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MORPHISMS AND ALGEBRAIC POINTS ON THE QUOTIENTS OF FERMAT CURVES AND HURWITZ CURVES

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ABSTRACT. In this paper we determine rational morphisms between the Hurwitz curves of affine equation : $u^n v^l + v^n + u^l = 0$ and the quotients of Fermat curves of affine equation $v^m = u^\lambda(u-1)$ where the integers $n > l \geq 1$ are coprime and $m = n^2 - ln + l^2$ and $\lambda \geq 1$. We also give a parametrization of the algebraic points of low degree on the quotient of Fermat curve : $v^7 = u(u-1)^2$. Using these morphisms, we explicitly determine the algebraic points of degree at most 3 on the Hurwitz curve $u^3v^2 + v^3 + u^2 = 0$ birationally isomorphic to the quotient of Fermat curve $v^7 = u^2(u-1)$.

Keywords: Hurwitz curve, Quotient of Fermat curve, Morphism, Degree of algebraic point

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1. Introduction

Let $\mathcal{C}$ be an algebraic curve defined over the rational number field $\mathbb{Q}$ and K an extension field of $\mathbb{Q}$. We denote by $\mathcal{C}(K)$ the set of rational points of $\mathcal{C}$ witch coordinates ares in K. A point $P \in \mathcal{C}(\overline{\mathbb{Q}})$ is said to be of degree d over $\mathbb{Q}$ if its field of definition L is an extension of $\mathbb{Q}$ of degree d. We denote by $\mathcal{C}^{(d)}(\mathbb{Q})$ the set of algebraic points of degree at most d on the curve $\mathcal{C}$ over $\mathbb{Q}$. A famous theorem of Faltings states that the number of rational points on an algebraic curve defined over a number field K is finite if the genus g of the curve is greater than 1. Currently, for a curve $\mathcal{C}$ of genus $g \geq 2$ defined over a number field K, there is no general method for computing the set $\mathcal{C}(K)$ or showing that $\mathcal{C}(K)$ is empty. But there are

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several methods for finding $\mathcal{C}(K)$ in special cases. These methods include the local method, the Chabauty elliptic method [4], the descent method [10], the Mordell-Weil Sieves method [3], the Sall-Fall method [9]. These methods are only applicable if the Mordell-Weil group $J_{\mathcal{C}}(\mathbb{Q})$ is known and is of a finite type. If $J_{\mathcal{C}}(\mathbb{Q})$ is finite, it is possible to determine $\mathcal{C}(\mathbb{Q})$ and to generalize to all number fields K and thus deduce $\mathcal{C}^{(d)}(\mathbb{Q})$ [7]. If we don't know the structure of the Mordell-Weil group, then we need to find a way to working around of it.

The purpose of this paper is to describe explicitly the morphisms between Hurwitz curves of affine equation $u^n v^l + v^n + u^l = 0$ and the Fermat quotient curves of affine equation $v^m = u^\lambda(u-1)$ where $n > l \geq 1$, $\gcd(n,l) = 1$, $\lambda \geq 1$ and $m = n^2 - ln + l^2$. Using these morphisms, we explicitly determine the algebraic points of degree at most 3 on the Hurwitz curves of affine equation $u^3v^2 + v^3 + u^2 = 0$, birationally isomorphic to the Fermat quotient curve of affine equation $v^7 = u^2(u-1)$. Our main results are Theorem 3.2 and Theorem 4.1.

2. Morphisms on Fermat curves and Hurwitz curves

2.1. **Fermat curves and quotients of Fermat curves.**

Let p be positive integer and K be an number field.

