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ON $\mathcal{I}$ CONCURRENT CLASS OF SEQUENCES

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Abstract. In this paper, we demonstrate the $\mathcal{I}$-concurrent relation between sequences and the equivalence relations produced from it. A few unique features of these equivalence classes are investigated. Finally, we show that the collection of all such equivalence classes of all $\mathcal{I}$-convergent sequences under the $\mathcal{I}$-concurrent relation generates a metric space that is isometric with the set of all constant sequences.

Keywords: $\mathcal{I}$-convergence, $\mathcal{I}$-Cauchy, $\mathcal{I}$-concurrent relation.

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1. Introduction

Natural density, additionally referred to as asymptotic density, is a fundamental notion in number theory and analysis that measures how large a subset of natural number is relative to the set of all natural numbers. For $M \subseteq \mathbb{N}$, the natural density of M is denoted by $\delta(M)$ and is defined as

$$\delta(M) = \lim_{n\to\infty} \frac{|k \leq n : k \in M|}{n}.$$

This notion holds significance not just in pure mathematics but also in disciplines such as statistical mechanics, probability theory and computer science, where understanding the distribution of numbers may reveal patterns and behaviors inside complex systems. Steinhaus

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[18] and Fast [7] in 1951, developed the idea of statistical convergence independently by implementing the idea of natural density (also known as asymptotic density). The statistical convergence of a sequence $\langle x_n \rangle$ to x_0 is attained if, for any $\varepsilon > 0$, the set $\{k \in \mathbb{N} : d(x_k, x_0) \geq \varepsilon\}$ has a density of zero. Numerous mathematicians, such as Fridy [8, 9], Salat [15], Rath and Tripathy [14], Bal [2], Sarkar et al. [17] have conducted extensive research on this convergence. In 2000, Kostyrko et al. [11] raised the concept of $\mathcal{I}$-convergence, while $\mathcal{I}$ Cauchy sequences was initially defined by Nabiev et al. [13]. $\mathcal{I}$-convergence is an extension of statistical convergence depending on the ideal's ($\mathcal{I}$'s) framework, where $\mathcal{I}$ is a family of subsets of the set of natural numbers. Although there have been a lot of generalizations of statistical convergence, we found $\mathcal{I}$-convergence the most interesting one, where $\mathcal{I}$ is an ideal. In the recent literature, there have been several publications on $\mathcal{I}$-convergence [3, 4, 5, 6, 10, 12, 3, 16], including some outstanding contributions by Bal [1].

In this study, we seek to establish a relationship between two sequences of the same nature by means of $\mathcal{I}$-convergence. In order to accomplish this, we introduce the $\mathcal{I}$-concurrent relation, which establishes an equivalence relation on the collection of all sequences in a metric space. Also, the collection of all equivalence classes produced by that equivalence relation on the set of all $\mathcal{I}$ -convergent sequences constructs a metric space.

2. Preliminaries

Prior to studying $\mathcal{I}$ concurrent sequences in depth, it is important to provide some basic definitions and notions. In this section, we briefly discuss the fundamental instruments and mathematical concepts required to comprehend the key findings.

Definition 2.1. [13] *A family $\mathcal{I}$ of subsets of a non empty set X is called an ideal if and only if $\emptyset \in \mathcal{I}$, $\mathcal{I}$ is closed under finite union and $\mathcal{I}$ is closed under subset. Also a family $\mathcal{F}$ of subsets of a non empty set X is called a filter if and only if $\emptyset \notin \mathcal{F}$, $\mathcal{F}$ is closed under finite intersection and $\mathcal{F}$ is closed under superset.*

If $X \notin \mathcal{I}$ and $\mathcal{I} \neq \emptyset$, then the ideal $\mathcal{I}$ is considered as an non-trivial ideal. If $\mathcal{I}$ is an ideal, then the collection $\mathcal{F} = \mathcal{F}(\mathcal{I}) = \{X \setminus M : M \in \mathcal{I}\}$ is a filter and called the dual filter of the ideal $\mathcal{I}$. If $\mathcal{I}$ is a non-trivial ideal which contains every singleton subset of X, then $\mathcal{I}$ is considered to be an admissible ideal. '$\mathcal{I}$' will represent an admissible ideal throughout the paper.

