International Journal of Maps in Mathematics

*Volume **8**, Issue **1**, 2025, Pages:**274-296***

E-ISSN: 2636-7467

www.simadp.com/journalmim

SHARP INEQUALITIES FOR QUASI HEMI-SLANT RIEMANNIAN SUBMERSIONS ($\mathcal{QHSS}$)

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Abstract. The purpose of this article, we obtain sharp inequalities involving the Ricci curvature and the scalar curvature on the horizontal and the vertical distributions for quasi-hemi-slant Riemannian submersions (briefly, $\mathcal{QHSS}$)) from complex space forms onto Riemannian manifolds and debate the equivalence posture the acquired inequality. Lastly, we adduce some examples for $\mathcal{QHSS}$.

Keywords: Riemannian submersion, quasi hemi-slant Riemannian submersion, Chen inequality, complex space form, vertical distribution

2010 Mathematics Subject Classification: Primary 53C15, 53B20.

1. Introduction

In 1990, the notion of slant submanifolds of almost Hermitian manifolds was introduced by [8]. It was a natural generalization of both holomorphic and real submanifolds. Inspired by this notion, several geometers have worked on several types of slant submanifolds (see: [1], [5], [6], [7], [27], [32], [33], [34], [35], [36], [38], [44], [45]).

In the 1960s, O'Neills [53] and Gray [20] studied separately Riemannian submersions. In 1976, Watson studied almost complex types of Riemannian submersions [54] and this invention revealed Hermitian submersions between almost Hermitian manifolds. After these

Received:2024.04.19 *Revised:2024.06.25* *Accepted:2024.12.16*

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studies, Şahin [47] introduced the semi-invariant submersions from almost Hermitian manifolds onto Riemannian manifolds that it was a generalization of holomorphic submersions and anti-invariant submersions [46] and slant submersions from almost Hermitian manifolds onto arbitrary Riemannian manifolds in 2013 [48]. Subsequently, different kinds of structures have been studied in several types of Riemannian submersions(see: [3], [16], [18], [22], [23], [25], [49]). Prasad, Shukla, and Kumar, as a natural generalization of hemi-slant submersions, semi-slant submersions, and bi-slant submersions, identified the notion of quasi bi-slant submersions from Kaehler manifold onto a Riemannian manifold [37]. Longwap, Massamba, and Homti [28]. introduced $\mathcal{QHSS}$ as a generalization of slant, semi-slant, and hemi-slant Riemannian submersions in 2019. On the contrary, Chen established Chen inequalities [9], [10], [11], [14] which as a solution "one of the basic problems in submanifold theory finds simple relationships between the extrinsic and intrinsic invariants of a submanifold". According to Chen [13], a generalization of this inequality was proved arbitrary submanifolds of an arbitrary Riemannian manifold in 2005. Subsequently, several authors investigated Chen-Ricci inequality of submersions and submanifolds (see: [2], [4], [15], [17], [19], [21], [24], [29], [30], [31], [39], [40], [41], [42], [43], [50], [51], [52], [55]). The main purpose of this article acquire some inequalities bearing Ricci curvatures and running Chen-Ricci inequality on the horizontal and the vertical distributions for $\mathcal{QHSS}$ from complex space forms onto Riemannian manifolds.

This article is organized as follows; in Section 2, we recall respectively some basic geometric properties of Riemannian submersions, O'Neill tensors, curvature relations, complex space form, and $\mathcal{QHSS}$. In Section 3, we attain Chen-Ricci inequalities on the horizontal the vertical distributions for $\mathcal{QHSS}$ from complex space forms onto Riemannian manifolds and dispute the equivalence case of the acquired inequality. Eventually, we ensure some examples for $\mathcal{QHSS}$.

2. Quasi hemi-slant Riemannian submersions($\mathcal{QHSS}$)

In this working, unless stated otherwise, all concepts such as manifolds, maps and so on, expressed will be considered differentiable. First let's give the following description.

Definition 2.1. *Let (M_1, g_1) and (M_2, g_2) be Riemannian manifolds, where $dim(M_1)$ is greater than $dim(M_2)$. A surjective mapping $\varphi : (M_1, g_1) \to (M_2, g_2)$ is called a Riemannian submersion if* *(i)* φ *has maximal rank, and*

(ii) φ_**, restricted to* $ker\varphi_*^{\perp}$ *is a linear isometry* [53].

Describe the O'Neill's tensors $\mathcal{T}$ and $\mathcal{A}$ by [53]:

$$\mathcal{T}_\xi \eta = \mathcal{V}\nabla_{\mathcal{V}\xi}\mathcal{H}\eta + \mathcal{H}\nabla_{\mathcal{V}\xi}\mathcal{V}\eta, \tag{2.1}$$

$$\mathcal{A}_\xi \eta = \mathcal{V}\nabla_{\mathcal{H}\xi}\mathcal{H}\eta + \mathcal{H}\nabla_{\mathcal{H}\xi}\mathcal{V}\eta \tag{2.2}$$

for any vector fields $\xi, \eta \in \Gamma(M_1)$, where ∇ is the Levi-Civita connection of g_1. Moreover, from (2.1) and (2.2), we have

$$\nabla_{V_1} V_2 = \mathcal{T}_{V_1} V_2 + \hat{\nabla}_{V_1} V_2, \tag{2.3}$$

$$\nabla_{V_1} X_1 = \mathcal{T}_{V_1} X_1 + \mathcal{H}\nabla_{V_1} X_1, \tag{2.4}$$

$$\nabla_{X_1} V_1 = \mathcal{A}_{X_1} V_1 + \mathcal{V}\nabla_{X_1} V_1, \tag{2.5}$$

$$\nabla_{X_1} X_2 = \mathcal{H}\nabla_{X_1} X_2 + \mathcal{A}_{X_1} X_2, \tag{2.6}$$

for $V_1, V_2 \in \Gamma(ker\varphi_*)$ and $X_1, X_2 \in \Gamma((ker\varphi_*)^{\perp})$ where $\hat{\nabla}_{V_1} V_2 = \mathcal{V}\nabla_{V_1} V_2$. It is not difficult to observe that $\mathcal{T}$ acts on the fibers as the second fundamental form, while $\mathcal{A}$ acts on the horizontal distribution and measures the obstruction to the integrability of this distribution [53].

Specify by R_1, R_2, R_3 and R_4 the Riemannian curvature tensor of Riemannian manifolds M_1, M_2, the vertical distribution $ker\varphi_*$ and the horizontal distribution $(ker\varphi_*)^{\perp}$, seriatim. Then the Gauss-Codazzi type equivalences are dedicated by

$$R_1(U_1, U_2, V_1, V_2) = R_3(U_1, U_2, V_1, V_2) + g_1(\mathcal{T}_{U_1} V_2, \mathcal{T}_{U_2} V_1) - g_1(\mathcal{T}_{U_2} V_2, \mathcal{T}_{U_1} V_1) \tag{2.7}$$

$$\begin{aligned} R_1(X_1, X_2, Y_1, Y_2) &= R_4(X_1, X_2, Y_1, Y_2) - 2g_1(\mathcal{A}_{X_1} X_2, \mathcal{A}_{Y_1} Y_2), \\ &+ g_1(\mathcal{A}_{X_2} Y_1, \mathcal{A}_{X_1} Y_2) - g_1(\mathcal{A}_{X_1} Y_1, \mathcal{A}_{X_2} Y_2), \end{aligned} \tag{2.8}$$

$$\begin{aligned} R_1(X_1, V_1, Y_1, U_1) &= g_1((\nabla_{X_1}\mathcal{T})(V_1, U_1), Y_1) + g_1((\nabla_{V_1}\mathcal{A})(X_1, Y_1), U_1), \\ &- g_1(\mathcal{T}_{V_1} X_1, \mathcal{T}_{U_1} Y_1) + g_1(\mathcal{A}_{Y_1} U_1, \mathcal{A}_{X_1} V_1), \end{aligned} \tag{2.9}$$

where

$$\varphi_*(R_4(X_1, X_2)Y_1)) = R_2(\varphi_* X_1, \varphi_* X_2)\varphi_* Y_1 \tag{2.10}$$

for all $U_1, U_2, V_1, V_2 \in \Gamma(ker\varphi_*)$ and $X_1, X_2, Y_1, Y_2 \in \Gamma(ker\varphi_*)^{\perp}$ [53].

Conversely, the mean curvature vector field $\acute{H}$ of any fibre of Riemannian submersion φ is dedicated by

$$\grave{N} = t\acute{H}, \grave{N} = \sum_{j=1}^{t} \mathcal{T}_{V_j} V_j \tag{2.11}$$

where $\{V_1, ..., V_t\}$ is an orthonormal basis of the vertical distribution $\mathcal{V}$. Additionally, φ has totally geodesic fibers if $\mathcal{T}$ vanishes on $ker\varphi_*$ and $(ker\varphi_*)^\perp$ [53].