Definition 2.1. *The Fermat curve of degree p over a number field K is given by the projective equation*

$$F_p : U^p + V^p + W^p = 0.$$

The affine equation of F_p is

$$F_p : u^p + v^p + 1 = 0.$$

The Fermat curve F_p is smooth when the characteristic $car(K)$ of K does not divide p and has genus

$$g = \frac{(p-1)(p-2)}{2}.$$

For a pair (r,s) of positive integers such that $1 \leq r, s, r+s < p$ and $\gcd(r,s,p) = 1$, we denote by $C_{r,s}(p)$ the quotient of F_p defined by the equation

$$v^p = u^r(u-1)^s$$

where the projection $F_p \longrightarrow C_{r,s}(p)$ is defined by

$$\begin{array}{rccc} \phi : & F_p & \longrightarrow & C_{r,s}(p) \\ & (u,v) & \longmapsto & (-u^p, (-u)^r v^s). \end{array}$$

Lemma 2.1. *Let $C_{r,s}(p)$ and $C_{r',s'}(p)$ be two quotients of Fermat curve F_P. If it exists three integers k, i and j such that :*

$$(r,s) = k(r',s') + p(i,j) \quad and \quad gcd(k,p) = 1,$$

then we have the birational equivalences

(1) $C_{r,s}(p) \cong C_{r',s'}(p)$,
(2) $C_{r,s}(p) \cong C_{s,r}(p)$,
(3) $C_{r,s}(p) \cong C_{p-s-r,s}(p)$.

Proof.

(1) Consider the following covering map:

$$\begin{array}{llll} f_{rs}: & C_{r',s'}(p) & \longrightarrow & C_{r,s}(p) \\ & (u,v) & \longmapsto & \left(u,\, v^k u^i (u-1)^j\right). \end{array}$$

We have the following successive equivalences :

$$\begin{aligned} \left(u,\, v^k u^i (u-1)^j\right) \in C_{r,s}(p) & \Leftrightarrow \left(v^k u^i (u-1)^j\right)^p - u^r (u-1)^s = 0 \\ & \Leftrightarrow v^{pk} u^{pi} (u-1)^{pj} - u^r (u-1)^s = 0 \\ & \Leftrightarrow u^{pi} (u-1)^{pj} \left(v^{pk} - u^{r-pi} (u-1)^{s-pj}\right) = 0 \\ & \Leftrightarrow u^{pi} (u-1)^{pj} \left(v^{pk} - u^{kr'} (u-1)^{ks'}\right) = 0 \\ & \Leftrightarrow u^{pi} (u-1)^{pj} \left(v^{p} - u^{r'} (u-1)^{s'}\right) \left(v^{p(k-1)} + \cdots\right) = 0. \end{aligned}$$

So

$$(u,v) \in C_{r',s'}(p) : v^p - u^{r'} (u-1)^{s'} = 0,$$

then $C_{r,s}(p)$ is isomorphic to $C_{r',s'}(p)$.

(2) Consider the following covering map:

$$\begin{array}{llll} f_{rs}: & C_{r,s}(p) & \longrightarrow & C_{s,r}(p) \\ & (u,v) & \longmapsto & (1-u, (-1)^{s+r} v). \end{array}$$

We have

$$\begin{aligned} (1-u, (-1)^{s+r} v) \in C_{s,r}(p) & \Longleftrightarrow ((-1)^{s+r} v)^p = (1-u)^s ((1-u)-1)^r \\ & \Longleftrightarrow (-1)^{r+s} v^p = (-1)^{r+s} u^r (u-1)^s \\ & \Longleftrightarrow (u,v) \in C_{r,s}(p). \end{aligned}$$

(3) Consider the following covering map:

$$\begin{array}{llll} f_{rs}: & C_{r,s}(p) & \longrightarrow & C_{p-r-s,s}(p) \\ & (u,v) & \longmapsto & \left(\dfrac{1}{u}, \dfrac{(-1)^s v}{u}\right). \end{array}$$

We have

$$\left(\frac{1}{u}, \frac{(-1)^s v}{u}\right) \in C_{p-r-s,s}(p) \iff \left(\frac{(-1)^s v}{u}\right)^p = \left(\frac{1}{u}\right)^{p-r-s}\left(\frac{1}{u}-1\right)^s$$
$$\iff (-1)^{sp} v^p = (-1)^s u^r (u-1)^s$$
$$\iff (u,v) \in C_{r,s}(p).$$

□

The following corollary is the consequence of the lemma 2.1.