Definition 2.2. [13] *Let $\mathcal{I}$ be an admissible ideal defined on the set $\mathbb{N}$ of natural numbers and (X, d) be a metric space. For a sequence $\langle x_n \rangle$, if for each $\varepsilon > 0$,*

$$\{k \in \mathbb{N} : d(x_k, x_0) \geq \varepsilon\} \in \mathcal{I},$$

then $\langle x_n \rangle$ is considered to be $\mathcal{I}$-convergent to x_0.

Definition 2.3. [13] *Let $\mathcal{I}$ be an admissible ideal defined on the set $\mathbb{N}$ of natural numbers and (X, d) be a metric space. For a sequence $\langle x_n \rangle$, if for every $\varepsilon > 0$ there exists a $m \in \mathbb{N}$ such that*

$$\{n \in \mathbb{N} : d\left(x_n, x_m\right) \geq \varepsilon\} \in \mathcal{I},$$

then $\langle x_n \rangle$ is considered to be an $\mathcal{I}$-Cauchy sequence in X.

3. On $\mathcal{I}$ Concurrent Sequences

Using the concept of $\mathcal{I}$-convergence, we want to create an equivalence relation on the set of all sequences that will separate the sequence space into disjoint equivalence classes. These classes of sequences will have sequences that are similar in nature, making it easier to study each one freely.

Theorem 3.1. *If the sequences $\langle x_n \rangle$ and $\langle y_n \rangle$ satisfy the $\mathcal{I}$-Cauchy criteria in a metric space (X, d), then the sequences $\langle z_n = d(x_n, y_n) : n \in \mathbb{N} \rangle$ will satisfy the $\mathcal{I}$-Cauchy criteria in a metric space (X, d_1) where $d_1(a, b) = |a - b|$.*

Proof. Since $\langle x_n \rangle$ and $\langle y_n \rangle$ satisfies $\mathcal{I}$-Cauchy criteria, therefore, $A_1 = \{n \in \mathbb{N} : d\left(x_n, x_{m_1}\right) < \frac{\varepsilon}{2}\} \in \mathcal{F}(\mathcal{I})$, for some $m_1 \in \mathbb{N}$ and $A_2 = \{n \in \mathbb{N} : d\left(y_n, y_{m_2}\right) < \frac{\varepsilon}{2}\} \in \mathcal{F}(\mathcal{I})$, for some $m_2 \in \mathbb{N}$.

Let $m = max\{m_1, m_2\}$. Also, $(A_1 \cap A_2) \in \mathcal{F}(\mathcal{I})$ and $\phi \notin \mathcal{F}(\mathcal{I})$, so $(A_1 \cap A_2) \neq \phi$ and for all $n \in (A_1 \cap A_2)$, we have

$d_1(z_n, z_m) \leq d(x_n, x_m) + d(y_n, y_m) < \varepsilon$,

i.e., $\{n \in \mathbb{N} : d_1(z_n, z_m) < \varepsilon\} \supseteq (A_1 \cap A_2) \in \mathcal{F}(\mathcal{I})$.

Therefore, $\langle z_n \rangle$ fulfills $\mathcal{I}$-Cauchy criteria in the metric space (X, d_1). □

Definition 3.1. *A sequence $\langle x_n \rangle$ is said to be $\mathcal{I}$ concurrent to another sequence $\langle y_n \rangle$ if the sequence $\langle z_n \rangle = \langle d(x_n, y_n) \rangle$ is such that $z_n \overset{\mathcal{I}\text{-}lim}{\longrightarrow} 0$. That is, $\{n \in \mathbb{N} : z_n = d(x_n, y_n) \geq \varepsilon\} \in \mathcal{I}$.*

