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ON GENERALIZED CLOSED $QTAG$-MODULES

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Abstract. This paper explores the concept of closed modules by utilizing the notion of h-topology within the context of $QTAG$-modules. In addition, we delve into the intricate relationships between different types of submodules and Ulm invariants, shedding light on their interconnected roles within the closures. This investigation aims to provide a deeper understanding of these algebraic structures and their dynamic interactions.

Keywords: QTAG-modules, Closures, Isotype submodules, Ulm invariants.

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1. Introduction

One of the rapidly developing areas of research in module theory is the study of TAG-modules. The idea was first introduced by Singh [15] in 1976. Moreover, module theory has also witnessed a surge of interest in recent research, with the TAG-module being an intriguing area of investigation, which is one of the variations of torsion Abelian groups in modules. Over time, many researchers have extensively studied torsion Abelian groups and its numerous variants, as evidenced by a range of notable studies found in [3, 11, 17].

Consider the following two conditions on a module M over an arbitrary (associative, unitary) ring R.

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"(i) Every finitely generated submodule of any homomorphic image of M is a direct sum of uniserial modules.

(ii) Given any two uniserial submodules U_1 and U_2 of a homomorphic image of M, for any submodule N of U_1, any non-zero homomorphism $\phi : N \to U_2$ can be extended to a homomorphism $\psi : U_1 \to U_2$, provided that the composition length $d(U_1/N) \leq d(U_2/\phi(N))$ holds."

When M over a ring R is a module and satisfies conditions (i) and (ii), it is called a TAG-module, and when M over a ring R has condition (i) only, it is called a $QTAG$-module. Following up on his investigation in [15], Singh [16] published a paper in 1987 titled "Abelian groups like modules," which naturally led to the introduction of the concept of $QTAG$-module, which has since generated interest in the field of module theory. The study was then followed by numerous developments on the topic. In recent years, this exploration for $QTAG$-modules has regained the interests of some authors, and a lot of interesting results on $QTAG$-modules of many torsion Abelian groups have been obtained during the course of this quest (see, for example, [1, 2, 9, 10] and the references cited therein). Many such advances in the theory of torsion Abelian groups exhibit characteristics of the earlier developments, which is not surprising. The current work contributes to the understanding of the structure of $QTAG$-modules and is a logical extension of the studies carried out in [18]. Another useful source on the explored subject is [4] (see [19], too) as well. For some other interesting generalizations of the topic mentioned here, the reader can see in [5, 6].

2. Preliminaries

Throughout the present paper, unless specified something else, let us assume that all rings R into consideration are associative with unity and modules M are unital $QTAG$-modules, written additively, as is the custom when studying them. All other not explicitly explained herein notions and notations are well-known and mainly follow those from [7] and [8]. A module M is called uniform if the intersection of any two of its nonzero submodules is nonzero. An element a in M is called uniform if aR is a nonzero uniform module. Standardly, the decomposition length of any module M with a unique decomposition series is denoted by $d(M)$. In addition, the exponent of a uniform element a of M, denoted by the symbol $e(a)$, is equal to $d(aR)$. As usual, for such a module M, we state the height of a in M as $H_M(a) = \sup\{d(bR/aR) : b \in M,\ a \in bR \text{ and } b \text{ uniform}\}$. Likewise, for $k \geq 0$, $H_k(M) = \{a \in M|\ H_M(a) \geq k\}$ represents the submodule of M that is generated by the elements that have at least k heights. The module M is h-divisible if $M = M^1 = \cap_{k=0}^{\infty} H_k(M)$, where M^1 is the submodule of M generated by uniform elements of M of infinite height. The module M is h-reduced if it does not contain any h-divisible submodule. The topology of M, which admits as a base of neighborhoods of zero, is known as the h-topology. This topology has the submodules $H_k(M)$ for some k. In this fashion, a submodule S of M is named the closure in M if $\overline{S} = \cap_{k=0}^{\infty}(S + H_k(M))$. With this in hand, we say that a submodule S of M is closed with respect to the h-topology provided that $\overline{S} = S$ and h-dense in M if $\overline{S} = M$. By closed module M, we mean those modules which do not have any element of infinite height and has a limit in M for every Cauchy sequence. Moreover, the sum of all the simple submodules of M is called the socle of M, denoted by $Soc(M)$.