Corollary 2.1. *Let $C_{r,s}(p)$ be the quotient curve of Fermat. We have*

(i) $C_{1,s}(p) \cong C_{s,1}(p) \cong C_{p-s-1,1}(p) \cong C_{1,p-s-1}(p)$.

(ii) *For $2s \leq p-1$, the curves $C_{1,1}(p)$, $C_{1,2}(p)$, $C_{1,3}(p,) \ldots, C_{1,\frac{p-1}{2}}(p)$ form a complete list (with repetition).*

(iii) *Any curve $C_{r,s}(7)$ is birationally isomorphic either to the hyperelliptic curve $C_{1,1}(7)$, or to the non-hyperelliptic curve $C_{1,2}(7)$ which is itself isomorphic to the Klein curve.*

2.2. **Hurwitz curves.**

Let n and l be positive integers $n > l \geq 1$ and K be an number field.

Definition 2.2. *The Hurwitz curve $H_{n,l}$ over K is given by the projective equation*

$$H_{n,l} : U^n V^l + V^n W^l + U^l W^n = 0.$$

The affine equation of $H_{n,l}$ is

$$H_{n,l} : u^n v^l + v^n + u^l = 0.$$

Let $m = n^2 - nl + l^2$. The Hurwitz curve $H_{n,l}$ has the following genus

$$g = \frac{m + 2 - 3\mathrm{gcd}(n,l)}{2}.$$

The curve $H_{n,l}$ is smooth when the characteristic car(K) of K is relatively prime to m.

Lemma 2.2. *Let n, l be two positive integers such that $gcd(n,l) = 1$. An integer $m > 3$ of the form $m = n^2 - nl + l^2$ is prime if and only if $m \equiv 1 \pmod 6$.*

Proof. See Bennama and Carbonne [2]. □

Lemma 2.3. *Let n and l be integers satisfying $1 \leq l < n$. The Hurwitz curve $H_{n,l}$ is covered by the Fermat curve F_m of degree m where $m = n^2 - nl + l^2$.*

Proof. Consider the following covering map is provided by [1]

$$\begin{array}{rccc} \phi_{n,l}: & F_m & \longrightarrow & H_{n,l} \\ & (u,v) & \longmapsto & \left(u^n v^{-l},\, u^l v^{n-l}\right). \end{array}$$

The image of the Fermat curve of affine equation $F_{n^2-ln+l^2}$ by the morphism $\phi_{n,l}$ include in $H_{n,l}$.

$$\begin{aligned} \left(u^n v^{-l},\, u^l v^{n-l}\right) \in H_{n,l} &\Longrightarrow (u^n v^{-l})^n (u^l v^{n-l})^l + (u^l v^{n-l})^n + (u^n v^{-l})^l = 0 \\ &\Longrightarrow (u^l v^{n-l})^n (u^{n^2-ln+l^2} + v^{n^2-ln+l^2} + 1) = 0 \\ &\Longrightarrow (u,v) \in F_{n^2-ln+l^2}. \end{aligned}$$

Therefore the Hurwitz curve $H_{n,l}$ is covered by the Fermat curve $F_{n^2-ln+l^2}$. □

In the Table 2.1 we have the following correspondence with Hurwitz curve $H_{n,l}$ and Fermat curve F_m where $m = n^2 - nl + l^2$.

TABLE 2.1. Covering map $\phi_{n,l} : F_m \to H_{n,l}$

n	l	m	Hurwitz curve $H_{n,l}$	Fermat curve F_m	Covering map
3	1	7	$H_{3,1}$	F_7	$\left(u^3v^{-1},\, uv^2\right)$
3	2	7	$H_{3,2}$	F_7	$\left(u^3v^{-2},\, u^2v\right)$
4	1	13	$H_{4,1}$	F_{13}	$\left(u^4v^{-1},\, uv^3\right)$
4	3	13	$H_{4,3}$	F_{13}	$\left(u^4v^{-3},\, u^3v\right)$
5	2	19	$H_{5,2}$	F_{19}	$\left(u^5v^{-2},\, u^2v^3\right)$
5	3	19	$H_{5,3}$	F_{19}	$\left(u^5v^{-3},\, u^3v^2\right)$
6	1	31	$H_{6,1}$	F_{31}	$\left(u^6v^{-1},\, uv^5\right)$
6	5	31	$H_{6,5}$	F_{31}	$\left(u^6v^{-5},\, u^5v\right)$