Example 3.1. *Let $X = \{0, 1\}$, equipped with the discrete metric $\sigma(x, y) = \begin{cases} 0, & if\ x = y, \\ 1, & otherwise, \end{cases}$. Consider the ideal $\mathcal{I}_\delta = \{A : \delta(A) = 0\}$ and the sequences $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ where*

$$a_n = \begin{cases} 0, & for\ n = k^2, k \in \mathbb{N}, \\ 1, & otherwise, \end{cases} \quad b_n = 1 \ for\ all\ n \in \mathbb{N}\ and\ c_n = \begin{cases} 1, & for\ n = k^2, k \in \mathbb{N}, \\ 0, & otherwise. \end{cases}$$

For every $\varepsilon > 0$, $\{n \in \mathbb{N} : \sigma(a_n, b_n) \geq \varepsilon\} = \{n = k^2 : k \in \mathbb{N}\} \in \mathcal{I}_\delta$, $\{n \in \mathbb{N} : \sigma(b_n, c_n) \geq \varepsilon\} = \{n \neq k^2 : k \in \mathbb{N}\} \notin \mathcal{I}_\delta$ and $\{n \in \mathbb{N} : \sigma(c_n, a_n) \geq \varepsilon\} = \mathbb{N} \notin \mathcal{I}_\delta$. Thus, $\{a_n\}$ and $\{b_n\}$ are $\mathcal{I}$ concurrent to each other; whereas $\{c_n\}$ is $\mathcal{I}$ concurrent neither to $\{a_n\}$ nor to $\{b_n\}$.

Theorem 3.2. *For two $\mathcal{I}$ concurrent sequences $\langle x_n \rangle$ and $\langle y_n \rangle$, if one sequence is $\mathcal{I}$ Cauchy, then the other also satisfies the $\mathcal{I}$-Cauchy criteria.*

Proof. Let $\langle x_n \rangle$ satisfy the $\mathcal{I}$ Cauchy criteria. Therefore, $A_1 = \{n \in \mathbb{N} : d(x_n, x_m) < \frac{\varepsilon}{3}\} \in \mathcal{F}(\mathcal{I})$. Since $\langle x_n \rangle$ and $\langle y_n \rangle$ are $\mathcal{I}$ concurrent, therefore $A_2 = \{n \in \mathbb{N} : d(x_n, y_n) < \frac{\varepsilon}{3}\} \in \mathcal{F}(\mathcal{I})$.

Since $(A_1 \cap A_2) \neq \phi$ and for all $n \in (A_1 \cap A_2)$ there exists a $m \in (A_1 \cap A_2)$ so that

$$d(y_n, y_m) \leq d(y_n, x_n) + d(x_n, x_m) + d(x_m, y_m) < \varepsilon,$$

$$\text{i.e., } \{n \in \mathbb{N} : d(y_n, y_m) < \varepsilon\} \in \mathcal{F}(\mathcal{I}) \text{ as } (A_1 \cap A_2) \in \mathcal{F}(\mathcal{I}).$$

Therefore, $\langle y_n \rangle$ also satisfies $\mathcal{I}$-Cauchy criteria. □

Example 3.2. *Two non $\mathcal{I}$-Cauchy sequences can be $\mathcal{I}$-Concurrent to each other. Let $X = [0, 2]$ and $d(a, b) = |(a - b)|$ for all $a, b \in X$. Then (X, d) forms a metric space. Also consider the sequences $\langle x_n \rangle$ and $\langle y_n \rangle$ of (X, d) where*

$$x_n = \frac{[1 + (-1)^{n+1}]}{2}$$

and

$$y_n = \begin{cases} 1, & for\ n\ is\ odd, \\ \frac{1}{n}, & for\ n\ is\ even. \end{cases}$$

Now take $\mathcal{I} = \mathcal{I}_{fin}$, the ideal containing all finite subsets of $\mathbb{N}$, then neither $\langle x_n \rangle$ nor $\langle y_n \rangle$ satisfy $\mathcal{I}$-Cauchy criteria.

But $\langle z_n \rangle = \langle d(x_n, y_n) \rangle$ where $d(x_n, y_n) = \begin{cases} 0, & for\ n\ is\ odd \\ \frac{1}{n}, & for\ n\ is\ even \end{cases}$ *is $\mathcal{I}$-convergent to 0.*

Since, $\{n \in \mathbb{N} : z_n \geq \varepsilon\} \in \mathcal{I}$

Therefore, $\langle x_n \rangle$ and $\langle y_n \rangle$ are not $\mathcal{I}$-Cauchy but $\mathcal{I}$-concurrent sequences.