Furthermore, we assemble some basic concepts which are crucial in the following development. For pertinent results related to these concepts, we refer the reader to [13, 14] (see [12] too). By analogy, for every ordinal σ, one can define the infinite height $H_\sigma(M)$ as follows: $H_\sigma(M) = H_1(H_{\sigma-1}(M))$ if σ is non-limit, or $H_\sigma(M) = \cap_{\gamma<\sigma} H_\gamma(M)$ otherwise. Usually, $H_\sigma(M)$ denotes the submodule consisting of all elements of M with height $\geq \sigma$. This submodule is also called σ^{th}-Ulm submodule of M. In particular, $H_\omega(M)$ will be the first Ulm submodule of M, i.e., the set of elements of infinite height. A submodule S of M is said to be σ-pure if, for all ordinal γ, there exists an ordinal σ (depending on S) such that $H_\gamma(M) \cap S = H_\gamma(S)$. Besides, a submodule S of M is termed isotype, if it is σ-pure for every ordinal σ. The cardinality of the minimal generating set of M is denoted by the symbol $g(M)$. For all ordinals σ, $f_M(\sigma) = g\big(Soc(H_\sigma(M))/Soc(H_{\sigma+1}(M))\big)$ is called the σ^{th}-Ulm invariant of M.

Finally, the project is organized as follows. In the previous section, we have explored the subject's background. The current section, i.e. here, looks at the topics's related notions. The study of generalized closed modules is discussed in the next section, and important results and distinctive properties of closures as well as Ulm invariants are presented. In the final section, we list some interesting left-open questions.

3. Main results

It is well-known that the direct sum of countably generated modules and the closed modules are determined up to isomorphism by their Ulm invariants. The latter type of modules can be characterized as the closed submodule of the closure of a direct sum of uniserial modules. This closure is considered with respect to an h-topology (cf. [1]) which is defined for modules without elements of infinite height. One of the main goals of this article is to extend the concept of h-topology, and thereby to include modules of arbitrary countable length for investigating the generalized closed modules.

The following notions are our major tools.

Let γ be an ordinal and M an h-reduced $QTAG$-module, we define a descending chain of submodules $H_\gamma(M)$ by

$$H_\gamma(M) = \begin{cases} H_1(H_\alpha(M)), \text{ if } \gamma = \alpha + 1 \\ \cap_{\alpha<\gamma} H_\alpha(M), \text{ if } \gamma \text{ is a limit ordinal.} \end{cases}$$

Since all modules are assumed to be h-reduced, there is an ordinal β such that $H_\gamma(M) = 0$ for $\gamma \geq \beta$. The smallest such β is usually referred to as the length of M. When length$(M) \leq \omega$, M is said to contain no elements of infinite height.

Let η be the first limit ordinal greater than or equal to the length of a $QTAG$-module M. Then a h-topology can be constructed using the submodules $H_\gamma(M)$, for $\gamma < \eta$, as a base for the neighborhoods of the identity. This extension of the h-topology is known as the natural topology.

We start here with a new useful criterion for a submodule to be isotype.

Proposition 3.1. *Suppose that γ is an ordinal. Then a submodule S is an isotype submodule of a $QTAG$-module M if $H_\gamma(M) \cap S \subset H_\gamma(S)$ implies $H_{\gamma+1}(M) \cap S \subset H_{\gamma+1}(M)$.*

Proof. The proof is by induction on γ in conjunction with $H_\gamma(M) \cap S \supset H_\gamma(S)$. Clearly, if γ is a limit ordinal and $H_\alpha(M) \cap S = H_\alpha(S)$ for all $\alpha < \gamma$, then

$$\begin{aligned} H_\gamma(M) \cap S &= (\cap_{\alpha<\gamma} H_\alpha(M)) \cap S, \\ &= \cap_{\alpha<\gamma} H_\alpha(S), \\ &= H_\gamma(S), \end{aligned}$$

which allows us to infer that S is isotype in M for each ordinal γ. □

In light of the previous construction, we obtain the following.

Proposition 3.2. *Let N be a submodule of a $QTAG$-module M of countable length β. Then there exists an isotype submodule S of M such that $N \subset S \subset M$ and $g(N) = g(S)$.*

Proof. Foremost, we construct inductively a chain of submodules S^k such that $S = \cup S^k$ for some positive integer k. Now, we set $S^o = N$, then there exist equations $x' = y$ with

$d(xR/x'R) = 1$ for some $x \in M$ and $y \in H_{\gamma+1}(M) \cap S^{k-1}$. However, we observe that these equations do not have a solution for $x \in S$. Among all such equations, let $T^{\gamma,k}$ be one solution of $H_\gamma(M)$, for each ordinal $\gamma < \beta$. In fact, denote by T^k the module generated by the elements of $\cup_{\gamma<\beta} T^{\gamma,k}$ and define $S^k = S^{k-1} + T^k$.