2.3. **Birational maps.**

Suppose that $1 \le l < n$ and $\gcd(n,l) = 1$. Then there exist integers δ and σ verifying

$$1 \le \delta \le l, \quad 1 \le \sigma \le n-1 \quad \text{and} \quad n\delta - \sigma l = 1.$$

Put $\lambda = \sigma n - \delta(n-l) = \sigma(n-l) + \delta l - 1$. We have $1 \le \lambda \le m-2$.

In [2], Bennama and Carbonne show the following proposition :

Proposition 2.1. *The Hurwitz curve $H_{n,l} : x^n y^l + y^n + x^l = 0$ is isomorphic to Fermat quotient curve $C_{\lambda,1}(m) : v^m = u^\lambda (u-1)$.*

Proof. The birational transformation is as follows

$$\begin{array}{rccc} f_{n,l}: & C_{\lambda,1}(m) & \longrightarrow & H_{n,l} \\ & (u,v) & \longmapsto & \left(\dfrac{\left((-1)^{\lambda} v\right)^n}{(-u)^{\sigma}}, \dfrac{\left((-1)^{\lambda} v\right)^l}{(-u)^{\delta}} \right) \end{array}$$

and

$$\begin{array}{rccc} g_{n,l}: & H_{n,l} & \longrightarrow & C_{\lambda,1}(m) \\ & (x,y) & \longmapsto & \left(\dfrac{-x^l}{y^n}, \dfrac{(-1)^{\lambda} x^{\delta}}{y^{\sigma}} \right). \end{array}$$

The composition of applications gives $(g_{n,l} \circ f_{n,l})(u,v) = (u,v)$ and $(f_{n,l} \circ g_{n,l})(x,y) = (x,y)$.

□

The following Table 2.2 shows the correspondence between Hurwitz curve $H_{n,l}$ and Fermat quotient curve $C_{\lambda,1}(m)$ where $m = n^2 - nl + l^2$ and $\lambda = \sigma n - \delta(n-l)$.

TABLE 2.2. Birational map $f_{n,l} : \mathcal{C}_{\lambda,1}(m) \longrightarrow H_{n,l}$

n	l	m	$H_{n,l}$	σ	δ	λ	$C_{\lambda,1}(m)$	$f_{n,l}(u,v)$
3	1	7	$H_{3,1}$	2	1	4	$C_{4,1}(7)$	$\left(\frac{v^3}{u^2}, -\frac{v}{u}\right)$
3	2	7	$H_{3,2}$	1	1	2	$C_{2,1}(7)$	$\left(-\frac{v^3}{u}, -\frac{v^2}{u}\right)$
4	1	13	$H_{4,1}$	3	1	9	$C_{9,1}(13)$	$\left(-\frac{v^4}{u^3}, \frac{v}{u}\right)$
4	3	13	$H_{4,3}$	1	1	3	$C_{3,1}(13)$	$\left(-\frac{v^4}{u}, \frac{v^3}{u}\right)$
5	2	19	$H_{5,2}$	2	1	7	$C_{7,1}(19)$	$\left(-\frac{v^5}{u^2}, -\frac{v^2}{u}\right)$
5	3	19	$H_{5,3}$	2	3	9	$C_{4,1}(19)$	$\left(-\frac{v^5}{u^2}, \frac{v^3}{u^3}\right)$

