\nabla^3_{X_1}X_1)+\lambda^2\vartheta_*(\nabla_{X_1}X_1)=0.\end{aligned} \tag{4.73}$$

Using Frenet-Serret equations in (4.73), we get

$$\begin{aligned}-(\tilde{\nabla}_{X_1}S)_{(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(X_1)-2S_{\left(\tilde{\nabla}_{X_1}(\nabla\vartheta_*)\right)(X_1,X_1)}\vartheta_*(X_1)\\ -5\kappa S_{(\nabla\vartheta_*)(X_1,W_1)}\vartheta_*(X_1)-S_{(\nabla\vartheta_*)(X_1,X_1)}Q\bar{\nabla}^{\vartheta}_{X_1}\vartheta_*(X_1)\\ +(\kappa''-\kappa^3-\kappa\tau^2)\vartheta_*(W_1)+\kappa\tau\kappa_3\vartheta_*(W_3)+\lambda^2\kappa\vartheta_*(W_1)\\ -3\kappa\kappa'\vartheta_*(X_1)+(2\kappa'\tau+\kappa\tau')\vartheta_*(U_1)=0.\end{aligned} \tag{4.74}$$

Taking the inner product of equation (4.74) with $\vartheta_*(X_1)$, we have

$$\begin{aligned}3\kappa\kappa'+g_N\Big((\tilde{\nabla}_{X_1}S)_{(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(X_1),\vartheta_*(X_1)\Big)\\ +2g_N\Big(S_{\left(\tilde{\nabla}_{X_1}(\nabla\vartheta_*)\right)(X_1,X_1)}\vartheta_*(X_1),\vartheta_*(X_1)\Big)+5g_N\Big(\kappa S_{(\nabla\vartheta_*)(X_1,W_1)}\vartheta_*(X_1),\vartheta_*(X_1)\Big)\\ +g_N\Big(S_{(\nabla\vartheta_*)(X_1,X_1)}Q\bar{\nabla}^{\vartheta}_{X_1}\vartheta_*(X_1),\vartheta_*(X_1)\Big)=0.\end{aligned} \tag{4.75}$$

Using (2.5) and equation (4.66) in (4.75), we get

$$\begin{aligned}\nabla^{\vartheta\perp}_{X_1}g_N\Big((\nabla\vartheta_*)(X_1,X_1),(\nabla\vartheta_*)(X_1,X_1)\Big)+2\kappa g_N\Big((\nabla\vartheta_*)(X_1,W_1),(\nabla\vartheta_*)(X_1,X_1)\Big)\\ +3\kappa\kappa'+g_N\Big((\tilde{\nabla}_{X_1}S)_{(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(X_1),\vartheta_*(X_1)\Big)=0.\end{aligned} \tag{4.76}$$

Using (2.5) and equation (4.66) in (2.9), we obtain

$$\begin{aligned}g_N\Big((\tilde{\nabla}_{X_1}S)_{(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(X_1),\vartheta_*(X_1)\Big)=-2\kappa g_N\Big((\nabla\vartheta_*)(X_1,W_1),(\nabla\vartheta_*)(X_1,X_1)\Big)\\ -\tfrac{1}{2}\nabla^{\vartheta\perp}_{X_1}g_N\Big((\nabla\vartheta_*)(X_1,X_1),(\nabla\vartheta_*)(X_1,X_1)\Big).\end{aligned} \tag{4.77}$$

Equation (4.76) and (4.77) together provides the required condition. □

Corollary 4.1. *Let $\vartheta : (M, g_M) \to (N, g_N)$ be a isotropic Riemannian map between Riemannian manifolds (M, g_M) and (N, g_N). If $\bar{\alpha}(s) = \vartheta \circ \alpha(s)$ is a helix on N, where α is a horizontal curve on M, then curvature of α is constant.*

Proof. Taking $||(\nabla\vartheta_*)(X_1, X_1)||^2 = constant$, in a Theorem 4.1, we get $\kappa = constant$. □

Theorem 4.2. [24] *Let $\vartheta : (M, g_M) \to (N, g_N)$ be a Riemannian map between Riemannian manifolds (M, g_M), $dimM \geq 2$ and (N, g_N). Then ϑ maps a horizontal helix α on M to a helix $\bar{\alpha} = \vartheta \circ \alpha$ on N iff ϑ is totally umbilical and the mean curvature vector field H satisfies the following equation*

$$(\nabla^{\vartheta\perp}_{X_1})^2 H_2 = -\tau^2 H_2.$$

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References

[1] Aquib, M., Lee, J. W., Vilcu G. E., & Yoon, B. W. (2019). Classification of Casorati ideal Lagrangian submanifolds in complex space forms. Differential Geometry and its Applications, 63, 30-49.