Next, assume that $H_\gamma(M) \cap S = H_\gamma(S)$ and choose $y \in H_{\gamma+1}(M) \cap S$ such that $y \in S^k$. Then by the definition of $T^{\gamma,k+1}$, there exists an element z such that $z \in T^{\gamma,k+1}$ and $z' = y$ where $d(zR/z'R) = 1$. By hypothesis on $H_\gamma(M)$, we have $T^{\gamma,k+1} \subset (H_\gamma(M) \cap S)$ and $H_\gamma(M) \cap S = H_\gamma(S)$. Therefore, we obviously observe that $y \in H_{\gamma+1}(S)$ and $H_{\gamma+1}(M) \cap S \subset H_{\gamma+1}(S)$. By appealing to the same reasoning as in Proposition 3.1, one may infer that the assertion follows. □

The following technicality is pivotal.

Proposition 3.3. *Let M be a QTAG-module. If $\overline{M}$ is the closure of M, then M is isotype in $\overline{M}$.*

Proof. Suppose that $H_\gamma(\overline{M}) \cap M = H_\gamma(M)$ and choose $x \in H_{\gamma+1}(\overline{M}) \cap M$. Then there exists a uniform element $y \in H_\gamma(\overline{M})$ such that $x = y'$ where $d(yR/y'R) = 1$. If $\{y_k\}$ is a sequence in M, then its limit y is also an element of M and $y_k - y$ is an element of $H_\gamma(\overline{M})$ for every k. This, in turn, implies that $y_k \in H_\gamma(\overline{M}) \cap M = H_\gamma(M)$. Therefore, $y'_k - x$ and y'_k are in $H_{\gamma+1}(M)$ such that $d(y_kR/y'_kR) = 1$. Hence, it consequently follows that $x \in H_{\gamma+1}(M)$, and the result follows from Proposition 3.1. □

The next statement is pretty simple but useful.

Proposition 3.4. *Let S be an isotype submodule of a QTAG-module M which is h-dense in M. Then S and M have equal lengths.*

Proof. Let β_1 and β_2 be the lengths of S and M, respectively. Clearly $\beta_1 \leq \beta_2$. Now, if x is a nonzero uniform element of $H_\gamma(M)$ for $\gamma < \beta_1$ and $\{y_k\}$ is a sequence in S converging to x, then $y_k - x \in H_\gamma(M)$ for every k. This gives that $y_k \in H_\gamma(M) \cap S = H_\gamma(S) = 0$ for every k and means that $H_\gamma(M) = 0$ for all $\gamma \geq \beta_1$. Consequently, as early checked, $\beta_1 = \beta_2$. The proof is over. □

We now will explore the closureness for the submodule classes.

Theorem 3.1. *Suppose M is a QTAG-module and γ is an ordianl. If $\overline{M}$ is the closure of M, then $\overline{H_\gamma(M)} = H_\gamma(\overline{M})$.*

Proof. Let x be a uniform element of $H_\gamma(\overline{M})$ with a sequence $\{x_k\}$ in M converging to x. Then there exists an integer t such that $x_k - x \in H_\gamma(\overline{M})$ for $k \geq t$. Setting $y_k = x_{t+k}$. Indeed, this gives a sequence in $H_\gamma(M)$ converging to x. Thereby, because of the closureness of M, it follows at one that $H_\gamma(\overline{M}) \subset \overline{H_\gamma(M)}$.

Turning to the opposite part-half, we shall prove at first by induction on γ. First, if $\gamma = 0$, it is obvious to assume the result holds for $\alpha < \gamma$. Now, we have two cases to consider. First, if γ is a limit ordinal, then

$$\begin{aligned}\overline{H_\gamma(M)} &= \overline{\cap_{\alpha<\gamma} H_\alpha(M)},\\ &\subset \cap_{\alpha<\gamma} \overline{H_\alpha(M)},\\ &= \cap_{\alpha<\gamma} H_\alpha(\overline{M}),\\ &= H_\gamma(\overline{M}),\end{aligned}$$