Let M_1 be an almost Hermitian manifold with an almost complex structure J_1 and a Hermitian metric g_1. If J_1 is parallel as far as concerns the Levi-Civita connection ∇ on M_1, that mean

$$(\nabla_{X_1} J_1) X_2 = 0$$

for all $X_1, X_2 \in \Gamma(TM_1)$, then (M_1, J_1, g_1, ∇) is yclepted a Kaehler manifold. A Kaehler manifold M_1 is named a complex space form if it has fixed holomorphic sectional curvature represented by $M_1(c_1)$. The curvature tensor of the complex space form $M_1(c)$ is dedicated by

$$\begin{aligned} R_5(X_1, X_2)Y_1 = \frac{c_1}{4}\{&g_1(X_2, Y_1)X_1 - g_1(X_1, Y_1)X_2 + g_1(J_1X_2, Y_1)J_1X_1 \\ &- g_1(J_1X_1, Y_1)J_1X_2 + 2g_1(X_1, J_1X_2)J_1Y_1\} \end{aligned} \tag{2.12}$$

for any $X_1, X_2, Y_1 \in \Gamma(TM_1)$.

Definition 2.2. *Let (M_1, g_1, J_1) be an almost Hermitian manifold and (M_2, g_2) be a Riemannian manifold. A Riemannian submersion $\varphi : (M_1, g_1, J_1) \to (M_2, g_2)$ is called a $\mathcal{QHSS}$ if there exist three mutually orthogonal distribution $\mathcal{D}$, $\mathcal{D}^\perp$ and $\mathcal{D}^\theta$ such that*
(i)$ker\varphi_* = \mathcal{D} \oplus \mathcal{D}^\perp \oplus \mathcal{D}^\theta$,
(ii)$J_1(\mathcal{D}) = \mathcal{D}$, $J_1\mathcal{D}^\perp \subseteq (ker\varphi_*)^\perp$
(iii) for any non-zero vector field $Z_1 \in \Gamma(\mathcal{D}^\theta_p)$, $p \in M_1$ the angle θ between $J_1(Z_1)$ and $\mathcal{D}^\theta_p$ is constant and independent of the choice of point p and Z_1 in $\mathcal{D}^\theta_p$ [28].

We name the angle θ a quasi hemi-slant angle. In this article, we will presume all horizontal vector fields as basic vector fields.

Let $\varphi : (M_1, g_1, J_1) \to (M_2, g_2)$ be a $\mathcal{QHSS}$, at present. Then [28], we have for all $V \in \Gamma(ker\varphi_*)$, we get

$$J_1V_1 = \psi V_1 + \omega V_1 \tag{2.13}$$

where $\psi V \in \Gamma(ker\varphi_*)$ and $\omega V_1 \in \Gamma(\omega\mathcal{D}^\theta \oplus \omega\mathcal{D}^\perp)$. For any $X_1 \in \Gamma((ker\varphi_*)^\perp)$, we get

$$J_1 X_1 = \mathcal{B}_1 X_1 + \mathcal{B}_2 X_1 \tag{2.14}$$

where $\mathcal{B}_1 X_1 \in \Gamma(ker\varphi_*)$ and $\mathcal{B}_2 X_1 \in \Gamma(\mathcal{V})$.

Theorem 2.1. [28] *Let M_1 be a 2m-dimensional almost Hermitian manifold with g_1 a Riemannian metric on M_1 and almost complex structure J_1, and M_2 be a Riemannian manifold with Riemannian metric g_2. Then there is a Riemannian submersion $\varphi : (M_1, g_1, J_1) \to (M_2, g_2)$ such that its vertical distribution* $\ker\varphi_*$ *admits three orthogonal distributions $\mathcal{D}, \mathcal{D}^\theta$ and $\mathcal{D}^\perp$ which are invariant, slant and anti-invariant respectively, i.e.*

$$\ker\varphi_* = \mathcal{D} \oplus \mathcal{D}^\perp \oplus \mathcal{D}^\theta,$$

with $J_1\mathcal{D} = \mathcal{D}$, the angle θ between $J_1\mathcal{D}^\theta$ and $\mathcal{D}^\theta$ being constant and $J_1\mathcal{D}^\perp \subseteq (\ker\varphi_*)^\perp$. If we denote the dimension of $\mathcal{D}, \mathcal{D}^\theta$ and $\mathcal{D}^\perp$ by m_1, m_2 and m_3, respectively, then we easily see the following particular cases:

(1) *If $m_1 = 0$, then M_1 is a hemi-slant submersion.*
(2) *If $m_2 = 0$, then M_1 is a semi-invariant submersion.*
(3) *If $m_3 = 0$, then M_1 is a semi-slant submersion.*

The submersion in Theorem 2.1 will be called $\mathcal{QHSS}$ and the angle θ is called the quasi hemi-slant angle of the submersion. This means that a $\mathcal{QHSS}$ is a generalization of hemi-slant, semi-invariant and semi-slant submersions.

We say that the $\mathcal{QHSS}$ $\varphi : (M_1, g_1, J_1) \to (M_2, g_2)$ is proper if $\mathcal{D} \neq \{0\}$, $\mathcal{D}^\perp \neq \{0\}$ and $\theta \neq 0, \frac{\pi}{2}$. From the above items, hemi-slant submersions, semi-invariant submersions, and semi-slant submersions are all examples of $\mathcal{QHSS}$. The undermentioned theorem is a characterization for $\mathcal{QHSS}$ of a complex space form. The proof of it completely identical with slant immersions see:[28]. Hence we omit its substantiation.

Theorem 2.2. [28] *Let φ be a Riemannian submersion from a complex manifold (M_1, g_1, J_1) onto a Riemannian manifold (M_2, g_2). Then, φ is a $\mathcal{QHSS}$ if and only if there exists a constant $\lambda \in [0, 1]$ such that*

$$\phi^2 U_1 = -\lambda U_1.$$

where $U_1 \in \Gamma(D^\theta)$. Furthermore, in such a case, if θ is the slant angle of φ, it satisfies that $\lambda = \cos^2\theta$.

Lemma 2.1. *Let $(M_1(c_1), g_1)$, (M_2, g_2) be a complex space form and a Riemannian manifold, seriatim and $\varphi : M_1(c_1) \to M_2$ a $\mathcal{QHSS}$. Then the undermentioned relations are current,*

$$g_1(\phi U_1, \phi V_1) = \cos^2\theta g_1(U_1, V_1),$$
$$g_1(\omega U_1, \omega V_1) = \sin^2\theta g_1(U_1, V_1),$$

for any $U_1, V_1 \in \Gamma\,(ker\varphi_*)$ [28].

Lemma 2.2. *If φ is a $\mathcal{QHSS}$ then we have*
i) $\phi^2 V = -(\cos^2\theta)V$,
ii) $g_1(\phi V_1, \phi V_2) = \cos^2\theta g_1(V_1, V_2)$,
iii) $g_1(\omega V_1, \omega V_2) = \sin^2\theta g_1(V_1, V_2)$
for all $V_1, V_2 \in \Gamma(D^\theta)$ [28].

3. Chen-Ricci inequality and Chen inequalities

Let $(M_1(c_1), g_1)$, (M_2, g_2) be a complex space form and a Riemannian manifold, seriatim and $\varphi : M_1(c_1) \to M_2$ a $\mathcal{QHSS}$. Additionally, let $\{V_1, ..., V_t, Y_1, ..., Y_n\}$ be an orthonormal basis of $T_pM_1(c_1)$ such that $\mathcal{V} = Sp\{V_1, ..., V_t\}$, $\mathcal{H} = Sp\{Y_1, ..., Y_n\}$ and $t = 2t_1 + 2t_2 + t_3$, where $dim\mathcal{D} = 2t_1$, $dim\mathcal{D}^\theta = 2t_2$ and $dim\mathcal{D}^\perp = t_3$. Then we may consider an adapted quasi hemi-slant orthonormal frames as follows:

$$V_1, V_2 = J_1V_1, ..., V_{2t_1-1}, V_{2t_1} = J_1V_{2t_1-1}, V_{2t_1+1},$$

$$V_{2t_1+2} = \sec\theta\psi V_{2t_1+1}, , ..., V_{2t_1+2t_2-1}, V_{2t_1+2r_2} = \sec\theta\psi V_{2t_1+2t_2-1},$$

$$V_{2t_1+2t_2+1}, ..., V_{2t_1+2t_2+t_3}.$$

Obviously, we obtain

$$g_1^2(J_1V_i, V_{i+1}) = \begin{cases} 1, & \text{for } i \in \{1, ..., 2t_1 - 1\}, \\ \cos^2\theta, & \text{for } i \in \{1, ..., 2t_1 + 2t_2 - 1\}, \\ 0, & \text{for } i \in \{2t_1 + 2t_2 + 1, ..., 2t_1 + 2t_2 + t_3 - 1\}, \end{cases}$$