Remark 2.1. *By combining the Lemma 2.1 and Proposition 2.1, we have*

$$C_{2,1}(7) \cong C_{4,1}(7) \Longrightarrow H_{3,2} \cong H_{3,1}.$$

3. Algebraic points on the curves $C_{1,2}(7)$

3.1. Auxiliary results.

For a divisor D on $\mathcal{C}_{1,2}(7)$, let $\mathcal{L}(D)$ denote the $\overline{\mathbb{Q}}-$vector space of all rational functions f on $\mathcal{C}_{1,2}(7)$ such that $f = 0$ or $\operatorname{div}(f) \geq -D$. Let $l(D)$ be the $\overline{\mathbb{Q}}-$dimension of $\mathcal{L}(D)$, u and v denote the rational functions on $\mathcal{C}_{1,2}(7)$ given by

$$u(U, V, W) = \frac{U}{W} \quad \text{and} \quad v(U, V, W) = \frac{V}{W}.$$

The projective equation of the curve $\mathcal{C}_{1,2}(7)$ is

$$\mathcal{C}_{1,2}(7) : V^7 = W^4 U(U - W)^2.$$

Let $Q_0 = (0, 0, 1)$, $Q_1 = (1, 0, 1)$, $Q_\eta = (\eta, \overline{\eta}, 1)$, $\overline{Q_\eta} = (\overline{\eta}, \eta, 1)$, $Q_\infty = (1, 0, 0)$ and $R_0 = -Q_\eta - \overline{Q_\eta} + 2Q_\infty$ where η is a primitive 6−th root of unity in $\overline{\mathbb{Q}}$ and $\overline{\eta}$ is the complex conjugate of η. The Abel-Jacobi map associated to Q_∞ is the embedding

$$\begin{array}{rccc} j : & \mathcal{C}_{1,2}(7) & \longrightarrow & J_{\mathcal{C}_{1,2}(7)}(\mathbb{Q}) \\ & P & \longmapsto & [P - Q_\infty] \end{array}$$

where $[P - Q_\infty]$ denotes the class of the divisor $P - Q_\infty$. The map j extends by linearity to the divisors of degree 0 : $Div^0(\mathcal{C}_{1,2}(7))$ to $J_{\mathcal{C}_{1,2}(7)}(\mathbb{Q})$ where

$$Div^0(\mathcal{C}_{1,2}(7)) = \left\{ \sum_{i=1}^{n} n_i P_i \;\middle|\; \sum_{i=1}^{n} n_i = 0,\ n \in \mathbb{N}^*,\ n_i \in \mathbb{Z},\ P_i \in \mathcal{C}_{1,2}(7) \right\}.$$

The Abel Jacobi theorem is an important result. A simple version is the following.

Theorem 3.1. *(Abel-Jacobi) The application j is surjective and its kernel is formed by the divisors of functions on $\mathcal{C}$. In other words, for a divisor $D \in Div^0(\mathcal{C})$, there exists $f \in K^*(\mathcal{C})$ such that $div(f) = D$.*

Proof. See Griffiths [6] □

Lemma 3.1. *Let $\mathcal{C}_{1,2}(7)$ be the curve of affine equation $v^7 = u(u-1)^2$. We have*

(1) $J_{\mathcal{C}_{1,2}(7)}(\mathbb{Q}) \cong \mathbb{Z}/7\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.
(2) $J_{\mathcal{C}_{1,2}(7)}(\mathbb{Q}) = \left\{ mj(Q_0) + pR_0 \;\middle|\; 0 \le m \le 6 \text{ and } 0 \le p \le 1 \right\}$.
(3) $div(u) = 7Q_0 - 7Q_\infty$, $div(u-1) = 7Q_1 - 7Q_\infty$ *and* $div(v) = Q_0 + 2Q_1 - 3Q_\infty$.
(4) $7j(Q_0) = 7j(Q_1) = 0$, $\quad j(Q_0) + 2j(Q_1) = 0$, $\quad 2j(R_0) = 0$.