so that $\overline{H_\gamma(M)} \subset H_\gamma(\overline{M})$, and we are done. For the remaining case, if γ is not a limit ordinal, then we write $\gamma = \alpha + 1$ and choose a sequence of ordinals γ_k with length of M as a supremum such that $\gamma_{k+1} > \gamma_k > \gamma$. Let $x \in \overline{H_\gamma(M)}$, we observe that a subsequence of sequence in $H_\gamma(M)$ converging to x, and we obtain a sequence $\{x_k\}$ in $H_\gamma(M)$ such that

$$\lim_{k\to\infty} x_k = x \text{ and } x_{k+1} - x_k \in H_{\gamma_{k+1}+1}(M)$$

for each k. Suppose $a \in H_\alpha(M)$ such that $a' = u$ where $d(aR/a'R) = 1$ and choose $z \in H_\beta(M)$ such that $z' = v - a'$ where $d(zR/z'R) = d(aR/a'R) = 1$. Setting $b = z + a$. This gives that $b' = v$, $b - a = z \in H_\beta(M)$, and $b \in H_\alpha(M)$ where $d(bR/b'R) = 1$. On continuing same process in this manner, one may see that there exists a sequence $\{z_k\}$ in $H_\alpha(M)$ such that $z'_k = x_n$ where $d(z_kR/z'_kR) = 1$. Let c be the limit of $\{z_k\}$ in $\overline{H_\alpha(M)}$. Then $c \in H_\alpha(\overline{M})$ and $c' = x \in H_\gamma(\overline{M})$ where $d(cR/c'R) = 1$. Thus, $\overline{H_\gamma(M)} \subset H_\gamma(\overline{M})$, and the result follows. □

The next two statements are worthy of noticing.

Corollary 3.1. *Suppose M is a QTAG-module and γ is an ordinal such that length of M is greater than γ. Then $\overline{M} = M + \overline{H_\gamma(M)}$.*

Proof. First, we take $x \in \overline{M}$, and let $\{x_n\}$ be a sequence in M which converges to x. Then there exists an integer t such that $x_t - x_k \in H_\gamma(M)$ for $k > t$ and length $(M) > \gamma$. By setting $y_k = x_t - x_{t+k}$, one may see that a sequence in $H_\gamma(M)$. Let y be the limit of $\{y_k\}$ in $\overline{H_\gamma(M)}$. Then $x = x_t - y$, and we are done. □

Corollary 3.2. *Suppose M is a $QTAG$-module and γ is an ordinal. Then M is closed if and only if $H_\gamma(M)$ is closed, provided length of $M > \gamma$.*

Proof. Assume that M is closed; i.e., $M = \overline{M}$. In accordance with Theorem 3.1, we subsequently deduce that

$$H_\gamma(M) = H_\gamma(\overline{M}) = \overline{H_\gamma(M)}$$

for each ordinal γ. This allows us to infer that $H_\gamma(M)$ is closed, thus completing the first half.

Conversely, we presume now that $H_\gamma(M)$ is closed for some $\gamma <$ length(M). Hence, $\overline{H_\gamma(M)} = H_\gamma(M)$ and so in conjunction with Corollary 3.1, we get

$$\overline{M} = M + \overline{H_\gamma(M)} = M + H_\gamma(M).$$

Consequently, it is plainly seen that M is closed, and we are finished. □

Our next example show that in the above corollaries, the requirement that length of $M > \gamma$ cannot be removed.

Example 3.1. Let

$$f : Soc(M/H_\gamma(M)) \to H_\gamma(M)/H_{\gamma+1}(M)$$

be a homomorphism such that $\ker(f) = \overline{M}/\overline{H_\gamma(M)}$. Let S be a submodule of $Soc(M/H_\gamma(M))$, then $f : Soc(M/H_\gamma(M)) \to H_\gamma(M)/H_{\gamma+1}(M)$ is an isomorphism. Obviously, $\overline{H_\gamma(M)} \in \overline{M}$, and $S = \Sigma_{\gamma \leq k < t}(x_t - y_k)$, for some integers t and k. Now, for every ordinal γ, let K_γ be a submodule of $\overline{H_\gamma(M)}$ and L_γ the image of K_γ in $\overline{M}/\overline{H_\gamma(M)}$. Then, $U = \Sigma_{\gamma+\omega \leq k < t}(x_t - y_k)$ is a direct sum of uniserial modules. Since $S \cap U = 0$, we have $S + U \leq \Sigma_{\gamma+t \leq \gamma+\omega}\ x_t$ and $\overline{S} + \overline{U} \leq \Sigma_{\gamma+k \leq \gamma+\omega}\ y_k$. Putting these inequalities together, we obtain the desired claim.