then

$$\sum_{i,j=1}^{t} g_1^2(J_1V_i, V_j) = 2(t_1 + t_2\cos^2\theta).$$

Furthermore, let $\{V_1, ..., V_t, Y_1, ..., Y_n\}$ be an orthonormal basis of $T_pM_1(c_1)$ such that $\mathcal{V} = Sp\{V_1, ..., V_t\}$, $\mathcal{H} = Sp\{Y_1, ..., Y_n\}$. Then Ric_1 and Ric_2 are dedicated by

$$Ric_1(V_1) = \sum_{i=1}^{t} R_3(V_1, V_i, V_i, V_1), \tag{3.15}$$

$$Ric_2(Y_1) = \sum_{s=1}^{n} R_2(Y_1, Y_j, Y_j, Y_1). \tag{3.16}$$

Furthermore, scalar curvature τ_1 and τ_2 are defined

$$\tau_1 = \sum_{1 \le i < j \le t} R_3(V_i, V_j, V_j, V_i), \tag{3.17}$$

$$\tau_2 = \sum_{1 \le i < j \le n} R_4(Y_i, Y_j, Y_j, Y_i). \tag{3.18}$$

Moreover, utilizing (2.7), (2.8) and (2.12), we get

$$\begin{aligned} R_3(V_1, V_2, V_3, V_4) = \frac{c_1}{4}\{&g_1(V_2, V_3)g_1(V_1, V_4) - g_1(V_1, V_3)g_1(V_2, V_4) + g_1(J_1V_2, V_3)g_1(J_1V_1, V_4) \\ &- g_1(J_1V_1, V_3)g_1(J_1V_2, V_4) + 2g_1(V_1, J_1V_2)g_1(J_1V_3, V_4)\} \\ &- g_1(\mathcal{T}_{V_1}V_4, \mathcal{T}_{V_2}V_3) + g_1(\mathcal{T}_{V_2}V_4, \mathcal{T}_{V_1}V_3), \end{aligned} \tag{3.19}$$

$$\begin{aligned} R_4(Y_1, Y_2, Y_3, Y_4) = \frac{c_1}{4}\{&g_1(Y_2, Y_3)g_1(Y_1, Y_4) - g_1(Y_1, Y_3)g_1(Y_2, Y_4) + g_1(J_1Y_2, Y_3)g_1(J_1Y_1, Y_4) \\ &- g_1(J_1Y_1, Y_3)g_1(J_1Y_2, Y_4) + 2g_1(Y_1, J_1Y_2)g_1(J_1Y_3, Y_4)\} \\ &+ 2g_1(\mathcal{A}_{Y_1}Y_2, \mathcal{A}_{Y_3}Y_4) - g_1(\mathcal{A}_{Y_2}Y_3, \mathcal{A}_{Y_1}Y_4) + g_1(\mathcal{A}_{Y_1}Y_3, \mathcal{A}_{Y_2}Y_4). \end{aligned} \tag{3.20}$$

Theorem 3.1. *Let $\varphi : M_1(c_1) \to M_2$ be a $\mathcal{QHSS}$ from a complex space form $(M_1(c_1), g_1)$ onto a Riemannian manifold (M_2, g_2). In that case, the undermentioned expressions are actual.*

i) If $V_1 \in \Gamma(\mathcal{D})$, in that case

$$Ric_1(V_1) \ge \frac{c_1}{4}(t+2) - tg_1(\mathcal{T}_{V_1}V_1, \acute{H}), \tag{3.21}$$

In case of (3.21)' equality holds for a unit vertical vector $V_1 \in \Gamma(\mathcal{D})$ if and only if each fiber is totally geodesic.

ii) If $V_1 \in \Gamma(\mathcal{D}^\theta)$, in that case

$$Ric_1(V_1) \ge \frac{c_1}{4}(t - 1 + 3\cos^2\theta) - tg_1(\mathcal{T}_{V_1}V_1, \acute{H}), \tag{3.22}$$

In case of (3.22)'s equality holds for a unit vertical vector $V_1 \in \Gamma(\mathcal{D}^\theta)$ if and only if each fiber is totally geodesic.

iii) If $V_1 \in \Gamma(\mathcal{D}^\perp)$, in that case

$$Ric_1(V_1) \ge \frac{c_1}{4}(t-1) - tg_1(\mathcal{T}_{V_1}V_1, \acute{H}), \tag{3.23}$$

In case of (3.23)'s equality holds for a unit vertical vector $V_1 \in \Gamma(\mathcal{D}^{\perp})$ if and only if each fiber is totally geodesic.

Proof. Using (3.15) and (3.19) we have,

$$Ric_1(V_1) = \frac{c_1}{4}(t - 1 + 3\sum_{i=1}^{t} g_1^2(J_1V_1, V_i)) - tg_1(\mathcal{T}_{V_1}V_1, \acute{H}) + \|\mathcal{T}_{V_1}V_i\|^2 . \quad (3.24)$$

In that case we have

$$\sum_{i=1}^{t} g_1^2(J_1V_1, V_i) = \begin{cases} 1, & \text{if } V_1 \in \Gamma(\mathcal{D}) \\ \cos^2\theta, & \text{if } V_1 \in \Gamma(\mathcal{D}^{\theta}) \\ 0, & \text{if } V_1 \in \Gamma(\mathcal{D}^{\perp}). \end{cases}$$

Using last equivalence in (3.24), we get (3.21), (3.22) and (3.23). □

Theorem 3.2. *Let $\varphi : M_1(c_1) \to M_2$ be a $\mathcal{QHSS}$ from a complex space form $(M_1(c_1), g_1)$ onto a Riemannian manifold (M_2, g_2). In that case*

$$2\tau_1 \geq \frac{c_1}{4}\{t(t-1) + 6(t_1 + t_2\cos^2\theta)\} - t^2 \left\|\acute{H}\right\|^2 . \quad (3.25)$$

The equivalence case of (3.25) holds if and only if each fiber is totally geodesic.

Proof. From (3.17) and (3.19) we have:

$$2\tau_1 = \frac{c_1}{4}(t(t-1) + 6(t_1 + t_2\cos^2\theta)) - t^2 \left\|\acute{H}\right\|^2 + \sum_{i,j=1}^{t} g_1(\mathcal{T}_{V_i}V_j, \mathcal{T}_{V_i}V_j). \quad (3.26)$$

Here we have use $\mathcal{T}$ is a symmetric operator. Hence from (3.26) the proof is completed. □

Since φ is $\mathcal{QHSS}$ and $\mathcal{A}$ is an anti-symmetric operator, from (3.18) and (3.20) we have

$$2\tau_2 = \frac{c_1}{4}(n(n-1) + 3\sum_{i,j=1}^{n} g_1(\mathcal{B}_2Y_i, Y_j)g_1(\mathcal{B}_2Y_i, Y_j)) - 3\sum_{i,j=1}^{n} g_1(\mathcal{A}_{Y_i}Y_j, \mathcal{A}_{Y_i}Y_j). \quad (3.27)$$

If we portray

$$\|\mathcal{B}_2\|^2 = \sum_{i=1}^{n} g_1^2(\mathcal{B}_2Y_i, Y_j), \quad (3.28)$$

In that case from (3.27) and (3.28) we get

$$2\tau_2 = \frac{c_1}{4}(n(n-1) + 3\|\mathcal{B}_2\|^2) - 3\sum_{i,j=1}^{n} g_1(\mathcal{A}_{Y_i}Y_j, \mathcal{A}_{Y_i}Y_j). \quad (3.29)$$

From (3.29) we have:

Theorem 3.3. *Let $\varphi : M_1(c_1) \to M_2$ be a $\mathcal{QHSS}$ from a complex space form $(M_1(c_1), g_1)$ onto a Riemannian manifold (M_2, g_2). In that case*

$$2\tau_2 \leq \frac{c_1}{4}(n(n-1) + 3\|\mathcal{B}_2\|^2). \tag{3.30}$$

In case of (3.30)'s equality holds if and only if $\mathcal{H}(M_1)$ is integrable.

