International Journal of Maps in Mathematics

*Volume **8**, Issue **1**, 2025, Pages:**208-226***

E-ISSN: 2636-7467

www.simadp.com/journalmim

ON LACUNARY $A^{\mathcal{I}}$-STATISTICAL CONVERGENCE OF FUZZY TRIPLE SEQUENCES OF ORDER γ

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Abstract. In this study, we propose the concepts of f-lacunary $A^{\mathcal{I}}$-statistical convergence of order γ and strongly f-lacunary $A^{\mathcal{I}}$-summability of order γ for triple sequences of fuzzy numbers. Additionally, we explore fundamental connections between these convergence notions. As a practical application, we apply this newly defined convergence to establish a fuzzy Korovkin-type approximation theorem concerning triple sequences of fuzzy positive linear operators. To highlight the effectiveness of our result, we provide an example that demonstrates the superiority of the established theorem over its classical counterpart.

Keywords: Fuzzy sequence, ideal, fuzzy type Korovkin-theorem, lacunary sequence, regular matrix, triple sequence, A-statistical convergence.

2010 Mathematics Subject Classification: 40G10, 41A36.

1. Introduction

The concept of statistical convergence for sequences, an extension of the usual notion of convergence, was initially introduced in [7, 39]. This concept has spurred extensive research across various spaces and has been influential in the fields of summability theory, functional analysis, and measure theory, among others (see [5, 6], [9], [14], [17], [20], [25], [26], [29]). In their 2008 study [40], Şahiner et al. investigated statistical convergence within the context of triple sequences. For a comprehensive understanding of optimal convergence in triple

Received:2024.04.16 *Revised:2024.08.16* *Accepted:2024.11.18*

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sequences, see [42] and other related sources. A significant advancement in convergence theory, including statistical convergence, was made by Kostyrko et al. [22], who introduced the notions of $\mathcal{I}$-convergence and $\mathcal{I}^*$-convergence in metric spaces using ideals based on natural numbers. Following Kostyrko et al.'s work, similar investigations have been conducted for function sequences in random 2-normed spaces [37] and other areas. Further studies have explored these concepts in metric spaces [28], 2-normed spaces [41], and for localized sequences in metric spaces [30], with additional references provided in [11, 12, 13, 19, 36, 45].

Recently, Aizpuru et al. [1] extended the concept of natural density by introducing the f-density of a subset of positive integers using an unbounded modulus function. In 2015, Bhardwaj and Dhawan [3] introduced the definitions of f-density and f-statistical convergence of order γ. Furthermore, Şengül and Et [44] advanced the field by proposing the concept of lacunary statistical convergence of order γ in 2018, employing the modulus function.

To address uncertainty and vagueness, Zadeh [46] introduced the concepts of fuzzy sets, fuzzy logic, and fuzzy numbers in 1965. Since then, fuzzy logic has found applications in various fields such as artificial intelligence, control systems, and decision-making processes. In 1986, Matloka [24] extended these ideas to sequence space theory. The concept of statistical convergence for sequences of fuzzy numbers was later explored by Savaş [34]. For additional details on fuzzy sequence spaces, see [4], [15], [18], [38], and the associated references.

Building on the previous research, we develop and examine the properties of f-lacunary $A^{\mathcal{I}}$-statistical convergence of order γ and strongly f-lacunary $A^{\mathcal{I}}$-summability of order γ for triple sequences of fuzzy numbers. We also explore the interrelationship between these newly defined concepts. Finally, we utilize lacunary triple sequences, the modulus function, and a regular matrix to establish a fuzzy Korovkin-type theorem for triple sequences of fuzzy numbers. As a result, our findings become specific cases of the results presented in [32].

2. Preliminaries

The sets of all natural numbers, all real numbers, and all complex numbers are represented by the letters $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$, respectively, throughout the text. Let $E \subseteq \mathbb{N}$ and $E(r) = \{i \in E : i \leq r\}$. Recall that the natural or asymptotic density of E is defined by $\delta(E) = \lim_{r\to\infty} \frac{|E(r)|}{r}$ if the limit exists.

$$\lim_{r\to\infty} \frac{1}{r} |\{j : j \leq r : |y_j - y| \geq \varepsilon\}| = 0, \text{ for all } \varepsilon > 0$$

indicates that the sequence (y_j) statistically converges to y [8]. Since then, the idea of ideal of subsets of $\mathbb{N}$ has been used to expand the concept of statistical convergence to include the idea of $\mathcal{I}$-convergence [22]. Let Z be a non-empty set and $\mathcal{P}(Z)$ be the family of all subsets of Z. An ideal, denoted as $\mathcal{I}$ $(\subset \mathcal{P}(Z))$ is a family of subsets of Z satisfying the following conditions: (a) $E, R \in \mathcal{I}$ imply $E \cup R \in \mathcal{I}$ (b) $R \in \mathcal{I}$, $E \subset R$ imply $E \in \mathcal{I}$, while an admissible ideal $\mathcal{I}$ of Z covers Z. If $Z \notin \mathcal{I}$, $\mathcal{I} \neq \emptyset$, the family of sets $\mathcal{F}(\mathcal{I}) = \{M \subset Z : Z \backslash M \in \mathcal{I}\}$ forms a filter of Z. By $\mathcal{I}_{fin}$ and $\mathcal{I}_\delta$, respectively, we indicate the ideal that is composed of all finite subsets and density zero subsets of $\mathbb{N}$. A sequence $a = (a_k)$ is said to be $\mathcal{I}$-convergent to $b \in \mathbb{R}$ provided for every $\varepsilon > 0$, the set $A(\varepsilon) = \{n \in \mathbb{N} : |a_k - b| \geq \varepsilon\}$ belongs to $\mathcal{I}$ [22]. When considering $\mathcal{I} = \mathcal{I}_{fin}$, $\mathcal{I}$-convergence of the sequence aligns with ordinary convergence, and when considering $\mathcal{I} = \mathcal{I}_\delta$, it aligns with statistical convergence. Furthermore, it is worth noting that [35] delves into the concept of $\mathcal{I}$-statistically convergence. A sequence (a_k) is deemed $\mathcal{I}$-statistically convergent to a if $\{n \in \mathbb{N} : 1/n \left|\{k \leq n : |a_k - a| \geq \varepsilon\}\right| \geq \delta\}$ belongs to $\mathcal{I}$ for each $\varepsilon, \delta > 0$. Then, a is the $\mathcal{I}$-statistical limit of the sequence (a_k) and $\mathcal{I}$-*st*-$\lim_{k\to\infty} a_k = a$.

The lacunary sequence $\theta = (k_r)$, $r \to \infty$, is a nonnegative integers sequence that increases where $k_0 = 0$, $h_r = (k_r - k_{r-1})$ and $h_r \to \infty$ (and $r \to \infty$). If the following limit holds for every $\varepsilon > 0$, then a sequence (y_k) is lacunary statistically convergent to y : $\lim_{r\to\infty} 1/h_r \left|\{k \in I_r : |y_k - y| \geq \varepsilon\}\right| = 0$, where $I_r = (k_{r-1}, k_r]$ and $q_r = \frac{k_r}{k_{r-1}}$. If the following limit holds for every $\varepsilon > 0$, then a sequence (y_k) is lacunary statistically convergent to y of order γ : $\lim_{r\to\infty} 1/h_r^\gamma \left|\{k \in I_r : |y_k - y| \geq \varepsilon\}\right| = 0$, where $(h_r^\gamma) = (h_1^\gamma, h_2^\gamma, \ldots, h_r^\gamma, \ldots)$ [43].

A modulus function $g : [0, \infty) \to [0, \infty)$ such that (i) $x = 0 \Leftrightarrow g(x) = 0$; (ii) the function g is increasing; (iii) for all $x, y \in [0, \infty)$, $g(x + y) \leq g(y) + g(x)$; (iv) the function g is continuous from the right at point 0 [31]. Therefore, the function g needs be continuous throughout the the interval $[0, \infty)$.

If the following limit holds for every $\varepsilon > 0$, a sequence (y_k), is f-lacunary statistically convergent to y of order γ : $\lim_{r\to\infty} 1/f(h_r^\gamma) f(\left|\{k \in I_r : |y_k - y| \geq \varepsilon\}\right|) = 0$, where $(h_r^\gamma) = (h_1^\gamma, h_2^\gamma, \ldots, h_r^\gamma, \ldots)$ [44].

Lemma 2.1 ([24]). $\lim_{t\to\infty} \frac{f(t)}{t} = \inf\left\{\frac{f(t)}{t} : t > 0\right\}$ *for any modulus function* f.

We now recall the following definitions which were given in [10, 16, 23, 24, 46].