We come now to our main theorem on closed modules.

Theorem 3.2. *Let M_i $(i \in I)$ be a system of $QTAG$-modules. Then $M = \oplus_{i \in I} M_i$ is closed if and only if there exists an ordinal $\gamma <$ length(M) such that the family $\mathcal{F} = \{i \in I :$ length$(M_i) > \gamma\}$ is of nonzero finite cardinality and for each $i \in \mathcal{F}$, M_i is closed.*

Proof. To prove necessity, let M be a closed module. If there is no γ such that $\mathcal{F}$ has nonzero finite cardinality, then there exists an increasing sequence $\beta_{i_k} <$ length(M_{i_k}) such that $\lim_{k \to \infty} \beta_{i_k} =$ length(M_{i_k}). Choose $0 \neq x_{i_k} \in Soc(H_{\beta_{i_k}}(M_{i_k}))$ and let $y_t = \oplus_{k=1}^{t} x_{i_k}$.

However, if $\{y_t\}$ is a sequence in M, then one sees that y is an element of M and $y = \oplus_{i\in J} x_i$, where J is a finite subset of I. Let r be an integer such that $i_r \in \mathcal{F}$. For $t > r$, we have

$$H_M(y - \oplus_{k=1}^{t} x_{i_k}) = \min\{H_{M_i}(\phi_i(y - \oplus_{k=1}^{t} x_{i_k}))\} \leq H_{M_{i_k}}(x_{i_k}),$$

where $\phi_i : M \to M_i$ is the projection map. Thus, $H_M(y - y_t) \leq H_{M_i}(x_{i_k})$ for some t, which is an absurd, so we pursued the contradiction. Therefore, there exists an ordinal γ such that $\mathcal{F}$ has the proper cardinality.

Let $\{x_{i_0}^k\}$ be a sequence in M_{i_0} for $i_0 \in \mathcal{F}$, and since M is closed, one verifies that the sequence has a limit in M, say $y = x_{i_0} + \oplus_{i\in I-i_0} x_i$. But $H_M(x - x_{i_0}^k) \leq H_M(x_i)$ implies $x_i = 0$ for $i \in I - i_0$, and besides, that $\lim_{k\to\infty} H_M(x - x_{i_0}^k) = H_M(0)$, which is greater than the height of any nonzero element of M. So, it follows that $x \in M_{i_0}$. This surely means that M_i is closed for $i \in \mathcal{F}$, as wanted.

To show now the truthfulness of sufficiency, let us assume that $\mathcal{F}$ has cardinality a positive integer with each M_i, for $i \in \mathcal{F}$, closed. In order to show that M is closed, it suffices to show that every bounded sequence in M has a limit in M. In order to do this, suppose $\{y_k\}$ is such a sequence and let $y_k = \oplus_{i\in I} x_i^k$, then $\{x_i^k\}$ is a sequence in the h-topology induced in M_i by the natural topology of M. In case that $i \in I - \mathcal{F}$, we can observe that the induced h-topology is discrete. So, for some k, we find $\{x_i^k\}$ is constant and has a limit x_i. But, in this case, $i \in \mathcal{F}$, we then can identify that the induced h-topology is either discrete or the natural topology of M_i. And since M_i is closed, there exists a limit x_i in M_i, and hence $y = \oplus_{i\in I} x_i$ is the limit of $\{y_k\}$. The proof is completed. □

The following gives a great deal of information about the *Ulm* invariants.

Theorem 3.3. *Let S be an isotype submodule of a QTAG-module M which is h-dense in M. Then M and S have the same Ulm invariants.*

Proof. Let the injection $S \to M$ induces a map

$$\phi : (H_\gamma(S) \cap Soc(S))/(H_{\gamma+1}(S) \cap Soc(S)) \to (H_\gamma(M) \cap Soc(M))/(H_{\gamma+1}(M) \cap Soc(M))$$

for every ordinal γ. Then

$$\begin{aligned}(H_\gamma(M) \cap Soc(S)) \cap H_{\gamma+1}(M) \cap Soc(M) &= (H_\gamma(S) \cap H_{\gamma+1}(M)) \cap Soc(S),\\ &= (S \cap H_{\gamma+1}(M)) \cap Soc(S),\\ &= H_{\gamma+1}(S) \cap Soc(S),\end{aligned}$$

and so ϕ is a monomorphism. This shows that $f_S(\gamma) \leq f_M(\gamma)$.