Let $(M_1(c_1), g_1)$ be a complex space form and (M_2, g_2) a Riemannian manifold. Assume that $\varphi : M_1(c_1) \to M_2$ is a $\mathcal{QHSS}$ and $\{V_1, ..., V_t, Y_1, ..., Y_n\}$ is an orthonormal basis of $TpM_1(c_1)$ such that $\mathcal{V}p(M_1) = Sp\{V_1, ..., V_t\}$, $\mathcal{H}p(M_1) = Sp\{Y_1, ..., Y_n\}$. Now, if we denote $\mathcal{T}_{ij}^s$ by

$$\mathcal{T}_{ij}^s = g_1(\mathcal{T}_{V_i} V_j, Y_s), \tag{3.31}$$

where $1 \leq i, j \leq t$ and $1 \leq s \leq n$ (see [17]). The same, if we denote $\mathcal{A}_{ij}^\alpha$ by

$$\mathcal{A}_{ij}^\alpha = g_1(\mathcal{A}_{Y_i} Y_j, V_\alpha), \tag{3.32}$$

where $1 \leq i, j \leq n$ and $1 \leq \alpha \leq t$. From [17], we use

$$\delta(\grave{N}) = \sum_{i=1}^{n} \sum_{k=1}^{t} g_1((\nabla_{Y_i} \mathcal{T})_{V_k} V_k, Y_i). \tag{3.33}$$

$$\sum_{s=1}^{n} \sum_{i,j=1}^{t} (\mathcal{T}_{ij}^s)^2 = \frac{1}{2} t^2 \left\| \acute{H} \right\|^2 + \frac{1}{2}(\mathcal{T}_{11}^s - \mathcal{T}_{22}^s - ... - \mathcal{T}_{tt}^s)^2$$
$$+ 2\sum_{s=1}^{n} \sum_{j=2}^{t} (\mathcal{T}_{1j}^s)^2 - 2\sum_{s=1}^{n} \sum_{2 \leq i < j \leq t} (\mathcal{T}_{ii}^s \mathcal{T}_{jj}^s - (\mathcal{T}_{ij}^s)^2). \tag{3.34}$$

The above equations, the Binomial theorem we have like equivalence between the tensor fields $\mathcal{T}$:

Theorem 3.4. *Let $\varphi : M_1(c_1) \to M_2$ be a $\mathcal{QHSS}$ from a complex space form $(M_1(c_1), g_1)$ onto a Riemannian manifold (M_2, g_2). In that case, the undermentioned statements are actual.*

i) If $V \in \Gamma(\mathcal{D})$, in that case

$$Ric_1(V) \geq \frac{c_1}{4}(t+2) - \frac{1}{4} t^2 \left\| \acute{H} \right\|^2. \tag{3.35}$$

ii) If $V \in \Gamma(\mathcal{D}^\theta)$, in that case

$$Ric_1(V) \geq \frac{c_1}{4}(t - 1 + 3\cos^2\theta) - \frac{1}{4} t^2 \left\| \acute{H} \right\|^2. \tag{3.36}$$

iii) If $V \in \Gamma(\mathcal{D}^{\perp})$, in that case

$$Ric_1(V) \geq \frac{c_1}{4}(t-1) - \frac{1}{4}t^2 \left\| \acute{H} \right\|^2. \tag{3.37}$$

In case of (3.35)'s, (3.36)'s and (3.37)'s equalities hold if and only if

$$\mathcal{T}^s_{11} = \mathcal{T}^s_{22} + ... + \mathcal{T}^s_{tt},$$

$$\mathcal{T}^s_{1j} = 0, \qquad j = 2, ..., r.$$

Proof. Let $\{V_1, ..., V_{2t_1}, V_{2t_1+1}, V_{2t_1+2}, ..., V_{2t_1+2t_2-1}, V_{2t_1+2t_2}, V_{2t_1+2t_2+1}, ..., V_{2t_1+2t_2+2t_3-1}, V_{2t_1+2t_2+2t_3}\}$ be an adapted quasi hemi-slant basis of $\mathcal{V}p(M_1)$.

i) Because in this case one can comprehend the concerted quasi hemi-slant basis such that $V_1 = V$, it suffices to prove (3.35) for $V = V_1$. Using (3.31) in (3.26) and the symmetry of $\mathcal{T}$, we get

$$2\tau_1 = \frac{c_1}{4}(t(t-1) + 6(t_1 + t_2\cos^2\theta)) - t^2 \left\| \acute{H} \right\|^2 + \sum_{s=1}^{n}\sum_{i,j=1}^{t}(\mathcal{T}^s_{ij})^2. \tag{3.38}$$

Thus using (3.34) in (3.38) we have

$$2\tau_1 = \frac{c_1}{4}(t(t-1) + 6(t_1 + t_2\cos^2\theta)) - \frac{1}{2}t^2 \left\| \acute{H} \right\|^2 + \frac{1}{2}(\mathcal{T}^s_{11} - \mathcal{T}^s_{22} - ... - \mathcal{T}^s_{tt})^2 + 2\sum_{s=1}^{n}\sum_{j=2}^{t}(\mathcal{T}^s_{1j})^2 - 2\sum_{s=1}^{n}\sum_{2\leq i<j\leq t}(\mathcal{T}^s_{ii}\mathcal{T}^s_{jj} - (\mathcal{T}^s_{ij})^2). \tag{3.39}$$

In that case from (3.39) we get

$$2\tau_1 \geq \frac{c_1}{4}(t(t-1) + 6(t_1 + t_2\cos^2\theta))) - \frac{1}{2}t^2 \left\| \acute{H} \right\|^2 - 2\sum_{s=1}^{n}\sum_{2\leq i<j\leq t}(\mathcal{T}^s_{ii}\mathcal{T}^s_{jj} - (\mathcal{T}^s_{ij})^2). \tag{3.40}$$

In addition to, letting $V_1 = V_2 = V_i$, $V_3 = V_4 = V_j$ in (3.19) and using (3.31), we have

$$2\sum_{2\leq i<j\leq t} R(V_i, V_j, V_j, V_i) = 2\sum_{2\leq i<j\leq t} R_3(V_i, V_j, V_j, V_i) + 2\sum_{s=1}^{n}\sum_{2\leq i<j\leq t}(\mathcal{T}^s_{ii}\mathcal{T}^s_{jj} - (\mathcal{T}^s_{ij})^2). \tag{3.41}$$

From (3.41) in (3.40), we have

$$2\tau_1 \geq \frac{c_1}{4}(t(t-1) + 6(t_1 + t_2\cos^2\theta)) - \frac{1}{2}t^2 \left\| \acute{H} \right\|^2 + 2\sum_{2\leq i<j\leq t} R_3(V_i, V_j, V_j, V_i) - 2\sum_{2\leq i<j\leq t} R(V_i, V_j, V_j, V_i). \tag{3.42}$$

In addition to, we know

$$2\tau_1 = 2\sum_{2\leq i<j\leq t} R_3(V_i, V_j, V_j, V_i) + 2\sum_{j=1}^{t} R_3(V_1, V_j, V_j, V_1). \tag{3.43}$$

Considering (3.43) in (3.42), we obtain

$$2Ric_1(V_1) \geq \frac{c_1}{4}(t(t-1)+6(t_1+t_2\cos^2\theta)) - \frac{1}{2}t^2\left\|\acute{H}\right\|^2 - 2\sum_{2\leq i<j\leq t} R(V_i,V_j,V_j,V_i). \tag{3.44}$$

Since M_1 is a complex space form, its curvature tensor R satisfies the equality (2.12), we have

$$\sum_{2\leq i<j\leq t} R_1(V_i,V_j,V_j,V_i) = \frac{c_1}{4}(\frac{(t-2)(t-1)}{2} + 3\sum_{2\leq i<j\leq t} g_1^2(J_1V_i,V_j)). \tag{3.45}$$

Taking $V_1 \in \Gamma(\mathcal{D})$ in (3.45), we get

$$\sum_{2\leq i<j\leq t} R_1(V_i,V_j,V_j,V_i) = \frac{c_1}{4}(\frac{(t-2)(t-1)}{2} + 3(t_1-1+t_2\cos^2\theta)). \tag{3.46}$$

Using last equation in (3.44) we have (3.35).

ii) Because in this case one can comprehend the concerted semi-slant basis $\{V_1,...,V_{2t_1},V_{2t_1+1},V_{2t_1+2},...,V_{2t_1+2t_2-1},V_{2t_1+2t_2},V_{2t_1+2t_2+1},...,V_{2t_1+2t_2+2t_3-1},V_{2t_1+2t_2+2t_3}\}$ such that $V_{2t_1+1} = V$, it suffices to prove (3.37) for $V = V_{2t_1+1}$.

With like arguments as in case i), we obtain

$$2Ric_1(V_{2t_1+1}) \geq \frac{c_1}{4}(t(t-1)+6(t_1+t_2\cos^2\theta)) - \frac{1}{2}t^2\left\|\acute{H}\right\|^2 - 2\sum_{1\leq k<s\leq t;\ k,s\neq 2t_1+1} R(V_k,V_s,V_s,V_k). \tag{3.47}$$

and

$$\sum_{1\leq k<s\leq t;\ k,s\neq 2t_1+1} R_1(V_k,V_s,V_s,V_k) = \frac{c_1}{4}(\frac{(t-2)(t-1)}{2} + \sum_{1\leq k<s\leq t;\ k,s\neq 2t_1+1} g_1^2(J_1V_k,V_s)). \tag{3.48}$$

As $V_{2t_1+1} \in \Gamma(\mathcal{D}^\theta)$, we acquire immediately

$$\sum_{1\leq k<s\leq t;\ k,s\neq 2t_1+1} g_1^2(J_1V_k,V_s) = t_1 + (t_2-1)\cos^2\theta_2$$

and therefore (3.48) can be written as

$$\sum_{1\leq k<s\leq t;\ k,s\neq 2t_1+1} R_1(V_k,V_s,V_s,V_k) = \frac{c_1}{4}\left[\frac{(t-2)(t-1)}{2} + 3(t_1+(t_2-1)\cos^2\theta)\right]. \tag{3.49}$$

Considering now the last equation in (3.47), we have

$$Ric_1(V_{2t_1+1}) \geq \frac{c_1}{4}(t-1+3\cos^2\theta) - \frac{1}{4}t^2\left\|\acute{H}\right\|^2$$

which implies (3.36).

iii) Because in this case one can comprehend the concerted semi-slant basis $\{V_1, ..., V_{2t_1}, V_{2t_1+1}$ $, V_{2t_1+2}, ..., V_{2t_1+2t_2-1}, V_{2t_1+2t_2}, V_{2t_1+2t_2+1}, ..., V_{2t_1+2t_2+2t_3-1}, V_{2t_1+2t_2+2t_3}\}$ such that $V_{2t_1+2t_2+1} = V$, it suffices to prove (3.37) for $V = V_{2t_1+2t_2+1}$.