A fuzzy number, denoted as $\tilde{a}$, is characterized as a fuzzy set of real numbers spanning the interval from $\mathbb{R}$ to $[0,1]$ and fulfilling the following properties:

F1. there is such a t in $\mathbb{R}$ such that $\tilde{a}(t) = 1$ i.e., $\tilde{a}$ is normal,

F2. $\tilde{a}(t) \geq \min\{\tilde{a}(d), \tilde{a}(c)\} = \tilde{a}(d) \wedge \tilde{a}(c)$ where $c < t < d$, that is $\tilde{a}$ is fuzzy convex

F3. $\tilde{a}$ is upper semi continuous,

F4. $supp(\tilde{a}) = \overline{\{t \in \mathbb{R} : \tilde{a}(t) > 0\}}$ is compact.

Also, for $\alpha \in (0,1]$, the α-level cut of $\tilde{a}$ can be defined as $[\tilde{a}]_\alpha = \{t \in \mathbb{R} : \tilde{a}(t) \geq \alpha\} = [\tilde{a}_\alpha^-, \tilde{a}_\alpha^+]$, the lower and upper boundaries of the α-level cut of the fuzzy number $\tilde{a}$ are demonstrated by $\tilde{a}_\alpha^-$ and $\tilde{a}_\alpha^+$, respectively. $\mathcal{F}_\mathbb{R}$ represents the set of all fuzzy numbers. For any $\lambda \in \mathbb{R}$ and $\tilde{a}, \tilde{b} \in \mathcal{F}_\mathbb{R}$, the scalar multiplication $\lambda \odot \tilde{a}$ and the sum $\tilde{a} \oplus \tilde{b}$ are defined in that : $\left(\tilde{a} \oplus \tilde{b}\right)_\alpha = \tilde{a}_\alpha \oplus \tilde{b}_\alpha$ and $(\lambda \odot \tilde{a})_\alpha = \lambda \tilde{a}_\alpha$. Now, d is the Hausdorff metric and $d : \mathcal{F}_\mathbb{R} \times \mathcal{F}_\mathbb{R} \to \mathbb{R}$ is given by

$$d\left(\tilde{a}, \tilde{b}\right) = \sup_{0<\alpha\leq 1} \max\left\{\left|\tilde{a}_\alpha^- - \tilde{b}_\alpha^-\right|, \left|\tilde{a}_\alpha^+ - \tilde{b}_\alpha^+\right|\right\} = \sup_{0<\alpha\leq 1} d\left([\tilde{a}]_\alpha, \left[\tilde{b}\right]_\alpha\right).$$

For every $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d} \in \mathcal{F}_\mathbb{R}$, we get

d1. the space $(\mathcal{F}_\mathbb{R}, d)$ is a metric space that is complete [33],

d2. $d\left(p\tilde{a}, p\tilde{b}\right) = |p|\, d\left(\tilde{a}, \tilde{b}\right); p \in \mathbb{C}$ (the set of all complex scalars),

d3. $d\left(\tilde{a}, \tilde{b}\right) = d\left(\tilde{a} \oplus \tilde{c}, \tilde{b} \oplus \tilde{c}\right)$,

d4. $d\left(\tilde{a} \oplus \tilde{c}, \tilde{b} \oplus \tilde{d}\right) \leq d\left(\tilde{a}, \tilde{b}\right) + d\left(\tilde{c}, \tilde{d}\right)$,

d5. $\left|d\left(\tilde{a}, \tilde{0}\right) - d\left(\tilde{b}, \tilde{0}\right)\right| \leq d\left(\tilde{a}, \tilde{b}\right) \leq d\left(\tilde{a}, \tilde{0}\right) + d\left(\tilde{b}, \tilde{0}\right)$, where $\tilde{0}$ is the additive identity element of $\mathcal{F}_\mathbb{R}$.

Let $\tilde{a} = (\tilde{a}_n)$ be a sequence of fuzzy real numbers and if

$$\lim_{r\to\infty} 1/r \left|\{n : n \leq r : d(\tilde{a}_n, \tilde{a}_0) \geq \epsilon\}\right| = 0$$

for every $\epsilon > 0$, then $(\tilde{a}_n)$ is statistically convergent to fuzzy number $\tilde{a}_0$.

Definition 2.1. *If there is a positive number M such that $d(\tilde{a}_{nkl}, \tilde{0}) < M$ for all n, k, l, then the triple sequence $\tilde{a} = (\tilde{a}_{nkl})$ of fuzzy numbers is said to be bounded. $\ell_\infty^{f,3}$ is the set that represents all bounded triple sequences of fuzzy numbers.*

Assume that $A = (a_{nkolpm})$ is a summability matrix with six-dimensions. If the series converges in the sense of Pringsheim for every $(n, o, p) \in \mathbb{N}^3$, the A-transform of a given triple sequence, $x = (x_{klm})$, is given by $Ax := \left\{(Ax)_{nop}\right\}$. Recall that a six dimensional matrix

$A = (a_{nkolpm})$ is said to be Robinson-Hamilton (RH)-regular if it maps every bounded P-convergent sequence with the same P-limit. The RH-conditions state that a six dimensional matrix $A = (a_{nkolpm})$ is RH-regular iff

RH1. For each $(k, l, m) \in \mathbb{N}^3$, $P\text{-}\lim_{n,o,p} a_{nkolpm} = 0$,

RH2. $P\text{-}\lim_{n,o,p} \sum_{k\in\mathbb{N}} a_{nkolpm} = 0$ for every $l \in \mathbb{N}, m \in \mathbb{N}$,

RH3. $P\text{-}\lim_{n,o,p} \sum_{l\in\mathbb{N}} a_{nkolpm} = 0$ for every $k \in \mathbb{N}, m \in \mathbb{N}$,

RH4. $P\text{-}\lim_{n,o,p} \sum_{m\in\mathbb{N}} a_{nkolpm} = 0$ for every $k \in \mathbb{N}, l \in \mathbb{N}$,

RH5. $\sum_{(k,l,m)\in\mathbb{N}^3} |a_{nkolpm}|$ is P-convergent for all $(n, o, p) \in \mathbb{N}^3$,

RH6. There exist finite positive integers B and C such that $\sum_{k,l,m>C} |a_{nkolpm}| < B$ holds for all $(n, o, p) \in \mathbb{N}^3$,

RH7. $P\text{-}\lim_{n,o,p} \sum_{(k,l,m)\in\mathbb{N}^3} a_{nkolpm} = 1$.

Now, assume that $K' \subset \mathbb{N}^3$ and $A = (a_{nkolpm})$ is non-negative RH-regular summability matrix. When the limit on the right-hand side exists in the sense of Pringsheim, the A-density of K' is then given by $\delta_3^A(K') := P - \lim_{n,o,p} \sum_{(k,l,m)\in K'} a_{nkolpm}$, where $K' := \left\{(k, l, m) \in \mathbb{N}^3 : |x_{klm} - \ell| \geq \varepsilon\right\}$. A real triple sequence $x = (x_{klm})$ is said to be A-statistically convergent to a number ℓ if $\delta_3^A(K') = 0$ for every $\varepsilon > 0$. $\left(A^3\text{-}stat\right)\text{-}\lim_{nop} x = \ell$ in this instance.

3. Main Results

This section introduces and investigates the concepts of strongly f-lacunary $A^{\mathcal{I}}$-summability of order γ and f-lacunary $A^{\mathcal{I}}$-statistical convergence of order γ for triple sequences of fuzzy numbers. Throughout this study, unless specified otherwise, we assume $0 < \gamma \leq 1$ and that f is an unbounded modulus function.

Definition 3.1. *Let f be an unbounded modulus function, $\theta_3 = \{(k_r, l_s, m_t)\}$ be a lacunary sequence and $\gamma \in (0, 1]$. A sequence $\tilde{a} = (\tilde{a}_{klm})$ of fuzzy numbers is f-lacunary $A^{\mathcal{I}_3}$-statistical convergent of order γ ($\gamma \in (0, 1]$) (or $A^{\mathcal{I}_3^f} - stat_{\gamma,\theta_3}$-convergent) to a fuzzy number $\tilde{a}_{000}$ if for every $\varepsilon > 0$, $\zeta > 0$,*

$$\left\{(r, s, t) \in \mathbb{N}^3 : \frac{1}{f\left(h_{rst}^{\gamma}\right)} f\left(\left|\left\{(k, l, m) \in I_{r,s,t} : d\left((A\tilde{a})_{klm}, \tilde{a}_{000}\right) \geq \varepsilon\right\}\right|\right) \geq \zeta\right\}$$

belongs to $\mathcal{I}_3$. In this case we write $\left(A^{\mathcal{I}_3^f}\text{-}stat_{\gamma,\theta_3}\right)\text{-}\lim_{k,l,m\to\infty} \tilde{a}_{klm} = \tilde{a}_{000}$. $\left(A^{\mathcal{I}_3^f}\text{-}stat_{\gamma,\theta_3}\right)$ represents the set of all f-lacunary $A^{\mathcal{I}_3^f}$-statistically convergent sequences of order γ.