On the other hand, if x is any uniform element of $H_\gamma(M) \cap Soc(M)$, there exists a sequence $\{x_k\}$ in S such that it has x as a limit. By adding terms and constructing subsequences, let us assume that all the elements of $\{x_k\}$ are in S such that $e(x_k) = 1$. Then, for some k, we have $x_k - x \in (H_{\gamma+1}(M) \cap Soc(M))$ which yields

$$\phi(x_k + (H_{\gamma+1}(S) \cap Soc(S))) = x + (H_{\gamma+1}(M) + Soc(M)),$$

and ϕ is an epimorphism. Thus, $f_S(\gamma) = f_M(\gamma)$ for each ordinal γ. □

Mimicking the method demonstrated above, we record the following consequence.

Corollary 3.3. *If S is an isotype, h-dense submodule of a $QTAG$-module M, then*

$$M/H_\gamma(M) \cong S/H_\gamma(S)$$

for all $\gamma < length(M)$.

We are now ready to give our desired example.

Example 3.2. Let U and V be the $QTAG$-modules having same Ulm invariants and length $\omega + 1$. In fact, as U and V are direct sum of uniserial modules, we have that $(Soc(U) + H_\omega(U))/H_\omega(U)$ is countably generated and $(Soc(V) + H_\omega(V))/H_\omega(V)$ is not countably generated. Indeed, there exists a countably generated module P of length ω^2 with $f_\gamma(P) = 1$ for all ordinals $\gamma < \omega^2$. Applying Corollary 3.3 appointed above, and the countability of P, we see that $\overline{P}/H_\omega(\overline{P})$ is countably generated, where $\overline{P}$ is the closure of P in U and V, respectively. Let us decompose $M = \overline{P} \oplus U$ and $S = \overline{P} \oplus V$. Then M and S are closed modules with same Ulm invariants. Therefore, $(Soc(M) + H_\omega(M))/H_\omega(M)$ is countably generated and $(Soc(S)+H_\omega(S))/H_\omega(S)$ is not countably generated. Consequently, $M \ncong S$, as claimed.

Remark 3.1. *Since an isomorphism between two closed modules M and S carried $Soc(H_\gamma(M))$ isomorphically to $Soc(H_\gamma(S))$ for each ordinal γ. Therefore, the natural topological structure is preserved by isomorphisms, and such maps are actually homomorpshims.*

We continue with the significant characterization of a closed module.

Theorem 3.4. *A closed $QTAG$-module containing a direct sum of countably generated modules which form an isotype, h-dense submodule is determined up to isomorphism by its Ulm invariants.*

Proof. Let M_1 and M_2 be two closed $QTAG$-modules having the same Ulm invariants and containing submodules S_1 and S_2 possessing the desired properties. By hypothesis and consulting with Theorem 3.3, we inspect that $f_{S_1}(\gamma) = f_{M_1}(\gamma) = f_{M_2}(\gamma) = f_{S_2}(\gamma)$ for each ordinal γ. However, it is easily verified that M_1 is isomorphic to M_2. In fact, since M_1 and M_2 are the direct sum of countably generated modules, we detect that S_1 is isomorphic to S_2, and we are done. □

The above theorem leads to the analysis of determining which closed modules contain a h-dense, isotype submodule, which is a direct sum of countably generated modules.

Analysis. In accordance with [1], we construct a closed module without elements of infinite height. In fact, for a closed module of length less than or equal to the first countable ordinal ω, any of its basic submodules is the h-dense, isotype submodule. However, this is not valid for closed modules of greater length. Letting M be a closed module of countable length β containing the h-dense, isotype submodule. According to Corollary 3.3, it is plainly seen that $M/H_\gamma(M)$ must be a direct sum of countably generated modules for all $\gamma < \beta$. If $M/H_\gamma(M)$ is countably generated for all $\gamma < \beta$, the situation is the following.

Theorem 3.5. *Let M be a $QTAG$-module such that $M/H_\gamma(M)$ is countably genrated for some ordinal γ. Then there exists a countably generated, h-dense, and isotype submodule S of M, provided length of $M > \gamma$.*

Proof. Let us assume that length$(M) = \beta$. If $\gamma \geq \beta$, we are done. For the remaining case $\gamma < \beta$, we choose a set of representatives $\{x_{\gamma,k}\}_{k\in Z^+}$ of the countably generated module $M/H_\gamma(M)$ and let $\mathcal{F} = \cup_{\gamma<\beta}\{x_{\gamma,k}\}_{k\in Z^+}$. Since β is countable, we obtain that $\mathcal{F}$ is countable, and then $\mathcal{F}$ generates a countably generated module in M. Having in mind Proposition 3.2, one infers that a countably generated isotype submodule S of M containing $\mathcal{F}$.