[2] Aydin, M. E., & Mihai, I. (2019). Wintgen inequality for statistical surfaces. Math. Inequal. Appl., 22, 123-132.

[3] Chen, B. Y. (1984). Total mean curvature and submanifolds of finite type. USA : World Scientific.

[4] Chen, B. Y. (1991). Some open problems and conjectures on submanifolds of finite type. Soochow J. Math., 17, 169-188.

[5] Eells J., & Sampson, J. H. (1964). Harmonic mappings of Riemannian manifolds. American Journal of Mathematics, 86, 109-160.

[6] Erdoğan, F. E., & Şahin, B. (2020). Isotropic Riemannian submersions. Turkish Journal of Mathematics, 44 (6), 2284-2296.

[7] Fischer, A. E. (1992). Riemannian maps between Riemannian manifolds. Contemporary Mathematics, 132, 331-366.

[8] Garay, O. (2012). Riemannian submanifolds shaped by the bending energy and its allies. Proceeding of the Sixteenth International Workshop on Diff. Geom., 16, 55-68.

[9] Ikawa, T. (1981). On some curves in Riemannian geometry. Soochow Journal of Mathematics, 7, 37-44.

[10] Jiang, G. Y. (1986). 2-harmonic maps and their first and second variational formulas. Chinese Ann. Math. Ser. A, 7, 389-402.

[11] Karakaş, G. K., & Şahin, B. (2024). Biharmonic curves along Riemannian maps. Filomat, 38 (1) , 227-239.

[12] Karakaş, G. K., & Şahin, B. (2024). Biharmonic curves along Riemannian Submersions. Miskolc Math. Notes, 25(2), 817-831.

[13] Lu, W. J. (2015). On f-bi-harmonic maps and bi-f-harmonic maps between Riemannian manifolds. Sci. China Math., 58 (2015), 1483-1498.

[14] Montaldo S., & Pampano, A. (2021). Triharmonic curves in 3-dimensional homogeneous spaces. Mediterranean Journal of Mathematics, 18, 1-17.

[15] Neill, B. O. (1966). The fundamental equations of a submersion. Michigan Mathematical Journal, 13 (4), 459-469.

[16] Nore, T. (1986). Second fundamental form of a map. Annali di Matematica Pura ed Applicata, 146, 281-310.

[17] Ou, Y. L. (2014). On f-biharmonic maps and f-biharmonic submanifolds. Pacific Journal of Mathematics, 271 (2), 461 - 477.

[18] Pal B., & Kumar, S. (2022). Characterization of proper curves and proper helix lying on $S_1^2(r)$. Hacettepe Journal of Mathematics and Statistics, 51 (5) 2022, 1288 -1303.

[19] Pal B., & Kumar S. (2022). Proper helix of order 6 and LC helix in pseudo-Euclidean space E_4^8. Jordan Journal of Mathematics and Statistics, 15 (4B), 1077-1092.

[20] Quakkas, S., Nasri R. & Djaa, M. (2010). On the f-harmonic and f-biharmonic maps. J. P. J. Geom. Topol., 10 (1), 11-27.

[21] Sa̧hin, B. Academic Press (2017). Riemannian submersions, Riemannian maps in Hermitian geometry, and their applications.

[22] Sa̧hin, B. (2017). *Circles along a Riemannian map and Clairaut Riemannian maps*. Bull. Korean Math. Soc., 54, 253-264.

[23] Sa̧hin, B. (2010). Invariant and anti-invariant Riemannian maps to Kähler manifolds. *International Journal of Geometric Methods in Modern Physics*, 9, 337-355 .

[24] Tükel, G. Ö., & Şahin, B., & and Turhan, T. (2022). Isotropic Riemannian Maps and Helices along Riemannian Maps. U.P.B. Sci. Bull., Series A, 84 (4), 89-100.

[25] Turhan, T., & Şahin, B. (2022). Hyperelastic curves along Riemannian maps. Turkish Journal of Mathematics, 46 , 1256-1267.

[26] Zhao C. L., & Lu, W. (2015). Bi-f-harmonic map equations on singly warped product manifolds. Applied Mathematics-A Journal of Chinese Universities, 30 (1), 111-126.

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