Definition 3.2. *A triple sequence $\tilde{a} = (\tilde{a}_{klm})$ of fuzzy numbers is strongly f-lacunary $A^{\mathcal{I}_3^f}$-summable of order γ(or ${}^{A^{\mathcal{I}_3^f}}W_{\gamma,\theta_3}$-summable) if there exists a fuzzy number $\tilde{a}_{000}$ such that*

$$\left\{(r,s,t) \in \mathbb{N}^3 : \frac{1}{f\left(h_{rst}^{\gamma}\right)} \sum_{(k,l,m)\in I_{r,s,t}} f\left(d\left((A\tilde{a})_{klm}, \tilde{a}_{000}\right)\right) \geq \varepsilon\right\} \in \mathcal{I}_3$$

for each $\varepsilon > 0$. $\left({}^{A^{\mathcal{I}_3^f}}W_{\gamma,\theta_3}\right)$ represents the set of all strongly f-lacunary $A^{\mathcal{I}_3^f}$-summable sequences of order γ.

Remark 3.1. $\left(A^{\mathcal{I}_3^f}\text{-}stat_{\gamma,\theta_3}\right)$*-convergence is well defined for $\gamma \in (0,1]$. It is not necessary to define it for $\gamma > 1$. To illustrate this, consider $(\tilde{g}_{nop})$ to be a sequence of fuzzy numbers defined as*

$$\tilde{g}_{nop}(t) = \begin{cases} t-3, & \text{if } n,o,p \text{ are odd}, \\ 1-(t-3), & \text{otherwise} \end{cases},$$

for $t \in [3,4]$, and the matrix $A = (a_{nkolpm})$ defined as

$$a_{nkolpm} = \begin{cases} 1, & \begin{array}{c}\text{if } n,o,p \text{ are a cube and} \\ k=n^3, l=o^3, m=p^3,\end{array} \\ 1, & \begin{array}{c}\text{if } n,o,p \text{ are a non cube and} \\ k=n^3+1, l=o^3+1, m=p^3+1,\end{array} \\ 0, & \text{otherwise.} \end{cases}$$

One can easily verify that

$$\begin{aligned} \left(A\tilde{g}(t)\right)_{nop} &= \sum_{k=1,l=1,m=1}^{\infty} a_{nkolpm}\tilde{g}_{klm} \\ &= \begin{cases} t-3 = (\tilde{a}), & \begin{array}{c} n,o,p \text{ is even non cube or} \\ n,o,p \text{ is odd cube,}\end{array} \\ 1-(t-3) = \left(\tilde{b}\right), & \begin{array}{c} n,o,p \text{ is an even cube or} \\ n,o,p \text{ is odd non cube.}\end{array} \end{cases} \end{aligned}$$

Therefore, we have

$$d\left(\left(A\tilde{g}(t)\right)_{nop}, \tilde{a}\right) := \begin{cases} 0; & \begin{array}{c} n,o,p \text{ are odd cubes or} \\ n,o,p \text{ are even non cubes,}\end{array} \\ 1; & \begin{array}{c} n,o,p \text{ are even cubes or} \\ n,o,p \text{ are odd non cubes,}\end{array} \end{cases}$$

and

$$d\left((A\tilde{g}(t))_{nop},\tilde{b}\right):=\begin{cases}1; & \begin{matrix} n,o,p \text{ are odd cubes or} \\ n,o,p \text{ are even non cubes,}\end{matrix} \\ 0; & \begin{matrix} n,o,p \text{ are even cubes or} \\ n,o,p \text{ are odd non cubes.}\end{matrix}\end{cases}$$

Assume that $\gamma>1$, $f(x)=x$ *and* $\theta_3=\{(j_r,k_s,l_t)\}=r^2s^2t^2$. *For* $\varepsilon>0$, $\zeta>0$ *we have*

$$\left\{(r,s,t)\in\mathbb{N}^3:\frac{1}{f(h_{rst}^{\gamma})}f\left(\left|\left\{(k,l,m)\in I_{r,s,t}:d\left((A\tilde{g})_{klm},\tilde{a}\right)\geq\varepsilon\right\}\right|\right)\geq\zeta\right\}$$
$$\subseteq\left\{(r,s,t)\in\mathbb{N}^3:\frac{h_{rst}}{h_{rst}^{\gamma}}\geq\zeta\right\}\in\mathcal{I}_3$$

and

$$\left\{(r,s,t)\in\mathbb{N}^3:\frac{1}{f(h_{rst}^{\gamma})}f\left(\left|\left\{(k,l,m)\in I_{r,s,t}:d\left((A\tilde{g})_{klm},\tilde{b}\right)\geq\varepsilon\right\}\right|\right)\geq\zeta\right\}$$
$$\subseteq\left\{(r,s,t)\in\mathbb{N}^3:\frac{h_{rst}}{h_{rst}^{\gamma}}\geq\zeta\right\}\in\mathcal{I}_3.$$

Thus, $(\tilde{g}_{nop})$ *is* f*-lacunary* $A^{\mathcal{I}_3}$*-statistically convergent to both* $\tilde{a}$ *and* $\tilde{b}$*, which is impossible.*

Theorem 3.1. *Let* $y=(\tilde{y}_{klm})$ *and* $g=(\tilde{g}_{klm})$ *be two triple fuzzy sequences and* $\gamma\in(0,1]$. *Then, the subsequent statements are valid:*

(a) If $\left(A^{\mathcal{I}_3^f}\text{-}stat_{\gamma,\theta_3}\right)\text{-}\lim_{k,l,m}\tilde{y}_{klm}=\tilde{y}_{000}$ *and* $z\in\mathbb{C}$*, then* $\left(A^{\mathcal{I}_3^f}\text{-}stat_{\gamma,\theta_3}\right)\text{-}\lim_{k,l,m}z\tilde{y}_{klm}=z\tilde{y}_{000}$.

(b) If $\left(A^{\mathcal{I}_3^f}\text{-}stat_{\gamma,\theta_3}\right)\text{-}\lim_{k,l,m}\tilde{y}_{klm}=\tilde{y}_{000}$ *and* $\left(A^{\mathcal{I}_3^f}\text{-}stat_{\gamma,\theta_3}\right)-\lim_{k,l,m}\tilde{g}_{klm}=\tilde{g}_{000}$*, then* $\left(A^{\mathcal{I}_3^f}\text{-}stat_{\gamma,\theta_3}\right)\text{-}\lim_{k,l,m}\left(\tilde{y}_{klm}+\tilde{g}_{klm}\right)=\tilde{y}_{000}+\tilde{g}_{000}$.

Proof. (a) For $z=0$, the result holds trivially. Let $z\neq0$, for given $\varepsilon>0$, we obtain

$$\left\{\left((k,l,m)\in I_{r,s,t}:d\left((Az\tilde{y})_{klm},z\tilde{y}_{000}\right)\geq\varepsilon\right)\right\}$$
$$=\left\{\left((k,l,m)\in I_{r,s,t}:|z|\,d\left((A\tilde{y})_{klm},\tilde{y}_{000}\right)\geq\varepsilon\right)\right\}$$
$$\subseteq\left\{\left((k,l,m)\in I_{r,s,t}:d\left((A\tilde{y})_{klm},\tilde{y}_{000}\right)\geq\frac{\varepsilon}{|z|}\right)\right\},$$

and, so we have

$$\left\{(r,s,t)\in\mathbb{N}^3:\frac{1}{f(h_{rst}^{\gamma})}f\left(\left|\left\{(k,l,m)\in I_{r,s,t}:d\left((Az\tilde{y})_{klm},z\tilde{y}_{000}\right)\geq\varepsilon\right\}\right|\right)\geq\zeta\right\}$$
$$\subseteq\left\{(r,s,t)\in\mathbb{N}^3:\frac{1}{f(h_{rst}^{\gamma})}f\left(\left|\left\{(k,l,m)\in I_{r,s,t}:d\left((A\tilde{y})_{klm},\tilde{y}_{000}\right)\geq\frac{\varepsilon}{|z|}\right\}\right|\right)\geq\zeta\right\}$$
$$\in\mathcal{I}_3,$$

for all $\zeta > 0$.