Now we choose $x \in M$ and constructing a sequence in $\mathcal{F}$ converging to x. After this, let us find a sequence $\{\gamma_t\}$ of ordinals whose limit is β. Then for each t choose the representative x_t from $\{x_{\gamma_t,k}\}_{k\in Z^+}$ such that x is in the same coset as x_t modulo M_{γ_t}. Since the ordinals γ_t converge to β, we have $\lim_{t\to\infty} x_t = x$ and the proof is completed. □

So, the leitmotif of this article is the utilization of the above material to explore the countability of quotient modules as follows: If the $QTAG$-module M has a countable length β and $M/H_\gamma(M)$ is countably generated, for some ordinal $\gamma < \beta$. This state is known as the countability property. Therefore, we have the following direct consequences of Theorems 3.4 and 3.5, respectively.

Corollary 3.4. *Closed $QTAG$-modules with the countability property are determined up to isomorphism by their Ulm invariants.*

Corollary 3.5. *If M is a closed $QTAG$-module with the countability property, then M is determined up to isomorphism by its Ulm invariants.*

We continue with an observation on the above two corollaries.

Example 3.3. Let M be a $QTAG$-module such that $M = \oplus_\gamma Soc(H_\gamma(M))$ is the decomposition of a closed module, then M is determined by its Ulm invariants if and only if $Soc(H_\gamma(M))$ is determined by its Ulm invariants. It is readily checked that for every submodule S of $Soc(H_\gamma(M))$, we get that $\mathrm{Hom}(S, \oplus_\gamma Soc(H_\gamma(M))) = 0$, which is an essential submodule of M. This means that countability property is not sufficient in order to find an isomorphism. In accordance with Theorem 3.5, one may see that there exists a countably generated, h-dense, and isotype submodule L of M such that it is a direct sum of uniserial modules, which is a closed $QTAG$-module, as required.

We will now argue the following theorem.

Theorem 3.6. *Suppose that M_1 is a $QTAG$-module with the countability property and that M_2 is a countably generated with $f_\gamma(M_1) = f_\gamma(M_2)$ for some ordinal γ. Then M_1 can be embedded as an isotype submodule S of $\overline{M_2}$ such that $M_1 \supset M_2$.*

Proof. The existence of a countably generated, h-dense, isotype submodule S of M_1 is guaranteed by Theorem 3.5, so hypothesis M_2 does exist. S and M_1 can considered submodules of $\overline{S}$ by means of the standard topological map that embeds a space in its closure. The h-denseness of M_1 in $\overline{S}$ follows from the fact that $S \subset M_1 \subset \overline{S}$. By applying Proposition 3.1, the isotype property of M_1 can be demonstrated. Due to the equality of Ulm invariants, S and M_2 are isomorphic, and this map can be extended to $\overline{S}$ and $\overline{M_2}$ to give the desired embedding map. □

The following lemma determines the cardinality of a closed module that meets the countability property.

Lemma 3.1. *Suppose M is a countably generated $QTAG$-module. If $\overline{M}$ is the closure of M, then $g(\overline{M}) = 2^{\aleph_0}$.*

Proof. Since $g(M) = \aleph_0$, we obtain the number of Cauchy sequences in $M \leq 2^{\aleph_0}$ and thus $g(\overline{M}) \leq 2^{\aleph_0}$. So, what remains to show is the inequality $g(\overline{M}) \geq 2^{\aleph_0}$. For this purpose, choose a sequence of ordinals γ_k with a length of M as a supremum such that $Soc(H_{\gamma_k}(M)) \subset Soc(H_{\gamma_k+1}(M))$ for some $k \geq 0$. Then, there exists a sequence $\{x_k\}$ of elements in M such that $x_k \in Soc(H_{\gamma_k}(M)) - Soc(H_{\gamma_k+1}(M))$.