Example 3.3. *Again, if two sequences are $\mathcal{I}$-Cauchy sequences, it does not imply that they are $\mathcal{I}$-concurrent. For example,* *Let $X = [0, 2]$ and $d(a, b) = |(a - b)|$ for all $a, b \in X$. Then (X, d) forms a metric space. Also consider the sequences $\langle x_n \rangle$ and $\langle y_n \rangle$ of (X, d) where*

$$x_n = \begin{cases} 2, & for\ n = k^2, k \in \mathbb{N}, \\ 1 + \frac{1}{n}, & otherwise \end{cases}$$

and

$$y_n = \begin{cases} 2, & for\ n = k^2, k \in \mathbb{N}, \\ \frac{1}{n}, & otherwise. \end{cases}$$

Now, if we take $I = I_\delta$, the class of all subsets of $\mathbb{N}$ whose natural density is 0, then $\langle x_n \rangle$ and $\langle y_n \rangle$ are $\mathcal{I}$-Cauchy sequences.

But $\langle z_n \rangle = \langle d(x_n, y_n) \rangle$ where $d(x_n, y_n) = \begin{cases} 0, & for\ n = k^2, k \in \mathbb{N}, \\ 1, & otherwise. \end{cases}$ is not $\mathcal{I}$-convergent to 0.

That is, $\langle x_n \rangle$ and $\langle y_n \rangle$ are $\mathcal{I}$-Cauchy but not $\mathcal{I}$-concurrent to each other.

Theorem 3.3. *Two sequences are $\mathcal{I}$ convergent to the same limit if and only if they are $\mathcal{I}$ concurrent sequences, one of them being $\mathcal{I}$ convergent.*

Proof. Let $\langle x_n \rangle$ be $\mathcal{I}$-convergent to the limit ℓ. Therefore, $A_1 = \{n \in \mathbb{N} : d(x_n, \ell) < \frac{\varepsilon}{2}\} \in \mathcal{F}(\mathcal{I})$.

Also, let $\langle x_n \rangle$ and $\langle y_n \rangle$ be $\mathcal{I}$-concurrent. Therefore, $A_2 = \{n \in \mathbb{N} : d(x_n, y_n) < \frac{\varepsilon}{2}\} \in \mathcal{F}(\mathcal{I})$.

Since $(A_1 \cap A_2) \neq \phi$ and for all $n \in (A_1 \cap A_2)$ we have

$$d(y_n, \ell) \leq d(y_n, x_n) + d(x_n, \ell) < \varepsilon,$$

$$\text{i.e., } \{n \in \mathbb{N} : d(y_n, \ell) < \varepsilon\} \in \mathcal{F}(\mathcal{I}) \text{ as } (A_1 \cap A_2) \in \mathcal{F}(\mathcal{I}).$$

Therefore, $\langle y_n \rangle$ is also $\mathcal{I}$-convergent to the same limit ℓ.

Conversely, let $\langle x_n \rangle$ and $\langle y_n \rangle$ be two $\mathcal{I}$ convergent sequences converging to the same limit ℓ. That is, $B_1 = \{n \in \mathbb{N} : d(x_n, \ell) < \frac{\varepsilon}{2}\} \in \mathcal{F}(\mathcal{I})$ and $B_2 = \{n \in \mathbb{N} : d(y_n, \ell) < \frac{\varepsilon}{2}\} \in \mathcal{F}(I)$. So $\forall\ n \in (A_1 \cap A_2) \subset \mathbb{N}$ we have

$$d(x_n, y_n) \leq d(x_n, \ell) + d(y_n, \ell) < \varepsilon,$$

$$\text{i.e., } \{n \in \mathbb{N} : d(x_n, y_n) < \varepsilon\} \supseteq (A_1 \cap A_2) \in \mathcal{F}(\mathcal{I}).$$

Therefore, $\langle x_n \rangle$ and $\langle y_n \rangle$ are $\mathcal{I}$-concurrent sequences. □