With similar arguments as in case i), we obtain

$$2Ric_1(V_{2t_1+2t_2+1}) \geq \frac{c_1}{4}(t(t-1) + 6(t_1 + t_2\cos^2\theta)) - \frac{1}{2}t^2 \left\|\acute{H}\right\|^2 -2 \sum_{1\leq k<s\leq t;\ k,s\neq 2t_1+2t_2+1} R_1(V_k, V_s, V_s, V_k) \tag{3.50}$$

and

$$\sum_{1\leq k<s\leq t;\ k,s\neq 2t_1+2t_2+1} R_1(V_k, V_s, V_s, V_k) = \frac{c_1}{4}(\frac{(t-2)(t-1)}{2} + \sum_{1\leq k<s\leq t;\ k,s\neq 2t_1+2t_2+1} g_1^2(J_1V_k, V_s)). \tag{3.51}$$

As $V_{2t_1+2t_2+1} \in \Gamma(\mathcal{D}^\perp)$, we obtain immediately

$$\sum_{1\leq k<s\leq t;\ k,s\neq 2t_1+2t_2+1} g_1^2(J_1V_k, V_s) = t_1 + t_2\cos^2\theta$$

and therefore (3.51) can be written as

$$\sum_{1\leq k<s\leq t;\ k,s\neq 2t_1+2t_2+1} R_1(V_k, V_s, V_s, V_k) = \frac{c_1}{4}\left[\frac{(t-2)(t-1)}{2} + 3(t_1 + t_2\cos^2\theta)\right]. \tag{3.52}$$

Thinking now the last equation in (3.50), we get

$$Ric_1(V_{2t_1+2t_2+1}) \geq \frac{c_1}{4}(t-1) - \frac{1}{4}t^2 \left\|\acute{H}\right\|^2$$

which implies (3.37). □

Theorem 3.5. *Let $\varphi : M_1(c_1) \to M_2$ be a $\mathcal{QHSS}$ from a complex space form $(M_1(c_1), g_1)$ onto a Riemannian manifold (M_2, g_2). In that case we have*

$$Ric_2(Y_1) \leq \frac{c_1}{4}(n - 1 + 3\|\mathcal{B}_2Y_1\|^2). \tag{3.53}$$

In case of (3.53)'s equality holds if and only if

$$\mathcal{A}_{1j}^\alpha = 0, \quad j = 2, ..., n.$$

Proof. Considering (3.29) and (3.32), we get

$$2\tau_2 = \frac{c_1}{4}(n(n-1) + 3\|\mathcal{B}_2\|^2) - 3\sum_{\alpha=1}^{t}\sum_{i,j=1}^{n}(\mathcal{A}_{ij}^\alpha)^2. \tag{3.54}$$

In that case (3.54) can be written as

$$2\tau_2 = \frac{c_1}{4}(n(n-1) + 3\left\|\mathcal{B}_2\right\|^2) - 6\sum_{\alpha=1}^{t}\sum_{j=2}^{n}(\mathcal{A}_{1j}^{\alpha})^2 - 6\sum_{\alpha=1}^{t}\sum_{2\leq i<j\leq n}(\mathcal{A}_{ij}^{\alpha})^2. \tag{3.55}$$

Besides, letting $X_1 = Y_2 = Y_i,\ X_2 = Y_1 = Y_j$ in (3.20) and considering (3.32), we derive

$$2\sum_{2\leq i<j\leq n} R(Y_i, Y_j, Y_j, Y_i) = 2\sum_{2\leq i<j\leq n} R_4(Y_i, Y_j, Y_j, Y_i) + 6\sum_{\alpha=1}^{t}\sum_{2\leq i<j\leq n}(\mathcal{A}_{ij}^{\alpha})^2. \tag{3.56}$$

Using (3.56) in (3.55), we get

$$\begin{aligned}2\tau_2 = &\frac{c_1}{4}(n(n-1) + 3\left\|\mathcal{B}_2\right\|^2) - 6\sum_{\alpha=1}^{t}\sum_{j=2}^{n}(\mathcal{A}_{1j}^{\alpha})^2 \\ &+ 2\sum_{2\leq i<j\leq n} R_4(Y_i, Y_j, Y_j, Y_i) - 2\sum_{2\leq i<j\leq n} R_1(Y_i, Y_j, Y_j, Y_i).\end{aligned} \tag{3.57}$$

Moreover, using (3.20) we have

$$\sum_{2\leq i<j\leq n} R_1(Y_i, Y_j, Y_j, Y_i) = \frac{c_1}{4}(\frac{(n-2)(n-1)}{2} + 3\sum_{2\leq i<j\leq n} g_1^2(\mathcal{B}_2 Y_i, Y_j)). \tag{3.58}$$

Taking into account that

$$\left\|\mathcal{B}_2\right\|^2 - 2\sum_{2\leq i<j\leq n} g_1^2(\mathcal{B}_2 Y_i, Y_j) = 2\left\|\mathcal{B}_2 Y_1\right\|^2. \tag{3.59}$$

and using (3.57), (3.58) and (3.59), we get

$$2Ric_2(Y_1) = \frac{c_1}{2}(n - 1 + 3\left\|\mathcal{B}_2 Y_1\right\|^2) - 6\sum_{\alpha=1}^{t}\sum_{j=2}^{n}(\mathcal{A}_{1j}^{\alpha})^2. \tag{3.60}$$

Hence the assertion follows. □

Now, we calculate the Chen-Ricci inequality between horizontal and the vertical distributions. For the scalar curvature τ of $M_1(c_1)$, we provide

$$2\tau = \sum_{s=1}^{n} Ric(Y_s, Y_s) + \sum_{k=1}^{t} Ric(V_k, V_k). \tag{3.61}$$

Additionally, we can write

$$\begin{aligned}2\tau = &\sum_{j,k=1}^{t} R_1(V_j, V_k, V_k, V_j) + \sum_{i=1}^{n}\sum_{k=1}^{t} R_1(Y_i, V_k, V_k, Y_i) \\ &+ \sum_{i,s=1}^{n} R_1(Y_i, Y_s, Y_s, Y_i) + \sum_{s=1}^{n}\sum_{j=1}^{t} R_1(V_j, Y_s, Y_s, V_j).\end{aligned} \tag{3.62}$$

Next, let us denote as usual (see [17]):

$$\left\|\mathcal{T}^{\mathcal{V}}\right\|^2=\sum_{i=1}^{n}\sum_{k=1}^{t} g_1(\mathcal{T}_{V_k}Y_i,\mathcal{T}_{V_k}Y_i), \tag{3.63}$$

$$\left\|\mathcal{T}^{\mathcal{H}}\right\|^2=\sum_{k,j=1}^{t} g_1(\mathcal{T}_{V_k}V_j,\mathcal{T}_{V_k}V_j), \tag{3.64}$$

$$\left\|\mathcal{A}^{\mathcal{V}}\right\|^2=\sum_{i,j=1}^{n} g_1(\mathcal{A}_{Y_i}Y_j,\mathcal{A}_{Y_i}Y_j), \tag{3.65}$$

$$\left\|\mathcal{A}^{\mathcal{H}}\right\|^2=\sum_{i=1}^{n}\sum_{k=1}^{t} g_1(\mathcal{A}_{Y_i}V_k,\mathcal{A}_{Y_i}V_k). \tag{3.66}$$