Let $A = (a_1, a_2, \dots)$ be the set of all $\aleph_0$ tuples, where a_k is 0 and 1. Now, define a map $f : A \to \overline{M}$ such that $f(A) = \lim y_k$ where $y_k = \oplus_{n=1}^{k} a_n x_n$. Let a and a' be two distinct elements of A with r the first n such that $a_n \neq a'_n$. Then, for $k > r$, we have

$$y_k - y'_k = x_r + \oplus_{n=r+1}^{k}(a_n x_k - a' k x_k) \notin H_{\gamma+1}(M).$$

Therefore, $f(a) \neq f(a')$. This gives that f is one-one and means that $2^{\aleph_0} = g(A) \leq g(\overline{M})$, as promised. □

We finish off with a statement which explores when a direct sum of countably generated modules has a length equal to ω_1, the first uncountable ordinal.

Theorem 3.7. *Let M_i $(i \in I)$ be a system of QTAG-modules, and let $M = \oplus_{i \in I} M_i$ be the direct sum of countably generated modules. Then M is a closed module under natural topology, provided length$(M) = \omega_1$.*

Proof. Let J be the set of countable ordinals, and let $\{x_\alpha\}_{\alpha \in J}$ be a Cauchy sequence. Then for each $i \in I$, one sees that $\{\phi_i(x_\alpha)\}_{\alpha \in J}$ is a Cauchy sequence, where $\phi_i : M \to M_i$ is the projection map. Therefore, for every ordinal γ_i, there will exist $\alpha_i \in J$ such that

$$\phi_i(x_\alpha) - \phi_i(x_\beta) \in M_{\gamma_i} \cap M_{i_{\gamma_i}} = 0$$

and $\gamma_i = \text{length}(M_i)$, for $\alpha, \beta > \alpha_i$.

Let us assume in a way of contradiction that the set $\mathcal{F} = \{i \in I : x_i \neq 0\}$ is not finite. Then there exists a sequence $\{i_k\}_{k \in Z^+}$ in $\mathcal{F}$ such that $\eta = \lim_{k \to \infty}\{\alpha_{i_k}\}$ where η is any countable ordinal. If, however, a countable ordinal $\alpha > \eta$, then $\phi_{i_k}(x_\alpha) = x_{i_k} \neq 0$ for some k, thus contradicting to our choice. So, $\mathcal{F}$ is a finite set. Letting σ be the countable ordinal with a countable ordinal α_σ such that $x_\alpha - x_\beta \in M_\sigma$ for $\alpha, \beta > \alpha_\sigma$. Thus, $\phi_i(x_\alpha) - \phi_i(x_\beta) \in M_\sigma$ for each $i \in I$. In case that $i \in \mathcal{F}$ and $\beta > \alpha_i$, we can observe that $\phi_i(x_\beta) = x_i$, and that $\phi_i(x_\alpha) - x_i \in M_\sigma$ for all $\alpha > \alpha_\sigma$. But in this case, $i \notin \mathcal{F}$ and $\beta > \alpha_i$, we may deduce that $\phi_i(x_\beta) = 0$, and that $\phi_i(x_{\alpha_i}) \in M_\sigma$ for all $\alpha > \alpha_\sigma$. Finally, in the remaining case, it can be inferred that $x_\alpha = \oplus_{i \in \mathcal{F}} x_i \in M_\sigma$ for all $\alpha > \alpha_\sigma$. This surely means that $\lim x_\alpha = \oplus_{i \in \mathcal{F}} x_i \in M$. Consequently, every Cauchy sequence in M converges in M, as formulated. □

4. Conclusion and Open problems

In this project, we examined different types of submodules and Ulm invariants via the notions of h-topology and closed modules. The intriguing properties of these notions and their interrelationships are explored, and some connections are investigated between the σ^{th}-submodules and the closures existing in the literature (see, for example, Theorem 3.1, etc.). We further revealed that a necessary and sufficient condition for a direct sum of $QTAG$-modules to be closed modules can be developed in terms of a direct summands, as detailed in Theorem 3.2. Moreover, we demonstrated that if a direct sum of countably generated modules has a length equal to ω_1, the first uncountable ordinal, then the module is a closed module under the natural topology, which occurred in Theorem 3.7.

In future work, we will study certain invariants by utilizing closed modules and h-topology via $QTAG$-modules. Also, we will generate a new countability property from $QTAG$-modules and other types of submodules in the literature. We close the work with certain challenging problems which are worthwhile for a further study.

Problem 4.1. Find the necessary (and sufficient) conditions under which a direct sum of a closed module is again a closed module?

Problem 4.2. Can closed modules be characterized by certain Ulm invariants?

Problem 4.3. Is it true that every $QTAG$-module of countable length γ with the countability property is isotype?

Problem 4.4. For a $QTAG$-module M of countable type, does it follow that $M \oplus M$ is a closed module?

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