Theorem 3.4. *Let S_X be the collection of all sequences on the metric space (X, d). Then the $\mathcal{I}$-concurrent relation $\approx_{\mathcal{I}\text{-}d}$ forms an equivalence relation on S_X.*

Proof. Since for any $\langle x_n \rangle \in S_X$, $d(x_n, x_n) = 0, \forall n \in \mathbb{N}$. Therefore, $\langle d(x_n, x_n) \rangle$ is $\mathcal{I}$-convergent to 0. So every sequence is $\mathcal{I}$-concurrent to itself. That is, the $\mathcal{I}$-concurrent relation $(\approx_{\mathcal{I}\text{-}d})$ is a reflexive relation on S_X.

Since for any $\langle x_n \rangle, \langle y_n \rangle \in S_X$, $d(x_n, y_n) = d(y_n, x_n), \forall n \in \mathbb{N}$. Therefore, if $\langle x_n \rangle$ is $\mathcal{I}$-concurrent to $\langle y_n \rangle$ then $\langle y_n \rangle$ is also $\mathcal{I}$-concurrent to $\langle x_n \rangle$. That is, the $\mathcal{I}$-concurrent relation $(\approx_{\mathcal{I}\text{-}d})$ is a symmetric relation on S_X.

Let $\langle x_n \rangle, \langle y_n \rangle, \langle v_n \rangle \in S_X$, Now, $d(x_n, v_n) \leq d(x_n, y_n) + d(y_n, v_n), \forall n \in \mathbb{N}$. It implies that if $\langle d(x_n, y_n) \rangle$ and $\langle d(y_n, v_n) \rangle$ are $\mathcal{I}$-convergent to 0, then $\langle d(x_n, v_n) \rangle$ is also $\mathcal{I}$-convergent to 0.

So if $(x_n) \approx_{\mathcal{I}\text{-}d} (y_n)$ and $(y_n) \approx_{\mathcal{I}\text{-}d} (v_n) \implies (x_n) \approx_{\mathcal{I}\text{-}d} (v_n)$, i.e., the $\mathcal{I}$-concurrent relation $(\approx_{\mathcal{I}\text{-}d})$ is a transitive relation on S_X.

$\therefore$ the $\mathcal{I}$-concurrent relation $(\approx_{\mathcal{I}\text{-}d})$ forms an equivalence relation on S_X. □

Corollary 3.1. *The set S_X of all sequences of the space (X, d) splits into disjoint equivalent classes under the $\mathcal{I}$-concurrent relation $(\approx_{\mathcal{I}\text{-}d})$, so that all the sequences of one class are*

(i) Either $\mathcal{I}$-convergent to the same limit or is not $\mathcal{I}$-convergent.

(ii) Either $\mathcal{I}$-Cauchy sequences or none of them are $\mathcal{I}$-Cauchy.

Proof. This can be easily verified from the previous theorems. □

Let C_X be the collection of all $\mathcal{I}$-Convergent sequences of the metric space (X, d). Also let $\langle \boldsymbol{x_n} \rangle^* = \{\langle y_n \rangle \in C_X : \langle y_n \rangle \approx_{\mathcal{I}\text{-}d} \langle x_n \rangle\}$, where $\langle \boldsymbol{x_n} \rangle^*$ denotes the equivalence class of $\langle x_n \rangle$ under the $\mathcal{I}$-concurrent relation $(\approx_{\mathcal{I}\text{-}d})$.

We define $\sigma(\langle \boldsymbol{x_n} \rangle^*, \langle \boldsymbol{y_n} \rangle^*) = d'(\langle x_n \rangle, \langle y_n \rangle) = \mathcal{I}\text{-}lim_{n \to \infty} d(x_n, y_n)$

Let $\boldsymbol{C_X}^*$ denote the set of all equivalence classes $\langle \boldsymbol{x_n} \rangle^*$, where $\langle x_n \rangle \in C_X$.