(b) It is derived from the fact that

$$\begin{aligned}
&\{(k,l,m) \in I_{r,s,t} : d\left(\left(A\left(\tilde{y}+\tilde{g}\right)\right)_{klm}, \tilde{y}_{000}+\tilde{g}_{000}\right) \geq \varepsilon\} \\
&\subseteq \{(k,l,m) \in I_{r,s,t} : d\left(\left(\left(A\left(\tilde{y}\right)\right)_{klm}, \tilde{y}_{000}\right) + \left(\left(A\left(\tilde{g}\right)\right)_{klm}, \tilde{g}_{000}\right)\right) \geq \varepsilon\} \\
&\subseteq \left\{(k,l,m) \in I_{r,s,t} : d\left(\left(A\left(\tilde{y}\right)\right)_{klm}, \tilde{y}_{000}\right) \geq \frac{\varepsilon}{2}\right\} \\
&\cup \left\{(k,l,m) \in I_{r,s,t} : d\left(\left(A\left(\tilde{g}\right)\right)_{klm}, \tilde{g}_{000}\right) \geq \frac{\varepsilon}{2}\right\}.
\end{aligned}$$

Additionally,

$$\begin{aligned}
&\left\{(r,s,t) \in \mathbb{N}^3 : \frac{1}{f\left(h_{rst}^{\gamma}\right)} f\left(\left|\{(k,l,m) \in I_{r,s,t} : d\left(\left(A\left(\tilde{y}+\tilde{g}\right)\right)_{klm}, \tilde{y}_{000}+\tilde{g}_{000}\right) \geq \varepsilon\}\right|\right) \geq \zeta\right\} \\
&\subseteq \left\{(r,s,t) \in \mathbb{N}^3 : \frac{1}{f\left(h_{rst}^{\gamma}\right)} f\left(\left|\left\{(k,l,m) \in I_{r,s,t} : d\left(\left(A\left(\tilde{y}\right)\right)_{klm}, \tilde{y}_{000}\right) \geq \frac{\varepsilon}{2}\right\}\right|\right) \geq \zeta\right\} \\
&\cup \left\{(r,s,t) \in \mathbb{N}^3 : \frac{1}{f\left(h_{rst}^{\gamma}\right)} f\left(\left|\left\{(k,l,m) \in I_{r,s,t} : d\left(\left(A\left(\tilde{g}\right)\right)_{klm}, \tilde{g}_{000}\right) \geq \frac{\varepsilon}{2}\right\}\right|\right) \geq \zeta\right\} \\
&\in \mathcal{I}_3,
\end{aligned}$$

is a consequence of this. Therefore, (b) follows. □

Theorem 3.2. *Let f be an unbounded modulus function such that $f(xy) \geq cf(x)f(y)$ for some positive constant $c, x, y \geq 0$ and $0 < \gamma \leq \delta \leq 1$. Then, we have* $\left(A^{\mathcal{I}_3^f} W_{\gamma,\theta_3}\right) \subseteq \left(A^{\mathcal{I}_3^f}\text{-}stat_{\gamma,\theta_3}\right)$.

Proof. Let $\tilde{y} \in \left(A^{\mathcal{I}_3} W_{\gamma,\theta_3 \cdot f}\right)$. Then, for $\varepsilon > 0$ and $\zeta > 0$

$$\left\{(r,s,t) \in \mathbb{N}^3 : \frac{1}{f\left(h_{rst}^{\gamma}\right)} \sum_{(k,l,m) \in I_{r,s,t}} f\left(d\left(\left(A\tilde{y}\right)_{klm}, \tilde{y}_{000}\right)\right) \geq \varepsilon\right\} \in \mathcal{I}_3, \tag{3.1}$$

and, so we get

$$\left\{(r,s,t)\in\mathbb{N}^3:\frac{1}{f\left(h_{rst}^{\gamma}\right)}\sum_{(k,l,m)\in I_{r,s,t}} f\left(d\left((A\tilde{y})_{klm},\tilde{y}_{000}\right)\right)\geq\varepsilon\right\}$$
$$\left\{(r,s,t)\in\mathbb{N}^3:\frac{1}{f\left(h_{rst}^{\gamma}\right)}\sum_{(k,l,m)\in I_{r,s,t}} f\left(d\left((A\tilde{y})_{klm},\tilde{y}_{000}\right)\right)\geq\varepsilon\right\}$$
$$\supseteq\left\{(r,s,t)\in\mathbb{N}^3:\frac{1}{f\left(h_{rst}^{\gamma}\right)}f\left(\sum_{(k,l,m)\in I_{r,s,t}} d\left((A\tilde{y})_{klm},\tilde{y}_{000}\right)\right)\geq\varepsilon\right\}$$
$$\supseteq\left\{(r,s,t)\in\mathbb{N}^3:\frac{1}{f\left(h_{rst}^{\delta}\right)}f\left(\sum_{\substack{(k,l,m)\in I_{r,s,t}\\ d\left((A(\tilde{y}))_{klm},\tilde{y}_0\right)\geq\varepsilon}} d\left((A\tilde{y})_{klm},\tilde{y}_{000}\right)\right.\right.$$
$$\left.\left.+\sum_{\substack{(k,l,m)\in I_{r,s,t}\\ d\left((A(\tilde{y}))_{klm},\tilde{y}_0\right)<\varepsilon}} d\left((A\tilde{y})_{klm},\tilde{y}_{000}\right)\right)\geq\varepsilon\right\}$$
$$\supseteq\left\{(r,s,t)\in\mathbb{N}^3:\frac{1}{f\left(h_{rst}^{\delta}\right)}f\left(\sum_{\substack{(k,l,m)\in I_{r,s,t}\\ d\left((A(\tilde{y}))_{klm},\tilde{y}_0\right)\geq\varepsilon}} d\left((A\tilde{y})_{klm},\tilde{y}_{000}\right)\right)\geq\varepsilon\right\}$$
$$\supseteq\left\{(r,s,t)\in\mathbb{N}^3:\frac{1}{f\left(h_{rst}^{\delta}\right)}f\left(\left|\left\{(k,l,m)\in I_{r,s,t}:d\left((A(\tilde{y}))_{klm},\tilde{y}_{000}\right)\geq\varepsilon\right\}\right|\varepsilon\right)\geq\zeta\right\}$$
$$\supseteq\left\{(r,s,t)\in\mathbb{N}^3:\left(\frac{c}{f\left(h_{rst}^{\delta}\right)}f\left(\left|\left\{(k,l,m)\in I_{r,s,t}:d\left((A(\tilde{y}))_{klm},\tilde{y}_{000}\right)\geq\varepsilon\right\}\right|\right)f(\varepsilon)\right)\geq\zeta\right\}$$

This implies

$$\left\{(r,s,t)\in\mathbb{N}^3:\frac{c}{f\left(h_{rst}^{\delta}\right)}f\left(\left|\left\{(k,l,m)\in I_{r,s,t}:d\left((A(\tilde{y}))_{klm},\tilde{y}_{000}\right)\geq\varepsilon\right\}\right|f(\varepsilon)\right)\geq\zeta\right\}$$
$$\subseteq\left\{(r,s,t)\in\mathbb{N}^3:\frac{1}{f\left(h_{rst}^{\gamma}\right)}\sum_{(k,l,m)\in I_{r,s,t}} f\left(d\left((A\tilde{y})_{klm},\tilde{y}_{000}\right)\right)\geq\varepsilon\right\}.$$

Using (3.1), we obtain $\tilde{y}\in\left(A^{\mathcal{I}_3^f}\text{-}stat_{\gamma,\theta_3}\right)$. □

Using Lemma 2.1, we can give the following theorem.

Theorem 3.3. *Let f be an unbounded modulus function such that $\lim_{t\to\infty}\frac{f(t)}{t}>0$, $\lim_{r,s,t\to\infty}\frac{f(h_{rst})}{f\left(h_{rst}^{\gamma}\right)}=1$ and $0<\gamma\leq\delta\leq1$. Then,*

$$\left(A^{\mathcal{I}_3^f}\text{-}stat_{\gamma,\theta_3}\right)\cap\ell_\infty^3(A)\subseteq\left(A^{\mathcal{I}_3^f}W_{\delta,\theta_3}\right)\cap\ell_\infty^3(A).$$

Proof. Assume that $\lim_{t\to\infty}\frac{f(t)}{t}=L$. Then, by Lemma 2.1, $L\leq\frac{f(t)}{t}$, for all $t>0$. Let $\tilde{y}\in\left(A^{\mathcal{I}_3^f}\text{-}stat_{\gamma,\theta_3}\right)\cap\ell_\infty^3(A)$. Then, there exists positive real number M such that

$d\left((A\tilde{y})_{klm}, \tilde{y}_{000}\right) \leq M$, for any $k, l, m \in \mathbb{N}$, and

$$\left\{(r,s,t) \in \mathbb{N}^3 : \frac{1}{f\left(h_{rst}^{\gamma}\right)} f\left(\left|\left\{(k,l,m) \in I_{r,s,t} : d\left((A(\tilde{y}))_{klm}, \tilde{y}_{000}\right) \geq \varepsilon\right\}\right|\right) \geq \zeta\right\} \in \mathcal{I}_3,$$

supplies for every $\varepsilon > 0$ and $\zeta > 0$.