Theorem 3.6. *Let $\varphi: M_1(c_1)\to M_2$ be a $\mathcal{QHSS}$ from a complex space form $(M_1(c_1),g_1)$ onto a Riemannian manifold (M_2,g_2).*

i) If $V_1\in\Gamma(\mathcal{D})$, in that case

$$\frac{c_1}{4}(nt+n+t+3(1+\|\mathcal{B}_1\|^2+\|\mathcal{B}_2Y_1\|^2))\leq Ric_1(V_1)+Ric_2(Y_1)$$
$$+\frac{1}{4}t^2\left\|\acute{H}\right\|^2+3\sum_{\alpha=1}^{t}\sum_{s=2}^{n}(\mathcal{A}^{\alpha}_{1s})^2-\delta(\grave{N})+\left\|\mathcal{T}^{\mathcal{V}}\right\|^2-\left\|\mathcal{A}^{\mathcal{H}}\right\|^2. \tag{3.67}$$

ii) If $V_1\in\Gamma(\mathcal{D}^{\theta})$, in that case

$$\frac{c_1}{4}(nt+n+t+3(\cos^2\theta+\|\mathcal{B}_1\|^2+\|\mathcal{B}_2Y_1\|^2))\leq Ric_1(V_1)+Ric_2(Y_1)$$
$$+\frac{1}{4}t^2\left\|\acute{H}\right\|^2+3\sum_{\alpha=1}^{t}\sum_{s=2}^{n}(\mathcal{A}^{\alpha}_{1s})^2-\delta(\grave{N})+\left\|\mathcal{T}^{\mathcal{V}}\right\|^2-\left\|\mathcal{A}^{\mathcal{H}}\right\|^2. \tag{3.68}$$

iii) If $V_1\in\Gamma(\mathcal{D}^{\perp})$, in that case

$$\frac{c_1}{4}(nt+n+t+3(t_1+t_2\cos^2\theta+\|\mathcal{B}_1\|^2+\|\mathcal{B}_2Y_1\|^2))\leq Ric_1(V_1)+Ric_2(Y_1)$$
$$+\frac{1}{4}t^2\left\|\acute{H}\right\|^2+3\sum_{\alpha=1}^{t}\sum_{s=2}^{n}(\mathcal{A}^{\alpha}_{1s})^2-\delta(\grave{N})+\left\|\mathcal{T}^{\mathcal{V}}\right\|^2-\left\|\mathcal{A}^{\mathcal{H}}\right\|^2. \tag{3.69}$$

In case of (3.67)'s, (3.68)'s and (3.69)'s equalties hold if and only if

$$\mathcal{T}^s_{11}=\mathcal{T}^s_{22}+...+\mathcal{T}^s_{tt},$$
$$\mathcal{T}^s_{1j}=0,\qquad j=2,...,t.$$

Proof. Since $M_1(c_1)$ is a complex space form, from (3.62) we have

$$2\tau=\frac{c_1}{4}[(n+t)(n+t-1)+6(t_1+t_2\cos^2\theta))+3(\|\mathcal{B}_2\|^2+2\sum_{i=1}^{n}\sum_{k=1}^{t}g_1^2(\mathcal{B}_1Y_i,V_k))]. \tag{3.70}$$

Now, we define

$$\|\mathcal{B}_1\|^2 = \sum_{i=1}^{n}\sum_{k=1}^{t} g^2(\mathcal{B}_1 Y_i, V_k). \tag{3.71}$$

Moreover, handling the Gauss-Codazzi type equations (2.7)-(2.9), we have

$$\begin{aligned} 2\tau &= 2\tau_1 + 2\tau_2 + t^2 \left\|\acute{H}\right\|^2 - \sum_{k,j=1}^{t} g_1(\mathcal{T}_{V_k}V_j, \mathcal{T}_{V_k}V_j) + 3\sum_{i,s=1}^{n} g_1(\mathcal{A}_{Y_i}X_s, \mathcal{A}_{Y_i}X_s) \\ &- \sum_{i=1}^{n}\sum_{k=1}^{t} g_1((\nabla_{Y_i}\mathcal{T})_{V_k}V_k, Y_i) + \sum_{i=1}^{n}\sum_{k=1}^{t}(g_1(\mathcal{T}_{V_k}Y_i, \mathcal{T}_{V_k}Y_i) - g_1(\mathcal{A}_{Y_i}V_k, \mathcal{A}_{Y_i}V_k)) \\ &- \sum_{s=1}^{n}\sum_{j=1}^{r} g_1((\nabla_{Y_s}\mathcal{T})_{V_j}V_j, Y_s) + \sum_{s=1}^{n}\sum_{j=1}^{t}(g_1(\mathcal{T}_{V_j}Y_s, \mathcal{T}_{V_j}Y_s) - g_1(\mathcal{A}_{Y_s}V_j, \mathcal{A}_{Y_s}V_j)). \end{aligned} \tag{3.72}$$

Thus considering (3.34) and (3.72), we get

$$\begin{aligned} 2\tau &= 2\tau_1 + 2\tau_2 + \frac{1}{2}t^2\left\|\acute{H}\right\|^2 - \frac{1}{2}(\mathcal{T}_{11}^s - \mathcal{T}_{22}^s - ... - \mathcal{T}_{tt}^s)^2 - 2\sum_{s=1}^{n}\sum_{j=2}^{t}(\mathcal{T}_{1j}^s)^2 \\ &+ 2\sum_{s=1}^{n}\sum_{2\le i<j\le t}(\mathcal{T}_{ii}^s\mathcal{T}_{jj}^s - (\mathcal{T}_{ij}^s)^2) + 6\sum_{\alpha=1}^{t}\sum_{s=2}^{n}(\mathcal{A}_{1s}^{\alpha})^2 + 6\sum_{\alpha=1}^{t}\sum_{2\le i<s\le n}(\mathcal{A}_{is}^{\alpha})^2 \\ &+ \sum_{i=1}^{n}\sum_{k=1}^{t}(g_1(\mathcal{T}_{V_k}Y_i, \mathcal{T}_{V_k}Y_i) - g_1(\mathcal{A}_{Y_i}V_k, \mathcal{A}_{Y_i}V_k)) - 2\delta(\grave{N}) \\ &+ \sum_{s=1}^{n}\sum_{j=1}^{t}(g_1(\mathcal{T}_{V_j}Y_s, \mathcal{T}_{V_j}Y_s) - g_1(\mathcal{A}_{Y_s}V_j, \mathcal{A}_{Y_s}V_j)). \end{aligned} \tag{3.73}$$

Considering (3.41), (3.56), (3.70) and (3.71) in (3.73), we get

$$\begin{aligned} &\frac{c_1}{4}[(n+t)(n+t-1) + 3(2(t_1 + t_2\cos^2\theta) + 2\|\mathcal{B}_1\|^2 + \|\mathcal{B}_2\|^2)] = 2Ric_1(V_1) \\ &+ 2Ric_2(Y_1) + \frac{1}{2}t^2\left\|\acute{H}\right\|^2 - \frac{1}{2}(\mathcal{T}_{11}^s - \mathcal{T}_{22}^s - ... - \mathcal{T}_{tt}^s)^2 - 2\sum_{s=1}^{n}\sum_{j=2}^{t}(\mathcal{T}_{1j}^s)^2 \\ &+ 6\sum_{\alpha=1}^{t}\sum_{s=2}^{n}(\mathcal{A}_{1s}^{\alpha})^2 + \sum_{i=1}^{n}\sum_{k=1}^{t}(g_1(\mathcal{T}_{V_k}Y_i, \mathcal{T}_{V_k}Y_i) - g_1(\mathcal{A}_{Y_i}V_k, A_{Y_i}V_k)) \\ &- 2\delta(\grave{N}) + \sum_{s=1}^{n}\sum_{j=1}^{r}(g_1(\mathcal{T}_{V_j}Y_s, \mathcal{T}_{V_j}Y_s) - g_1(\mathcal{A}_{Y_s}V_j, \mathcal{A}_{Y_s}V_j)) \\ &+ \sum_{2\le i<j\le t} R_1(V_i, V_j, V_j, V_i) + \sum_{2\le i<j\le n} R_1(Y_i, Y_j, Y_j, Y_i). \end{aligned} \tag{3.74}$$

If we take $V_1 \in \Gamma(\mathcal{D})$, considering (3.46), (3.58), (3.59), (3.63) and (3.66) in (3.74) we obtain (3.67). If we take $V_1 \in \Gamma(\mathcal{D}^{\theta})$, considering (3.49), (3.58), (3.59), (3.63) and (3.66) in (3.74) we obtain (3.68). Similarly, if we take $V_1 \in \Gamma(\mathcal{D}^{\perp})$, considering (3.52), (3.58), (3.59), (3.63) and (3.66) in (3.74) we obtain (3.69). This completes the proof. □

Considering (3.70), (3.71) and (3.72) we have

$$\frac{c_1}{4}[(n+t)(n+t-1)+3(2(t_1+t_2\cos^2\theta)+2\|\mathcal{B}_1\|^2+\|\mathcal{B}_2\|^2)]=2\tau_1+2\tau_2$$
$$+t^2\left\|\acute{H}\right\|^2-\|\mathcal{T}^{\mathcal{H}}\|^2+3\|\mathcal{A}^{\mathcal{V}}\|^2-2\delta(\grave{N})+2\|\mathcal{T}^{\mathcal{V}}\|^2-2\|\mathcal{A}^{\mathcal{H}}\|^2. \tag{3.75}$$

Considering (3.75) we get the following theorem.