Proof. See Sall [7] □

Lemma 3.2. *The $\overline{\mathbb{Q}}$−basis of the $\mathcal{L}(mQ_\infty)$ on the curve $\mathcal{C}_{1,2}(7)$ for $1 \le m \le 11$ are*

- $\mathcal{L}(Q_\infty) = \mathcal{L}(2Q_\infty) =< 1 >$,
- $\mathcal{L}(3Q_\infty) = \mathcal{L}(4Q_\infty) =< 1, v >$,
- $\mathcal{L}(5Q_\infty) =< 1, v, \frac{v^4}{u-1} >$,
- $\mathcal{L}(6Q_\infty) =< 1, v, \frac{v^4}{u-1}, v^2 >$,
- $\mathcal{L}(7Q_\infty) =< 1, v, \frac{v^4}{u-1} v^2, u >$,
- $\mathcal{L}(8Q_\infty) =< 1, v, \frac{v^4}{u-1}, v^2, u, \frac{v^5}{u-1} >$,
- $\mathcal{L}(9Q_\infty) =< 1, v, \frac{v^4}{u-1}, v^2, u, \frac{v^5}{u-1}, v^3 >$,
- $\mathcal{L}(10Q_\infty) =< 1, v, \frac{v^4}{u-1}, v^2, u, \frac{v^5}{u-1}, v^3, uv >$,
- $\mathcal{L}(11Q_\infty) =< 1, v, \frac{v^4}{u-1}, v^2, u, \frac{v^5}{u-1}, v^3, uv, \frac{v^6}{u-1} >$.

Proof. See Sall [7] □

3.2. **The Main result on $\mathcal{C}_{1,2}(7)$.**

The main result in the curve $\mathcal{C}_{1,2}(7)$ is the following theorem.

Theorem 3.2. *The set of algebraic points of degree at most 3 on the quotient of Fermat curve $\mathcal{C}_{1,2}(7)$ over $\mathbb{Q}$ is $\mathcal{C}_{1,2}(7)^{(3)}(\mathbb{Q}) = \mathcal{M}_1 \cup \mathcal{M}_2 \cup \mathcal{M}_3$ where*

(1) *The set of rational points is $\mathcal{M}_1 = \{Q_0,\ Q_1,\ Q_\infty\}$.*
(2) *The set of quadratic points is $\mathcal{M}_2 = \{Q_\eta,\ \overline{Q_\eta}\}$.*
(3) *The set of cubic points is $\mathcal{M}_3 = \mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}_3 \cup \mathcal{P}_4$ with*

$$\mathcal{P}_1 = \left\{(u,\theta) \;\middle|\; u^3 - 2u^2 + u - \theta^7 = 0,\ \theta \in \mathbb{Q}^*\right\},$$
$$\mathcal{P}_2 = \left\{\left(1+\theta v^{2+\alpha}, v\right) \;\middle|\; v^{3-2\alpha} - \theta^3 v^{2+\alpha} - \theta^2 = 0,\ \theta \in \mathbb{Q}^*,\ \alpha \in \{0,1\}\right\},$$
$$\mathcal{P}_3 = \left\{\left(1+\alpha v^4 + (\alpha-1)v, v\right) \;\middle|\; v^3 + v^2 - 1 = 0,\ \alpha \in \{0,1\}\right\},$$
$$\mathcal{P}_4 = \left\{\left(\alpha v^3 - v^2 + \alpha, v\right) \;\middle|\; v^3 + (-1)^\alpha(2v^2+v) + 1 = 0,\ \alpha \in \{0,1\}\right\}.$$

Proof. Let P be an algebraic point on $\mathcal{C}_{1,2}(7)$ of degree $d \leq 3$ over $\mathbb{Q}$; if $d \leq 2$ these points are described by Faddeev ([5]) and Sall ([7]), so we can assume that $d = 3$. Let P_1, P_2, P_3 be the Galois conjugates of P. Then none of the points P_i is equal to the algebraic points on $\mathcal{C}_{1,2}(7)$ of degree ≤ 2 over $\mathbb{Q}$. We have

$$[P_1 + P_2 + P_3 - 3Q_\infty] \in J(\mathcal{C}_{1,2}(7))(\mathbb{Q})$$

and Lemma 3.1 gives

$$[P_1 + P_2 + P_3 - 3Q_\infty] = mj(Q_0) + pj(R_0) \ \text{ with } \ 0 \leq m \leq 6 \text{ and } 0 \leq p \leq 1. \tag{3.1}$$

The possible combinations for m and p are given in the Table 3.3

TABLE 3.3. combinations for m and p

m	0	1	2	3	4	5	6	0	1	2	3	4	5	6
p	0	0	0	0	0	0	0	1	1	1	1	1	1	1

We distinguish 14 cases to study.