Now,

$$\begin{aligned}
&\left\{(r,s,t) \in \mathbb{N}^3 : \tfrac{1}{f\left(h_{rst}^{\delta}\right)} \sum_{(k,l,m)\in I_{r,s,t}} f\left(d\left((A\tilde{y})_{klm}, \tilde{y}_{000}\right)\right) \geq \varepsilon\right\} \\
&\subseteq \left\{(r,s,t) \in \mathbb{N}^3 : \tfrac{1}{f\left(h_{rst}^{\gamma}\right)} \sum_{(k,l,m)\in I_{r,s,t}} f\left(d\left((A\tilde{y})_{klm}, \tilde{y}_{000}\right)\right) \geq \varepsilon\right\} \\
&= \left\{(r,s,t) \in \mathbb{N}^3 : \tfrac{1}{f\left(h_{rst}^{\gamma}\right)} \left[\sum_{\substack{(k,l,m)\in I_{r,s,t} \\ d\left((A(\tilde{y}))_{klm}, \tilde{y}_0\right) \geq \varepsilon}} f\left(d\left((A\tilde{y})_{klm}, \tilde{y}_{000}\right)\right)\right.\right. \\
&\quad \left.\left. + \sum_{\substack{(k,l,m)\in I_{r,s,t} \\ d\left((A(\tilde{y}))_{klm}, \tilde{y}_0\right) < \varepsilon}} f\left(d\left((A\tilde{y})_{klm}, \tilde{y}_{000}\right)\right)\right] \geq \varepsilon\right\} \\
&= \left\{(r,s,t) \in \mathbb{N}^3 : \tfrac{1}{f\left(h_{rst}^{\gamma}\right)} \left[\sum_{\substack{(k,l,m)\in I_{r,s,t} \\ d\left((A(\tilde{y}))_{klm}, \tilde{y}_{000}\right) \geq \varepsilon}} f(M) + \sum_{\substack{(k,l,m)\in I_{r,s,t} \\ d\left((A(\tilde{y}))_{klm}, \tilde{y}_{000}\right) < \varepsilon}} f(\varepsilon)\right]\right\} \\
&= \left\{(r,s,t) \in \mathbb{N}^3 : \tfrac{1}{f\left(h_{rst}^{\gamma}\right)} f(M) \left|\left\{(k,l,m) \in I_{r,s,t} : d\left((A(\tilde{y}))_{klm}, \tilde{y}_{000}\right) \geq \varepsilon\right\}\right| \geq \zeta\right\} \\
&+ \left\{(r,s,t) \in \mathbb{N}^3 : \tfrac{h_{rst}}{f\left(h_{rst}^{\gamma}\right)} f(\varepsilon) \geq \zeta\right\} \\
&\subseteq \left\{(r,s,t) \in \mathbb{N}^3 : \tfrac{L^{-1}}{f\left(h_{rst}^{\gamma}\right)} f(M) f\left(\left|\left\{(k,l,m) \in I_{r,s,t} : d\left((A(\tilde{y}))_{klm}, \tilde{y}_{000}\right) \geq \varepsilon\right\}\right|\right) \geq \zeta\right\} \\
&+ \left\{(k,l,m) \in I_{r,s,t} : \tfrac{L^{-1} f(h_{rst})}{f\left(h_{rst}^{\gamma}\right)} f(\varepsilon) \geq \zeta\right\}.
\end{aligned} \tag{3.2}$$

Using $\lim_{r,s,t\to\infty} \frac{f(h_{rst})}{f\left(h_{rst}^{\gamma}\right)} = 1$, we have $\tilde{y} \in \left(A^{\mathcal{I}_3^f} W_{\delta,\theta_3}\right) \cap \ell_\infty(A)$. □

3.1. **Fuzzy Korovkin-type theorems.** One notable theorem in mathematics is Korovkin's theorem, named after the mathematician Korovkin [21]. This theorem addresses how a sequence of positive linear operators can uniformly approximate continuous functions defined on compact metric spaces. Over time, the theorem's importance has grown across various mathematical disciplines. Researchers have explored its applications in numerous settings and have proposed several extensions in areas such as functional analysis, measure theory, probability theory, and summability theory (see [2], [16], [27], [32]). In this section, we apply lacunary triple sequences, modulus functions, and regular matrices to establish a fuzzy Korovkin-type theorem specifically for triple fuzzy number sequences.

A fuzzy valued function $\tilde{f}:[a,b]\times[a,b]\times[a,b]\to\mathcal{F}_{\mathbb{R}}$ is fuzzy continuous at $(u_{000},y_{000},z_{000})$ in $([a,b])^3=[a,b]\times[a,b]\times[a,b]$ if $(u_{klm},y_{klm},z_{klm})\to(u_{000},y_{000},z_{000})$, then

$$d^*\left(\tilde{f}\left(u_{klm},y_{klm},z_{klm}\right),\tilde{f}\left(u_{000},y_{000},z_{000}\right)\right)\to 0,\ \text{as } k,l,m\to\infty,$$

where

$$\begin{aligned}&d^*\left(\tilde{f}\left(u_{klm},y_{klm},z_{klm}\right),\tilde{f}\left(u_{000},y_{000},z_{000}\right)\right)\\ &=\sup_{(u,y,z)\in([a,b])^3} d\left(\tilde{f}\left(u_{klm},y_{klm},z_{klm}\right),\tilde{f}\left(u_{000},y_{000},z_{000}\right)\right).\end{aligned}$$

If $\tilde{f}$ is fuzzy continuous at every point in $[a,b]$, then $\tilde{f}$ is fuzzy continuous on $([a,b])^3$. The set of all fuzzy continuous functions on the interval $([a,b])^3$ is denoted by $C_{\mathcal{F}}\left(([a,b])^3\right)$, and $C\left(([a,b])^3\right)$ represents the space of all continuous functions on $([a,b])^3$.

An operator $\tilde{\mathcal{T}}:C_{\mathcal{F}}\left(([a,b])^3\right)\to C_{\mathcal{F}}\left(([a,b])^3\right)$ is fuzzy linear, if

$$\tilde{\mathcal{T}}\left(\lambda_1\odot\tilde{f}_1\oplus\lambda_2\odot\tilde{f}_2;u,y,z\right)=\lambda_1\odot\tilde{\mathcal{T}}\left(\tilde{f}_1;u,y,z\right)\oplus\lambda_2\odot\tilde{\mathcal{T}}\left(\tilde{f}_2;u,y,z\right),$$

for every $\lambda_1,\lambda_2\in\mathbb{R}$ and $\tilde{f}_1,\tilde{f}_2\in C_{\mathcal{F}}\left(([a,b])^3\right)$. Furthermore, $\tilde{\mathcal{T}}$ is fuzzy positive linear operator, if it is fuzzy linear and

$$\tilde{\mathcal{T}}\left(\tilde{f}_1;u,y,z\right)\leq\tilde{\mathcal{T}}\left(\tilde{f}_2;u,y,z\right)$$

for all $\tilde{f}_1,\tilde{f}_2\in C_{\mathcal{F}}\left(([a,b])^3\right)$, and for any $(u,y,z)\in([a,b])^3$, and with $\tilde{f}_1(u,y,z)\leq\tilde{f}_2(u,y,z)$.

Theorem 3.4. *Assume that* $\left(\tilde{\mathcal{T}}_{klm}\right)$ *be a triple sequence of positive linear operators from* $C_{\mathcal{F}}\left(([a,b])^3\right)$ *to* $C_{\mathcal{F}}\left(([a,b])^3\right)$. *Suppose that there is a sequence* $(\mathcal{T}_{klm})$ *of positive linear operators from* $C\left(([a,b])^3\right)$ *into* $C\left(([a,b])^3\right)$ *such that*

$$\left\{\tilde{\mathcal{T}}_{klm}\left(\tilde{f};u,y,z\right)\right\}_{\alpha}^{\pm}=\mathcal{T}_{klm}\left(\tilde{f}_{\alpha}^{\pm};u,y,z\right),\quad(k,l,m\in\mathbb{N})\tag{3.3}$$

for each $\tilde{f}\in C_{\mathcal{F}}\left(([a,b])^3\right)$, $\alpha\in[0,1]$ *and* $(u,y,z)\in([a,b])^3$. *Then, if*

$$\left\{(k,l,m)\in I_{r,s,t}:\left\|\mathcal{T}_{klm}(g_i)-g_i\right\|\geq\varepsilon\right\}\in\mathcal{I}_3;\ \left(i=\overline{0,4}\right),\tag{3.4}$$

where $g_0=1,\ g_1=u,\ g_2=y,\ g_3=z,\ g_4=u^2+y^2+z^2$, *we have*

$$\left\{(k,l,m)\in I_{r,s,t}:d^*\left(\tilde{\mathcal{T}}_{klm}\left(\tilde{f}\right),\tilde{f}\right)\geq\varepsilon\right\}\in\mathcal{I}_3;\ \forall\tilde{f}\in C_{\mathcal{F}}[a,b],\tag{3.5}$$

for every $\varepsilon>0$.