Theorem 3.7. *Let $\varphi : M_1(c_1) \to M_2$ be a $\mathcal{QHSS}$ from a complex space form $(M_1(c_1), g_1)$ onto a Riemannian manifold (M_2, g_2). In that case we have*

$$2\tau_1+2\tau_2\leq\frac{c_1}{4}[(n+t)(n+t-1)+3(2(t_1+t_2\cos^2\theta)+2\|\mathcal{B}_1\|^2+\|\mathcal{B}_2\|^2)]$$
$$-t^2\left\|\acute{H}\right\|^2+\|\mathcal{T}^{\mathcal{H}}\|^2+2\delta(\grave{N})-2\|\mathcal{T}^{\mathcal{V}}\|^2+2\|\mathcal{A}^{\mathcal{H}}\|^2, \tag{3.76}$$

$$2\tau_1+2\tau_2\geq\frac{c_1}{4}[(n+t)(n+t-1)+3(2(t_1+t_2\cos^2\theta)+2\|\mathcal{B}_1\|^2+\|\mathcal{B}_2\|^2)]$$
$$-t^2\left\|\acute{H}\right\|^2+\|\mathcal{T}^{\mathcal{H}}\|^2-3\|\mathcal{A}^{\mathcal{V}}\|^2+2\delta(\grave{N})-2\|\mathcal{T}^{\mathcal{V}}\|^2. \tag{3.77}$$

In case of (3.76)'s and (3.77)'s equalities hold for all $p \in M_1$ if and only if horizontal distribution $\mathcal{H}$ is integrable.

Considering (3.75) we have the following theorem.

Theorem 3.8. *Let $\varphi : M_1(c_1) \to M_2$ be a $\mathcal{QHSS}$ from a complex space form $(M_1(c_1), g_1)$ onto a Riemannian manifold (M_2, g_2). In that case we have*

$$2\tau_1+2\tau_2\geq\frac{c_1}{4}[(n+t)(n+t-1)+3(2(t_1+t_2\cos^2\theta)+2\|\mathcal{B}_1\|^2+\|\mathcal{B}_2\|^2)]$$
$$-t^2\left\|\acute{H}\right\|^2+2\delta(\grave{N})-2\|\mathcal{T}^{\mathcal{V}}\|^2+2\|\mathcal{A}^{\mathcal{H}}\|^2-3\|\mathcal{A}^{\mathcal{V}}\|^2, \tag{3.78}$$

$$2\tau_1+2\tau_2\leq\frac{c_1}{4}[(n+t)(n+t-1)+3(2(t_1+t_2\cos^2\theta)+2\|\mathcal{B}_1\|^2+\|\mathcal{B}_2\|^2)]$$
$$-t^2\left\|\acute{H}\right\|^2+\|\mathcal{T}^{\mathcal{H}}\|^2+2\delta(\grave{N})+2\|\mathcal{A}^{\mathcal{H}}\|^2-3\|\mathcal{A}^{\mathcal{V}}\|^2. \tag{3.79}$$

In case of (3.78)'s and (3.79)'s equalities hold for all $p \in M_1$ if and only if the fiber through p of φ is a totally geodesic submanifold of M_1.

Lemma 3.1. *Let k ve l be non-negative real number, In that case*

$$\frac{k+l}{2}\geq\sqrt{kl}$$

with equality iff $k = l$.

Applying Lemma 3.1 in (3.75), we have.

Theorem 3.9. *Let $\varphi : M_1(c_1) \to M_2$ be a $\mathcal{QHSS}$ from a complex space form $(M_1(c_1), g_1)$ onto a Riemannian manifold (M_2, g_2). In that case we have*

$$\frac{c_1}{4}[(n+t)(n+t-1)+3(2(t_1+t_2\cos^2\theta)+2\|\mathcal{B}_1\|^2+\|\mathcal{B}_2\|^2)] \leq 2\tau_1+2\tau_2$$
$$+t^2\left\|\acute{H}\right\|^2+2\left\|\mathcal{T}^{\mathcal{V}}\right\|^2+3\left\|\mathcal{A}^{\mathcal{V}}\right\|^2-2\delta(\grave{N})-2\sqrt{2}\left\|\mathcal{A}^{\mathcal{H}}\right\|\left\|\mathcal{T}^{\mathcal{H}}\right\|. \tag{3.80}$$

In case of (3.80)'s equality hold for all $p \in M_1$ if and only if $\left\|\mathcal{A}^{\mathcal{H}}\right\| = \left\|\mathcal{T}^{\mathcal{H}}\right\|$.

Theorem 3.10. *Let $\varphi : M_1(c_1) \to M_2$ be a $\mathcal{QHSS}$ from a complex space form $(M_1(c_1), g_1)$ onto a Riemannian manifold (M_2, g_2). In that case we get*

$$\frac{c_1}{4}[(n+t)(n+t-1)+3(2(t_1+t_2\cos^2\theta)+2\|\mathcal{B}_1\|^2+\|\mathcal{B}_2\|^2)] \geq 2\tau_1+2\tau_2$$
$$+t^2\left\|\acute{H}\right\|^2-\left\|\mathcal{T}^{\mathcal{H}}\right\|^2-2\delta(\grave{N})-2\left\|\mathcal{A}^{\mathcal{H}}\right\|^2+2\sqrt{6}\left\|\mathcal{A}^{\mathcal{V}}\right\|\left\|\mathcal{T}^{\mathcal{V}}\right\|. \tag{3.81}$$

In case of (3.81)'s equality hold for all $p \in M_1$ if and only if $\left\|\mathcal{A}^{\mathcal{V}}\right\| = \left\|\mathcal{T}^{\mathcal{V}}\right\|$.

Lemma 3.2. [50] *Let $k_1, k_2, ..., k_n$, be $n-$real number $(n > 1)$, In that case*

$$\frac{1}{n}(\sum_{i=1}^{n} k_i)^2 \leq \sum_{i=1}^{n} k_i^2$$

with equality iff $k_1 = k_2 = ... = k_n$.

Theorem 3.11. *Let $\varphi : M_1(c_1) \to M_2$ be a $\mathcal{QHSS}$ from a complex space form $(M_1(c_1), g_1)$ onto a Riemannian manifold (M_2, g_2). In that case we get*

$$\frac{c_1}{4}[(n+t)(n+t-1)+3(2(t_1+t_2\cos^2\theta)+2\|\mathcal{B}_1\|^2+\|\mathcal{B}_2\|^2)] \leq 2\tau_1+2\tau_2$$
$$+t(t-1)\left\|\acute{H}\right\|^2+3\left\|\mathcal{A}^{\mathcal{V}}\right\|^2-2\delta(\grave{N})+2\left\|\mathcal{T}^{\mathcal{V}}\right\|^2-2\left\|\mathcal{A}^{\mathcal{H}}\right\|^2. \tag{3.82}$$

In case of (3.82)'s equality holds for all $p \in M_1$ if and only if we get statements:

i) φ is a Riemannian submersion that has a totally umbilical fiber.

ii) $\mathcal{T}_{ij} = 0$, for $i \neq j \in \{1, 2, ..., r\}$.

Proof. Using (3.75) we get

$$\frac{c_1}{4}[(n+t)(n+t-1)+3(2(t_1+t_2\cos^2\theta)+2\|\mathcal{B}_1\|^2+\|\mathcal{B}_2\|^2)]$$
$$=2\tau_1+2\tau_2+t^2\left\|\acute{H}\right\|^2-\sum_{j=1}^{t}(\mathcal{T}_{jj}^s)^2-\sum_{i=1}^{n}\sum_{j\neq k}^{t}(\mathcal{T}_{jk}^s)^2$$
$$+3\left\|\mathcal{A}^{\mathcal{V}}\right\|^2-2\delta(\grave{N})+2\left\|\mathcal{T}^{\mathcal{V}}\right\|^2-2\left\|\mathcal{A}^{\mathcal{H}}\right\|^2. \tag{3.83}$$

Applying Lemma 3.2 in (3.83), we have

$$\frac{c_1}{4}[(n+t)(n+t-1)+3(2(t_1+t_2\cos^2\theta)+2\|\mathcal{B}_1\|^2+\|\mathcal{B}_2\|^2)]$$
$$\leq 2\tau_1+2\tau_2+t^2\left\|\acute{H}\right\|^2-\frac{1}{t}(\sum_{j=1}^{t}\mathcal{T}_{jj}^s)^2-\sum_{s=1}^{n}\sum_{j\neq k}^{t}(\mathcal{T}_{jk}^s)^2$$
$$+3\left\|\mathcal{A}^{\mathcal{V}}\right\|^2-2\delta(\grave{N})+2\left\|\mathcal{T}^{\mathcal{V}}\right\|^2-2\left\|\mathcal{A}^{\mathcal{H}}\right\|^2. \tag{3.84}$$