Theorem 3.5. *The set of all equivalence classes $\boldsymbol{C_X}^*$ forms a metric space with the metric σ such that $\sigma(\langle \boldsymbol{x_n} \rangle^*, \langle \boldsymbol{y_n} \rangle^*) = \mathcal{I}\text{-}lim_{n \to \infty} d(x_n, y_n)$.*

Proof. Let $\langle \boldsymbol{x_n} \rangle^*, \langle \boldsymbol{y_n} \rangle^*, \langle \boldsymbol{z_n} \rangle^* \in \boldsymbol{C_X}^*$. So there is a $\langle x_n \rangle \in \langle \boldsymbol{x_n} \rangle^*, \langle y_n \rangle \in \langle \boldsymbol{y_n} \rangle^*, \langle z_n \rangle \in \langle \boldsymbol{z_n} \rangle^*$.

Now since d is a metric, $\forall n \in \mathbb{N}$.

$d(x_n, y_n) \geq 0 \implies \mathcal{I}\text{-}lim_{n \to \infty} d(x_n, y_n) \geq 0$

$\implies \sigma(\langle \boldsymbol{x_n} \rangle^*, \langle \boldsymbol{y_n} \rangle^*) \geq 0$ for all $\langle \boldsymbol{x_n} \rangle^*, \langle \boldsymbol{y_n} \rangle^*, \in \boldsymbol{C_X}^*$

Now $\sigma(\langle \boldsymbol{x_n} \rangle^*, \langle \boldsymbol{y_n} \rangle^*) = 0$

$\iff \mathcal{I}\text{-}lim_{n\to\infty} d(x_n, y_n) = 0$

$\iff \langle x_n\rangle$ and $\langle y_n\rangle$ are $\mathcal{I}$-concurrent to each other. So they belong to the same equivalence class.

That is, $\langle \boldsymbol{x_n}\rangle^* = \langle \boldsymbol{y_n}\rangle^*$.

Again, $\sigma(\langle \boldsymbol{x_n}\rangle^*, \langle \boldsymbol{y_n}\rangle^*) = \mathcal{I}\text{-}lim_{n\to\infty} d(x_n, y_n) = \mathcal{I}\text{-}lim_{n\to\infty} d(y_n, x_n) = \sigma(\langle \boldsymbol{y_n}\rangle^*, \langle \boldsymbol{x_n}\rangle^*)$ (Since d is symmetric).

$\therefore \sigma$ is symmetric.

Since d is metric, we have $d(x_n, y_n) \le d(x_n, z_n) + d(z_n, y_n)$, $\forall n \in \mathbb{N}$.

$\therefore \mathcal{I}\text{-}lim_{n\to\infty} d(x_n, y_n) \le \mathcal{I}\text{-}lim_{n\to\infty} d(x_n, z_n) + \mathcal{I}\text{-}lim_{n\to\infty} d(z_n, y_n)$.

$\implies \sigma(\langle \boldsymbol{x_n}\rangle^*, \langle \boldsymbol{y_n}\rangle^*) \le \sigma(\langle \boldsymbol{x_n}\rangle^*, \langle \boldsymbol{z_n}\rangle^*) + \sigma(\langle \boldsymbol{z_n}\rangle^*, \langle \boldsymbol{y_n}\rangle^*)$

Hence $(\boldsymbol{C_X}^*, \sigma)$ forms a metric space. □

Theorem 3.6. *Let $C'_X = \{\{x_n : x_n = x, \forall n \in \mathbb{N}\} : x \in X\}$. Then (C'_X, d') and $(\boldsymbol{C_X}^*, \sigma)$ are isometry.*

Proof. Let $f : C'_X \to \boldsymbol{C_X}^*$ is a function defined by $f(\langle x_n\rangle) = \langle \boldsymbol{x_n}\rangle^*$ where $\langle \boldsymbol{x_n}\rangle^* = \{\{z_n\} \in C_X : \langle z_n\rangle \approx_{\mathcal{I}\text{-}d} \langle x_n\rangle\}$. Now $\sigma(f(\langle x_n\rangle), f(\langle y_n\rangle)) = \sigma(\langle \boldsymbol{x_n}\rangle^*, \langle \boldsymbol{y_n}\rangle^*) = d'(\langle x_n\rangle, \langle y_n\rangle)$. Therefore, f is an isometry.