Theorem 3.5. *Consider a fuzzy sequence* $\left(\tilde{\mathcal{T}}_{klm}\right)$ *of positive linear operators from* $C_{\mathcal{F}}\left(([a,b])^3\right)$ *into* $C_{\mathcal{F}}\left(([a,b])^3\right)$*. Suppose there exists a sequence* $(\mathcal{T}_{klm})$ *of positive linear operators from* $C\left(([a,b])^3\right)$ *into* $C\left(([a,b])^3\right)$ *such that equation (3.3) holds. If*

$$\left(A^{\mathcal{I}_3^f}\text{-}stat_{\gamma,\theta_3}\right)\text{-}\left\|\mathcal{T}_{klm}\left(g_i\right)-g_i\right\|=0,\ \left(i=\overline{0,4}\right), \tag{3.6}$$

where $g_0=1, g_1=u, g_2=y, g_3=z$ *and* $g_4=u^2+y^2+z^2$*, then we have*

$$\left(A^{\mathcal{I}_3^f}\text{-}stat_{\gamma,\theta_3}\right)\text{-}d^*\left(\tilde{\mathcal{T}}_{klm}\left(\tilde{f}\right),\tilde{f}\right)=0,\ \forall\tilde{f}\in C_{\mathcal{F}}\left[a,b\right]. \tag{3.7}$$

Proof. Let $\tilde{f}\in C_{\mathcal{F}}\left(([a,b])^3\right)$ and $(u,y,z)\in([a,b])^3$. Since $\tilde{f}_\alpha^{\pm}$ is continuous on $([a,b])^3$, for every $\varepsilon>0$ there exists $\delta>0$ such that $\left|\tilde{f}_\alpha^{\pm}(e,f,h)-\tilde{f}_\alpha^{\pm}(u,y,z)\right|<\varepsilon$, whenever $|e-u|<\delta, |f-y|<\delta, |h-z|<\delta$. Since $\tilde{f}$ is fuzzy bounded, we have $\left|\tilde{f}_\alpha^{\pm}(u,y,z)\right|\leq\mathcal{K}_\alpha^{\pm}$ for all $(u,y,z)\in([a,b])^3$. Thus, we get

$$\left|\tilde{f}_\alpha^{\pm}(e,f,h)-\tilde{f}_\alpha^{\pm}(u,y,z)\right|\leq\varepsilon+\frac{2\mathcal{K}_\alpha^{\pm}}{\delta^2}\left|(e-u)^2+(f-y)^2+(h-z)^2\right| \tag{3.8}$$

for every $(e,f,h),(u,y,z)\in([a,b])^3$.

Applying $(A\mathcal{T}(g_0;u,y,z))_{klm}$ on both the sides for a fixed (u,y,z) and by the monotonicity and linearity of $(A\mathcal{T}(g_0;u,y,z))_{klm}$, we have

$$\begin{aligned}
&\left|\left(A\mathcal{T}\left(\tilde{f}_\alpha^{\pm}(e,f,h);u,y,z\right)\right)_{klm}-\left(A\mathcal{T}\left(\tilde{f}_\alpha^{\pm}(u,y,z);u,y,z\right)\right)_{klm}\right|\\
&\leq\left|\varepsilon\left(A\mathcal{T}(1;u,y,z)\right)_{klm}+\tfrac{2\mathcal{K}_\alpha^{\pm}}{\delta^2}\left(A\mathcal{T}\left((e-u)^2+(f-y)^2+(h-z)^2;u,y,z\right)\right)_{klm}\right|\\
&=\left|\varepsilon\left(A\mathcal{T}(1;u,y,z)\right)_{klm}+\tfrac{2\mathcal{K}_\alpha^{\pm}}{\delta^2}\left(\left(A\mathcal{T}\left(e^2+f^2+h^2;u,y,z\right)\right)_{klm}\right.\right.\\
&-2u\left(A\mathcal{T}(e;u,y,z)\right)_{klm}-2y\left(A\mathcal{T}(f;u,y,z)\right)_{klm}|\\
&-2z\left(A\mathcal{T}(h;u,y,z)\right)_{klm}+\left(u^2+y^2+z^2\right)\left(A\mathcal{T}(1;u,y,z)\right)_{klm}|.
\end{aligned} \tag{3.9}$$

Using (3.8) and (3.9), we have

$$
\begin{aligned}
&\left|\left(A\mathcal{T}\left(\tilde{f}_{\alpha}^{\pm}(e,f,h);u,y,z\right)\right)_{klm}-\tilde{f}_{\alpha}^{\pm}(u,y,z)\right| \\
&=\left|\left(A\mathcal{T}\left(\tilde{f}_{\alpha}^{\pm}(e,f,h);u,y,z\right)\right)_{klm}-\tilde{f}_{\alpha}^{\pm}(u,y,z)\left(A\mathcal{T}(1;u,y,z)\right)_{klm}\right. \\
&\left.+\tilde{f}_{\alpha}^{\pm}(u,y,z)\left(A\mathcal{T}(1;u,y,z)\right)_{klm}-\tilde{f}_{\alpha}^{\pm}(u,y,z)\right| \\
&=\left|\left(A\mathcal{T}\left(\tilde{f}_{\alpha}^{\pm}(e,f,h);u,y,z\right)\right)_{klm}-\left(A\mathcal{T}\left(\tilde{f}_{\alpha}^{\pm}(u,y,z);u,y,z\right)\right)_{klm}\right. \\
&\left.+\tilde{f}_{\alpha}^{\pm}(u,y,z)\left(\left(A\mathcal{T}(1;u,y,z)\right)_{klm}-1\right)\right| \\
&\leq\left|\left(A\mathcal{T}\left(\tilde{f}_{\alpha}^{\pm}(e,f,h);u,y,z\right)\right)_{klm}-\left(A\mathcal{T}\left(\tilde{f}_{\alpha}^{\pm}(u,y,z);u,y,z\right)\right)_{klm}\right| \\
&+\left|\tilde{f}_{\alpha}^{\pm}(u,y,z)\left(\left(A\mathcal{T}(1;u,y,z)\right)_{klm}-1\right)\right| \\
&\leq\left|\varepsilon\left(A\mathcal{T}(1;u,y,z)\right)_{klm}+\frac{2\mathcal{K}_{\alpha}^{\pm}}{\delta^{2}}\left[\left(A\mathcal{T}\left(\left(e^{2}+f^{2}+h^{2}\right);u,y,z\right)\right)_{klm}\right.\right. \\
&-2u\left(A\mathcal{T}(e;u,y,z)\right)_{klm}-2y\left(A\mathcal{T}(f;u,y,z)\right)_{klm} \\
&\left.\left.-2z\left(A\mathcal{T}(h;u,y,z)\right)_{klm}+\left(u^{2}+y^{2}+z^{2}\right)\left(A\mathcal{T}(1;u,y,z)\right)_{klm}\right]\right| \\
&+\left|\tilde{f}_{\alpha}^{\pm}(u,y,z)\left(\left(A\mathcal{T}(1;u,y,z)\right)_{klm}-1\right)\right| \\
&\leq\left|\varepsilon\left(A\mathcal{T}(1;u,y,z)\right)_{klm}+\frac{2\mathcal{K}_{\alpha}^{\pm}}{\delta^{2}}\left[\left(\left(A\mathcal{T}\left(\left(e^{2}+f^{2}+h^{2}\right);u,y,z\right)\right)_{klm}-\left(u^{2}+y^{2}+z^{2}\right)\right)\right.\right. \\
&-2u\left(\left(A\mathcal{T}(e;u,y,z)\right)_{klm}-u\right)-2y\left(\left(A\mathcal{T}(f;u,y,z)\right)_{klm}-y\right) \\
&\left.\left.-2z\left(\left(A\mathcal{T}(h;u,y,z)\right)_{klm}-z\right)+\left(u^{2}+y^{2}+z^{2}\right)\left(\left(A\mathcal{T}(1;u,y,z)\right)_{klm}-1\right)\right]\right| \\
&+\left|\tilde{f}_{\alpha}^{\pm}(u,y,z)\left(\left(A\mathcal{T}(1;u,y,z)\right)_{klm}-1\right)\right| \\
&\leq\left|\varepsilon+\varepsilon\left(A\mathcal{T}(1;u,y,z)\right)_{klm}-\varepsilon\right|+\frac{2\mathcal{K}_{\alpha}^{\pm}}{\delta^{2}}\left[\left|\left(A\mathcal{T}\left(\left(e^{2}+f^{2}+h^{2}\right);u,y,z\right)\right)_{klm}-\left(u^{2}+y^{2}+z^{2}\right)\right|\right. \\
&+|2u|\left|\left(A\mathcal{T}(e;u,y,z)\right)_{klm}-u\right|+|2y|\left|\left(A\mathcal{T}(f;u,y,z)\right)_{klm}-y\right| \\
&\left.+|2z|\left|\left(A\mathcal{T}(h;u,y,z)\right)_{klm}-z\right|+\left(u^{2}+y^{2}+z^{2}\right)\left|\left(A\mathcal{T}(1;u,y,z)\right)_{klm}-1\right|\right] \\
&+\left|\tilde{f}_{\alpha}^{\pm}(u,y,z)\left(\left(A\mathcal{T}(1;u,y,z)\right)_{klm}-1\right)\right| \\
&\leq\varepsilon+\varepsilon\left|\left(A\mathcal{T}(1;u,y,z)\right)_{klm}-1\right|+\frac{2\mathcal{K}_{\alpha}^{\pm}}{\delta^{2}}\left|\left(A\mathcal{T}\left(\left(e^{2}+f^{2}+h^{2}\right);u,y,z\right)\right)_{klm}-\left(u^{2}+y^{2}+z^{2}\right)\right| \\
&+\frac{4\mathcal{K}_{\alpha}^{\pm}}{\delta^{2}}|u|\left|\left(A\mathcal{T}((e+f+h);u,y,z)\right)_{klm}-u\right|+\frac{4\mathcal{K}_{\alpha}^{\pm}}{\delta^{2}}|y|\left|\left(A\mathcal{T}((e+f+h);u,y,z)\right)_{klm}-y\right| \\
&+\frac{4\mathcal{K}_{\alpha}^{\pm}}{\delta^{2}}|z|\left|\left(A\mathcal{T}((e+f+h);u,y,z)\right)_{klm}-z\right|+\frac{2\mathcal{K}_{\alpha}^{\pm}}{\delta^{2}}\left(u^{2}+y^{2}+z^{2}\right)\left|\left(A\mathcal{T}(1;u,y,z)\right)_{klm}-1\right| \\
&+\left|\tilde{f}_{\alpha}^{\pm}(u,y,z)\right|\left|\left(\left(A\mathcal{T}(1;u,y,z)\right)_{klm}-1\right)\right| \\
&\leq\varepsilon+\left(\varepsilon+\frac{2\mathcal{K}_{\alpha}^{\pm}\left(B^{2}+C^{2}+D^{2}\right)}{\delta^{2}}+\mathcal{K}_{\alpha}^{\pm}\right)\left|\left(A\mathcal{T}(1;u,y,z)\right)_{klm}-1\right| \\
&+\frac{4\mathcal{K}_{\alpha}^{\pm}B}{\delta^{2}}\left|\left(A\mathcal{T}((e+f+h);u,y,z)\right)_{klm}-u\right|+\frac{4\mathcal{K}_{\alpha}^{\pm}C}{\delta^{2}}\left|\left(A\mathcal{T}((e+f+h);u,y,z)\right)_{klm}-y\right| \\
&+\frac{4\mathcal{K}_{\alpha}^{\pm}D}{\delta^{2}}\left|\left(A\mathcal{T}((e+f+h);u,y,z)\right)_{klm}-z\right| \\
&+\frac{2\mathcal{K}_{\alpha}^{\pm}}{\delta^{2}}\left|\left(A\mathcal{T}\left(\left(e^{2}+f^{2}+h^{2}\right);u,y,z\right)\right)_{klm}-\left(u^{2}+y^{2}+z^{2}\right)\right| \\
&\leq\varepsilon+\mathcal{M}_{\alpha}^{\pm}\left(\left|\left(A\mathcal{T}(g_{0};u,y,z)\right)_{klm}-g_{0}\right|+\left|\left(A\mathcal{T}(g_{1};u,y,z)\right)_{klm}-g_{1}\right|\right. \\
&\left.+\left|\left(A\mathcal{T}(g_{2};u,y,z)\right)_{klm}-g_{2}\right|+\left|\left(A\mathcal{T}(g_{3};u,y,z)\right)_{klm}-g_{3}\right|+\left|\left(A\mathcal{T}(g_{4};u,y,z)\right)_{klm}-g_{4}\right|\right)
\end{aligned}
$$