From this we have (3.82). In case of (3.82)'s equality holds for all $p\in M_1$ if and only if

$$\mathcal{T}_{11}=\mathcal{T}_{22}=...=\mathcal{T}_{tt} \text{ and } \sum_{s=1}^{n}\sum_{j\neq k}^{t}(\mathcal{T}_{jk}^s)^2=0.$$

This completes proof of the theorem. □

By comparing the proof of Theorem 3.11, we have

Theorem 3.12. *Let $\varphi : M_1(c_1)\to M_2$ be a $\mathcal{QHSS}$ from a complex space form $(M_1(c_1), g_1)$ onto a Riemannian manifold (M_2, g_2). In that case we have*

$$\frac{c_1}{4}[(n+t)(n+t-1)+3(2(t_1+t_2\cos^2\theta)+2\|\mathcal{B}_1\|^2+\|\mathcal{B}_2\|^2)]\geq 2\tau_1+2\tau_2$$
$$+t^2\left\|\acute{H}\right\|^2-\left\|\mathcal{T}^{\mathcal{H}}\right\|^2+\frac{3}{n}(\mathcal{A}^{\mathcal{V}})^2-2\delta(\grave{N})+2\left\|\mathcal{T}^{\mathcal{V}}\right\|^2-2\left\|\mathcal{A}^{\mathcal{H}}\right\|^2. \tag{3.85}$$

Equality case of (3.85) holds for all $p\in M_1$ if and only if $\mathcal{A}_{11}=\mathcal{A}_{22}=...=\mathcal{A}_{nn}$ and $\mathcal{A}_{ij}=0$, for $i\neq j\in\{1,2,...,n\}$.

4. Examples

In this section, we provide examples of $\mathcal{QHSS}$, illustrating the main results stated above and the examples of $\mathcal{QHSS}$ satisfying the equality case of all inequalities established in the above section.

Example 4.1. *Let $(\mathbb{R}^8, g_{\mathbb{R}^8}, J_1)$ be an almost Hermitian manifold which $J_1(x_1,...,x_8)=(-x_6,x_5,-x_7,x_8,-x_2,x_1,x_3,-x_4)$ be a complex structure and $(\mathbb{R}^3, g_{\mathbb{R}^3})$ be a Riemanniann manifold.*
$\varphi : (\mathbb{R}^8, g_{\mathbb{R}^8}, J_1)\to(\mathbb{R}^3, g_{\mathbb{R}^3})$ be a map defined by

$$\varphi(x_1,x_2,...,x_8)=(\frac{1}{\sqrt{2}}x_1-\frac{1}{\sqrt{2}}x_5, x_2, x_3cos\alpha-x_7sin\alpha)$$

where $\theta\in(0,\frac{\pi}{2})$. In that case φ is a $\mathcal{QHSS}$ (where rank $\varphi_*=3$) such that

$$V_1=\frac{1}{\sqrt{2}}\frac{\partial}{\partial x_1}+\frac{1}{\sqrt{2}}\frac{\partial}{\partial x_5}, V_2=sin\alpha\frac{\partial}{\partial x_3}+cos\alpha\frac{\partial}{\partial x_7}, V_3=\frac{\partial}{\partial x_4}, V_4=\frac{\partial}{\partial x_6}, V_5=\frac{\partial}{\partial x_8},$$

$ker\varphi_* = \mathcal{D} \oplus \mathcal{D}^\perp \oplus \mathcal{D}^\theta$, *where*

$$\mathcal{D} = \left\langle V_3 = \frac{\partial}{\partial x_4}, V_5 = \frac{\partial}{\partial x_8} \right\rangle$$

$$\mathcal{D}^\perp = \left\langle V_2 = sin\alpha \frac{\partial}{\partial x_3} + cos\alpha \frac{\partial}{\partial x_7} \right\rangle$$

$$\mathcal{D}^\theta = \left\langle V_1 = \frac{1}{\sqrt{2}} \frac{\partial}{\partial x_1} + \frac{1}{\sqrt{2}} \frac{\partial}{\partial x_5}, V_4 = \frac{\partial}{\partial x_6} \right\rangle$$

and

$$(\ker\varphi_*)^\perp = \left\langle H_1 = \frac{1}{\sqrt{2}} \frac{\partial}{\partial x_1} - \frac{1}{\sqrt{2}} \frac{\partial}{\partial x_5}, H_2 = \frac{\partial}{\partial x_2} H_3 = cos\alpha \frac{\partial}{\partial x_3} - sin\alpha \frac{\partial}{\partial x_3} \right\rangle$$

which $\mathcal{D} = \langle V_3, V_5 \rangle$ *is invariant,* $\mathcal{D}^\perp = \langle V_2 \rangle$ *is anti-invariant and* $\mathcal{D}^\theta = \langle V_1, V_7 \rangle$ *slant with slant angle* $\theta = \frac{\pi}{4}$.

Example 4.2. *Let* $(\mathbb{R}^{10}, g_{\mathbb{R}^{10}}, J_1)$ *be an almost Hermitian manifold which* $J_1(x_1, ..., x_{10}) = (-x_7, -x_9, -x_6, x_{10}, x_8, x_3, x_1, -x_5, x_2, -x_4)$ *be a complex structure and* $(\mathbb{R}^5, g_{\mathbb{R}^3})$ *be a Riemanniann manifold.*

$F : (\mathbb{R}^{10}, g_{\mathbb{R}^{10}}, J_1) \to (\mathbb{R}^5, g_{\mathbb{R}^5})$ *be a map defined by*

$$F(x_1, x_2, ..., x_{10}) = (cos\alpha x_1 - sin\alpha x_{10}, \frac{1}{\sqrt{2}}x_2 - \frac{1}{\sqrt{2}}x_6, \frac{1}{\sqrt{2}}x_3 + \frac{1}{\sqrt{2}}x_9, x_4, \frac{\sqrt{3}}{\sqrt{2}}x_5 - \frac{1}{2}x_8)$$

where $\theta \in (0, \frac{\pi}{2})$. *In that case* F *is a* $\mathcal{QHSS}$ *(where rank* $F_* = 5$ *) such that*

$$V_1 = -sin\alpha \frac{\partial}{\partial x_1} + cos\alpha \frac{\partial}{\partial x_{10}}, V_2 = \frac{1}{\sqrt{2}} \frac{\partial}{\partial x_2} + \frac{1}{\sqrt{2}} \frac{\partial}{\partial x_6}, V_3 = -\frac{1}{\sqrt{2}} \frac{\partial}{\partial x_3} + \frac{1}{\sqrt{2}} \frac{\partial}{\partial x_9},$$

$$V_4 = \frac{1}{2} \frac{\partial}{\partial x_5} - \frac{\sqrt{3}}{2} \frac{\partial}{\partial x_8}, V_5 = \frac{\partial}{\partial x_7},$$

$kerF_* = \mathcal{D} \oplus \mathcal{D}^\perp \oplus \mathcal{D}^\theta$, *where*

$$\mathcal{D} = \left\langle V_2 = \frac{1}{\sqrt{2}} \frac{\partial}{\partial x_2} + \frac{1}{\sqrt{2}} \frac{\partial}{\partial x_6}, V_3 = -\frac{1}{\sqrt{2}} \frac{\partial}{\partial x_3} + \frac{1}{\sqrt{2}} \frac{\partial}{\partial x_9} \right\rangle$$

$$\mathcal{D}^\perp = \left\langle V_4 = \frac{1}{2} \frac{\partial}{\partial x_5} + \frac{\sqrt{3}}{2} \frac{\partial}{\partial x_8} \right\rangle$$

$$\mathcal{D}^\theta = \left\langle V_1 = -sin\alpha \frac{\partial}{\partial x_1} - cos\alpha \frac{\partial}{\partial x_{10}}, V_5 = \frac{\partial}{\partial x_7} \right\rangle$$

and

$$(kerF_*)^\perp = \langle H_1 = -cos\alpha \frac{\partial}{\partial x_1} + sin\alpha \frac{\partial}{\partial x_{10}}, H_2 = \frac{1}{\sqrt{2}} \frac{\partial}{\partial x_2} - \frac{1}{\sqrt{2}} \frac{\partial}{\partial x_6}, H_3 = \frac{1}{\sqrt{2}} \frac{\partial}{\partial x_3} + \frac{1}{\sqrt{2}} \frac{\partial}{\partial x_9},$$

$$H_4 = \frac{\sqrt{3}}{2} \frac{\partial}{\partial x_5} - \frac{1}{2} \frac{\partial}{\partial x_8}, H_5 = \frac{\partial}{\partial x_4} \rangle$$

which $\mathcal{D} = \langle V_2, V_3 \rangle$ *is invariant,* $\mathcal{D}^\perp = \langle V_4 \rangle$ *is anti-invariant and* $\mathcal{D}^\theta = \langle V_1, V_5 \rangle$ *slant with slant angle* $\theta = arccos(sin\alpha)$.

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