Case 1 : $m = 0$ and $p = 0$.

The formula (3.1) becomes $[P_1 + P_2 + P_3 - 3Q_\infty] = 0$. The Abel-Jacobi Theorem 3.1 implies the existence of a rational function f defined over $\mathbb{Q}$ such that

$$\text{div}(f) = P_1 + P_2 + P_3 - 3Q_\infty.$$

So $f \in \mathcal{L}(3P_\infty)$, hence $f = a_0 + a_1 v$ with $a_i \neq 0$. At points P_i, we have $a_0 + a_1 v = 0$ so $v = -\frac{a_0}{a_1} \in \mathbb{Q}^*$. By putting $v = \theta$ in the equation $v^7 = u(u-1)^2$ we have

$$u^3 - 2u^2 + u - \theta^7 = 0.$$

So we have the set of family cubic points

$$\mathcal{P}_1 = \left\{(u, \theta) \mid u^3 - 2u^2 + u - \theta^7 = 0,\, \theta \in \mathbb{Q}^*\right\}.$$

Case 2 : $m = 1$ and $p = 0$:

The relation (3.1) becomes $[P_1 + P_2 + P_3 - 3Q_\infty] = [Q_0 - Q_\infty] = -6[Q_0 - Q_\infty]$. This means

$$[P_1 + P_2 + P_3 + 6Q_0 - 9Q_\infty] = 0.$$

There exists a function f such that

$$\operatorname{div}(f) = P_1 + P_2 + P_3 + 6Q_0 - 9Q_\infty.$$

Therefore $f \in \mathcal{L}(9Q_\infty)$, hence

$$f = a_0 + a_1 v + a_2 \frac{v^4}{u-1} + a_3 v^2 + a_4 u + a_5 \frac{v^5}{u-1} + a_6 v^3.$$

The function f is of order 6 at the point Q_0, so $a_0 = a_1 = a_2 = a_3 = a_5 = a_6 = 0$, thus $f = a_4 u$. At points P_i, $a_4 u = 0$, hence $a_4 = 0$ or $u = 0$ which is absurd.

Cases 1 to 14 : By similar reasoning to the two previous cases, the results obtained can be summarized in the Table 3.4.

TABLE 3.4. Summary of solutions for all cases

m	p	Set of cubic points	m	p	Set of cubic points
0	0	$\mathcal{P}_1$	0	1	$\mathcal{P}_3$ with $\alpha = 0$
1	0	Absurd	1	1	$\mathcal{P}_3$ with $\alpha = 1$
2	0	$\mathcal{P}_2$ with $\alpha = 0$	2	1	$\mathcal{P}_4$ with $\alpha = 0$
3	0	$\mathcal{P}_2$ with $\alpha = 1$	3	1	Absurd
4	0	Absurd	4	1	Absurd
5	0	absurd	5	1	$\mathcal{P}_4$ with $\alpha = 1$
6	0	Absurd	6	1	Absurd

□

4. Algebraic points of low degree on Hurwitz curve

In this section we use birational maps to give algebraic points of low degree on $H_{3,2}$.

4.1. **Preliminary results.**

Lemma 4.1. *If two curves $\mathcal{X}$ and $\mathcal{Y}$ defined over a number field K are birationally equivalent then $\mathcal{X}$ is isomorphic to $\mathcal{Y}$ and $\mathcal{X}(K) \cong \mathcal{Y}(K)$.*

Proof. See Perrin [8]. □

4.2. **Main result on the Hurwitz curve $H_{3,2}$.**

Let $P_0 = (0,0)$, $\infty_- = (1,0)$ and $\infty_+ = (0,1)$. The main result is the following theorem :

Theorem 4.1. *Let $H_{3,2}^{(3)}(\mathbb{Q})$ be the set of algebraic points of degree at most 3 on the Hurwitz curves $H_{3,2}$ over $\mathbb{Q}$, then $H_{3,2}^{(3)}(\mathbb{Q}) = \mathcal{L}_1 \cup \mathcal{L}_2 \cup \mathcal{L}_3$ where*