where

$$\mathcal{M}_\alpha^{\pm}=\max\left\{\varepsilon+\tfrac{2\mathcal{K}_\alpha^{\pm}\left(B^2+C^2+D^2\right)}{\delta^2}+\mathcal{K}_\alpha^{\pm},\tfrac{4\mathcal{K}_\alpha^{\pm}(B+C+D)}{\delta^2},\tfrac{2\mathcal{K}_\alpha^{\pm}}{\delta^2}\right\},$$

$$B=\max\{|u|\},\ C=\max\{|y|\}\ \text{ and } D=\max\{|z|\}.$$

Then, taking supremum over $(u,y,z)\in([a,b])^3$, we obtain

$$\left\|\left(A\mathcal{T}\left(\tilde{f}_\alpha^{\pm}\right)\right)_{klm}-\tilde{f}_\alpha^{\pm}(u,y,z)\right\|\le\varepsilon+\mathcal{M}_\alpha^{\pm}\sum_{i=0}^{4}\left\|\left(A\mathcal{T}(g_i)\right)_{klm}-g_i\right\|. \tag{3.10}$$

Using the definition of $d^*(.,.)$ and the relation (3.3), we have

$$d^*\left(\left(A\tilde{\mathcal{T}}\left(\tilde{f}\right)\right)_{klm},\tilde{f}\right)=\sup_{(u,y,z)\in([a,b])^3}d\left(\left(A\mathcal{T}\left(\tilde{f};u,y,z\right)\right)_{klm},\tilde{f}(u,y,z)\right) \tag{3.11}$$

$$=\sup_{(u,y,z)\in([a,b])^3}\sup_{\alpha\in[0,1]}\max\left\{\left|\left\{\left(A\tilde{\mathcal{T}}\left(\tilde{f};u,y,z\right)\right)_{klm}\right\}_\alpha^{-}-\left\{\tilde{f}(u,y,z)\right\}_\alpha^{-}\right|,\right.$$

$$\left.\left|\left\{\left(A\tilde{\mathcal{T}}\left(\tilde{f};u,y,z\right)\right)_{klm}\right\}_\alpha^{+}-\left\{\tilde{f}(u,y,z)\right\}_\alpha^{+}\right|\right\}$$

$$=\sup_{(u,y,z)\in([a,b])^3}\sup_{\alpha\in[0,1]}\max\left\{\left|\left(A\mathcal{T}\left(\tilde{f}_\alpha^{-};u,y,z\right)\right)_{klm}-\tilde{f}_\alpha^{-}(u,y,z)\right|,\right.$$

$$\left.\left|\left(A\mathcal{T}\left(\tilde{f}_\alpha^{+};u,y,z\right)\right)_{klm}-\tilde{f}_\alpha^{+}(u,y,z)\right|\right\}$$

$$=\sup_{\alpha\in[0,1]}\max\left\{\left\|\left(A\mathcal{T}\left(\tilde{f}_\alpha^{-}\right)\right)_{klm}-\tilde{f}_\alpha^{-}\right\|,\left\|\left(A\mathcal{T}\left(\tilde{f}_\alpha^{+}\right)\right)_{klm}-\tilde{f}_\alpha^{+}\right\|\right\}.$$

From (3.10) and (3.11), we have

$$d^*\left(\left(A\tilde{\mathcal{T}}\left(\tilde{f}\right)\right)_{klm},\tilde{f}\right)\le\varepsilon+\mathcal{M}_\alpha\sum_{i=0}^{4}\left\|\left(A\mathcal{T}(g_i)\right)_{klm}-g_i\right\|,$$

where $\mathcal{M}_\alpha=\sup_{\alpha\in[0,1]}\max\{\mathcal{M}_\alpha^{-},\mathcal{M}_\alpha^{+}\}$.

For a given $t>0$, choose $\epsilon>0$ such that $t>\epsilon$. Then, let

$$\mathcal{D}_{r,s,t}=\left\{(k,l,m)\in I_{r,s,t}:d^*\left(\left(A\tilde{\mathcal{T}}\left(\tilde{f}\right)\right)_{klm},\tilde{f}\right)\ge\epsilon\right\}$$

and

$$\mathcal{D}_{r,s,t;i}=\left\{(k,l,m)\in I_{r,s,t}:\left\|\left(A\mathcal{T}(g_i)\right)_{klm}-g_i\right\|\ge\frac{t-\epsilon}{3\mathcal{M}_\alpha}\right\},$$

where $i=\overline{0,4}$ and $(r,s,t)\in\mathbb{N}^3$. Therefore, $\mathcal{D}_{r,s,t}\subseteq\cup_{i=0}^{4}\mathcal{D}_{r,s,t;i}$. This implies

$$\left|\left\{(k,l,m)\in I_{r,s,t}:d^*\left(\left(A\tilde{\mathcal{T}}\left(\tilde{f}\right),\tilde{f}\right)\right)_{klm}\ge\epsilon\right\}\right|$$

$$\le\sum_{i=0}^{4}\left|\left\{(k,l,m)\in I_{r,s,t}:\left\|\left(A\mathcal{T}(g_i)\right)_{klm}-g_i\right\|\ge\frac{t-\epsilon}{3\mathcal{M}_\alpha}\right\}\right|$$

and using (3.6) for $\zeta > 0$ we get

$$\left\{(r,s,t) \in \mathbb{N}^3 : \frac{1}{f\left(h_{rst}^{\gamma}\right)} f\left(\left|\left\{(k,l,m) \in I_{r,s,t} : d^*\left(\left(A\tilde{\mathcal{T}}\left(\tilde{f}\right)\right)_{klm}, \tilde{f}\right) \geq \epsilon\right\}\right|\right) \geq \zeta\right\} \tag{3.12}$$
$$\subseteq \left\{(r,s,t) \in \mathbb{N}^3 : \frac{1}{f\left(h_{rst}^{\gamma}\right)} \sum_{i=0}^{4} f\left(\left|\left\{(k,l,m) \in I_{r,s,t} : \left\|\left(A\mathcal{T}\left(g_i\right)\right)_{klm} - g_i\right\| \geq \frac{t-\epsilon}{3\mathcal{M}_\alpha}\right\}\right|\right) \geq \zeta\right\}$$

belongs to $\mathcal{I}_3$. Therefore, we have

$$\left(A^{\mathcal{I}_3^f}\text{-}stat_{\gamma,\theta_3}\right)\text{-}d^*\left(\tilde{\mathcal{T}}_{klm}\left(\tilde{f}\right), \tilde{f}\right) = 0, \ \forall \tilde{f} \in C_{\mathcal{F}}[a,b].$$