Again $f(\langle x_n\rangle) = f(\langle y_n\rangle) \implies \langle x_n\rangle = \langle y_n\rangle$ and for any equivalence class $\langle \boldsymbol{w_n}\rangle^* \in \boldsymbol{C_X}^*$ there exist a constant sequences $\langle w_n\rangle \in C'_X$, i.e., f is a bijective mapping. Hence (C'_X, d') and $(\boldsymbol{C_X}^*, \sigma)$ are isometry. □

4. Conclusion

An equivalence relation that splits the sequence space into disjoint equivalence classes has been discovered on the set of all sequences. Sequences in these categories are of the same kind with respect to $\mathcal{I}$-convergence and $\mathcal{I}$-Cauchy criteria. Further, a metric space is obtained for the collection of all these equivalence classes. It is possible to study the classes of point-wise convergence, uniform convergence, etc. independently if this idea is extended to the sequences of functions.

Declaration on Data Availability and Financial Support:

No data set has been produced or analyzed for this article. Therefore, the sharing of data is not relevant here. The individuals who wrote it don't own any proprietary or financial stake in any of the content covered in this piece of writing.

References

[1] Bal, P. (2023). On the class of $\mathcal{I}$-γ-open cover and $\mathcal{I}$-St-γ-open cover. *Hacettepe Journal of Mathematics and Statistics*, 52(3), 630-639.

[2] Bal, P., & Rakshit, D. (2023). A variation of the class of statistical γ covers. *Topological Algebra and its Applications*, 11(1), 20230101.

[3] Das, P., & Ghosal, S. K. (2010). Some further results on $\mathcal{I}$-Cauchy sequences and condition (AP). *Computers & Mathematics with Applications*, 59(8), 2597-2600.

[4] Debnath, S., & Rakshit, D. (2018). On $\mathcal{I}$-statistical convergence. *Iranian Journal of Mathematical Sciences and Informatics*, 13(2), 101-109.

[5] Demirci, K. (2001). $\mathcal{I}$-limit superior and limit inferior. *Mathematical communications*, 6(2), 165-172.

[6] Dems, K. (2004). On $\mathcal{I}$-Cauchy sequences. *Real analysis exchange*, 30(1), 123-128.

[7] Fast, H. (1951). Sur la convergence statistique. *In Colloquium mathematicae*, 2(3-4), 241-244.

[8] Fridy, J. A. (1993). Statistical limit points. *Proceedings of the American mathematical society*, 118(4), 1187-1192.

[9] Fridy, J. A. (1985). On statistical convergence. *Analysis*, 5(4), 301-314.

[10] Katětov, M. (1968). Products of filters. *Commentationes Mathematicae Universitatis Carolinae*, 9(1), 173-189.

[11] Kostyrko, P., Wilczyński, W., & Šalát, T. (2000). $\mathcal{I}$-convergence. *Real Anal. Exchange*, 26(2), 669-686.

[12] Mursaleen, M., Debnath, S., & Rakshit, D. (2017). $\mathcal{I}$-statistical limit superior and I-statistical limit inferior. *Filomat*, 31(7), 2103-2108.

[13] Nabiev, A., Pehlivan, S., & Gürdal, M. (2007). On $\mathcal{I}$-Cauchy sequences. *Taiwanese Journal of Mathematics*, 11(2), 569-576.

[14] Rath, D., & Tripathy, B. C. (1994). On statistically convergent and statistically Cauchy sequences. *Indian Journal of Pure and Applied Mathematics*, 25, 381-381.

[15] Šalát, T. (1980). On statistically convergent sequences of real numbers. *Mathematica slovaca*, 30(2), 139-150.

[16] Savas, E., & Das, P. (2011). A generalized statistical convergence via ideals. *Applied mathematics letters*, 24(6), 826-830.

[17] Sarkar, S., Bal, P., & Rakshit, D. (2023). On Statistical Compactness. *Iranian Journal of Mathematical Sciences and Informatics*, (accepted).

[18] Steinhaus, H. (1951). Sur la convergence ordinaire et la convergence asymptotique. *In Colloq. math*, (Vol. 2, No. 1, pp. 73-74).

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