□

Example 3.1. *Let $(\tilde{y}_{nop})$ be a triple fuzzy sequence defined by*

$$\tilde{y}_{nop}(t) = \begin{cases} 1, & \text{if } n,o,p \text{ are squares,} \\ 0, & \text{otherwise,} \end{cases} \quad \forall t \in [0,1].$$

Also, consider the matrix $A = (a_{nkolpm})$ defined by

$$a_{nkolpm} = \begin{cases} 1, & \text{if } klm = (nop)^2, \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$(A\tilde{y}(t))_{nop} = \sum_{k=1,l=1,m=1}^{\infty} a_{nkolpm}\tilde{y}_{klm} = \begin{cases} 1, & \text{if } n,o,p \text{ are squares,} \\ 0, & \text{otherwise, } \forall t \in [0,1]. \end{cases}$$

Now, assume $f(x) = x$, we have for $\varepsilon > 0, \zeta > 0$

$$\left\{(r,s,t) \in \mathbb{N}^3 : \frac{1}{h_{rst}^{\alpha}} \left|\left\{(k,l,m) \in I_{r,s,t} : d\left((A\tilde{y})_{klm}, 0\right) \geq \varepsilon\right\}\right| \geq \zeta\right\}$$
$$\subseteq \left\{(r,s,t) \in \mathbb{N}^3 : \frac{1}{h_{rst}^{\alpha}} \geq \zeta\right\} \in \mathcal{I}_3.$$

This implies $(\tilde{y}_{nop})$ is f-lacunary A-statistical convergent to 0 but it is not convergent to 0. Let $\tilde{y} \in C_{\mathcal{F}}([0,1])^3$, $(a,b,c) \in ([0,1])^3$ and consider the fuzzy Bernstein operators

$$\tilde{\mathcal{B}}_{nop}(\tilde{y};a,b,c)$$
$$= \bigoplus_{i=0,j=0,k=0}^{n,o,p} \binom{n}{i}\binom{o}{j}\binom{p}{k} a^i(1-a)^{n-i} b^j(1-b)^{o-j} c^k(1-c)^{p-k} \odot \tilde{y}\left(\frac{i}{n},\frac{j}{o},\frac{k}{p}\right).$$

This implies

$$\left\{\tilde{\mathcal{B}}_{nop}\left(\tilde{y};a,b,c\right)\right\}_{\alpha}^{\pm}$$
$$=\mathcal{B}_{nop}\left(\tilde{y}_{\alpha}^{\pm};a,b,c\right)$$
$$=\sum_{i=0,j=0,k=0}^{n,o,p}\binom{n}{i}\binom{o}{j}\binom{p}{k}a^{i}\left(1-a\right)^{n-i}b^{j}\left(1-b\right)^{o-j}c^{k}\left(1-c\right)^{p-k}\tilde{y}_{\alpha}^{\pm}\left(\frac{i}{n},\frac{j}{o},\frac{k}{p}\right),$$

where $\tilde{y}_{\alpha}^{\pm}\in C\left([0,1]\right)^{3}$ *and* $\alpha\in[0,1]$. *We define the sequence of fuzzy positive linear operators on* $C_{\mathcal{F}}\left([0,1]\right)^{3}$ *as follows:*

$$A\tilde{\mathcal{T}}_{nop}\left(\tilde{y}\left(x\right);a,b,c\right)=\left(\left(A\tilde{y}\right)_{nop}+1\right)\odot\tilde{\mathcal{B}}_{nop}\left(\tilde{y};a,b,c\right),$$

using these polynomials. Currently,

$$A\mathcal{T}_{nop}\left(\tilde{y}_{\alpha}^{\pm};a,b,c\right)$$
$$=\left(\left(A\tilde{y}\right)_{nop}+1\right)\sum_{i=0,j=0,k=0}^{n,o,p}\binom{n}{i}\binom{o}{j}\binom{p}{k}a^{i}\left(1-a\right)^{n-i}b^{j}\left(1-b\right)^{o-j}c^{k}\left(1-c\right)^{p-k}\tilde{y}_{\alpha}^{\pm}\left(\frac{i}{n},\frac{j}{o},\frac{k}{p}\right). \tag{3.13}$$

Then, we calculate

$$\left(A\mathcal{T}\left(g_{0};u,y,z\right)\right)_{nop}=\left(\left(A\tilde{y}\right)_{nop}+1\right)g_{0}\left(u,y,z\right),$$
$$\left(A\mathcal{T}\left(g_{1};u,y,z\right)\right)_{nop}=\left(\left(A\tilde{y}\right)_{nop}+1\right)g_{1}\left(u,y,z\right),$$
$$\left(A\mathcal{T}\left(g_{2};u,y,z\right)\right)_{nop}=\left(\left(A\tilde{y}\right)_{nop}+1\right)g_{2}\left(u,y,z\right),$$
$$\left(A\mathcal{T}\left(g_{3};u,y,z\right)\right)_{nop}=\left(\left(A\tilde{y}\right)_{nop}+1\right)g_{3}\left(u,y,z\right),$$
$$\left(A\mathcal{T}\left(g_{4};u,y,z\right)\right)_{nop}=\left(\left(A\tilde{y}\right)_{nop}+1\right)\left(g_{4}\left(u,y,z\right)+u^{2}+y^{2}+z^{2}+\frac{u-u^{2}}{n}+\frac{y-y^{2}}{o}+\frac{z-z^{2}}{p}\right).$$

Since $\left(A^{\mathcal{I}_3^f}\text{-}stat_{\gamma,\theta_3}\right)$-$\lim_{nop}\tilde{y}_{nop}=0$, *we conclude thatwhere* $g_0=1, g_1=u, g_2=y, g_3=z, g_4=u^2+y^2+z^2$.

$$\left(A^{\mathcal{I}_3^f}\text{-}stat_{\gamma,\theta_3}\right)-\lim_{nop}\left(A\mathcal{T}\left(g_{0};u,y,z\right)\right)_{nop}=1,$$
$$\left(A^{\mathcal{I}_3^f}\text{-}stat_{\gamma,\theta_3}\right)-\lim_{nop}\left(A\mathcal{T}\left(g_{1};u,y,z\right)\right)_{nop}=u,$$
$$\left(A^{\mathcal{I}_3^f}\text{-}stat_{\gamma,\theta_3}\right)-\lim_{nop}\left(A\mathcal{T}\left(g_{2};u,y,z\right)\right)_{nop}=y,$$
$$\left(A^{\mathcal{I}_3^f}\text{-}stat_{\gamma,\theta_3}\right)-\lim_{nop}\left(A\mathcal{T}\left(g_{3};u,y,z\right)\right)_{nop}=z,$$
$$\left(A^{\mathcal{I}_3^f}\text{-}stat_{\gamma,\theta_3}\right)-\lim_{nop}\left(A\mathcal{T}\left(g_{3};u,y,z\right)\right)_{nop}=u^{2}+y^{2}+z^{2}.$$

So, by using Theorem 3.5, we have

$$\left(A^{\mathcal{I}_3^f}\text{-}stat_{\gamma,\theta_3}\right)-\lim_{nop}d^{*}\left(\left(A\tilde{\mathcal{T}}\left(\tilde{y}\right)\right)_{nop},\tilde{y}\right)=0$$

However, since $(\tilde{y}_{nop})$ is not convergent, Theorem 3.4 does not work for operator defined by (3.13). This demonstrates the superiority of our Theorem 3.5 over Theorem 3.4.

4. Conclusion

In this study, we explore the concepts of strongly f-lacunary A-summability of order γ and f-lacunary $A^{\mathcal{I}_3}$-statistical convergence of order γ for sequences of fuzzy numbers. We also establish that for $0 < \gamma \leq 1$, the f-lacunary $A^{\mathcal{I}_3}$-statistical convergence of order γ is well-defined. Moreover, we investigate the relationships between newly defined spaces and show that, under certain conditions, these spaces are interconnected. As a significant application, we prove a fuzzy Korovkin-type theorem and provide an example that highlights the advantages of our result over the classical version. By utilizing f-lacunary $A^{\mathcal{I}_3}$-statistical convergence, this paper offers a new perspective on the fuzzy Korovkin-type approximation theorem. Further exploration is needed to fully understand these concepts and the results pertaining to double sequences of fuzzy numbers.

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