1. *The set of rational points is $\mathcal{L}_1 = \{P_0, \infty_-, \infty_+\}$*
2. *The set of quadratic points is $\mathcal{L}_2 = \{(-\eta, -\overline{\eta}), (-\overline{\eta}, -\eta)\}$*
3. *The set of cubic points is $\mathcal{L}_3 = \mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3 \cup \mathcal{G}_4$ with*

$$\mathcal{G}_1 = \left\{ \left(\frac{\theta^3}{1-u}, \frac{\theta^2}{u-1} \right) \mid u^3 - 2u^2 + u - \theta^7 = 0,\ \theta \in \mathbb{Q}^* \right\},$$
$$\mathcal{G}_2 = \left\{ \left(-\frac{v^{1-\alpha}}{\theta}, \frac{v^{-\alpha}}{\theta} \right) \mid v^{3-2\alpha} - \theta^3 v^{2+\alpha} - \theta^2 = 0,\ \theta \in \mathbb{Q}^*,\ \alpha \in \{0,1\} \right\},$$
$$\mathcal{G}_3 = \left\{ \left(\frac{v^2}{1-\alpha-\alpha v^3}, -\frac{v}{1-\alpha-\alpha v^3} \right) \mid v^3 + v^2 - 1 = 0,\ \alpha \in \{0,1\} \right\},$$
$$\mathcal{G}_4 = \left\{ \left(\frac{v^3}{1-\alpha+v^2-\alpha v^3}, -\frac{v^2}{1-\alpha+v^2-\alpha v^3} \right) \mid v^3 + (-1)^\alpha (2v^2 + v) + 1 = 0,\ \alpha \in \{0,1\} \right\}.$$

Proof.

- The Remark 2.1 gives $H_{3,2} \cong C_{2,1}(7)$ and by using theorem 3.2, we have $\#H_{3,2}(\mathbb{Q}) = 3$. An elementary search give us the set

$$\mathcal{L}_1 = \{P_0, \infty_-, \infty_+\}.$$

- We use birational maps to determine the quadratic and cubic points on the curve $H_{3,2}$. Let

$$\begin{array}{rccc} \varphi : & C_{1,2}(7) & \longrightarrow & C_{2,1}(7) \\ & (u,v) & \longmapsto & (1-u, -v) \end{array}$$

and

$$\begin{array}{rccc} \psi : & C_{2,1}(7) & \longrightarrow & H_{3,2} \\ & (u,v) & \longmapsto & \left(-\frac{v^3}{u}, -\frac{v^2}{u}\right). \end{array}$$

Thus, we have

$$\begin{array}{rccc} \psi\circ\varphi: & C_{1,2}(7) & \longrightarrow & H_{3,2} \\ & (u,v) & \longmapsto & (\frac{v^3}{1-u}, -\frac{v^2}{1-u}). \end{array}$$

(a) The set of quadratic points on $H_{3,2}$ are given by

$$\mathcal{L}_2 = (\psi\circ\varphi)(\mathcal{M}_2).$$

We obtain

$$\mathcal{L}_2 = \Big\{(\psi\circ\varphi)(\eta,\overline{\eta}),\ (\psi\circ\varphi)(\overline{\eta},\eta)\Big\} = \Big\{(-\eta,-\overline{\eta}),\ (-\overline{\eta},-\eta)\Big\}.$$

(b) The set of cubic points on $H_{3,2}$ are given by $\mathcal{G}_1\cup\mathcal{G}_2\cup\mathcal{G}_3\cup\mathcal{G}_4$ with

$$\mathcal{G}_i = (\psi\circ\varphi)(\mathcal{P}_i), \qquad \text{for } i\in\{1,\,2,\,3,\,4\}.$$

We obtain

$$\mathcal{G}_1 = \Big\{\Big(\frac{\theta^3}{1-u},\frac{\theta^2}{u-1}\Big)\,|\,u^3-2u^2+u-\theta^7=0,\ \theta\in\mathbb{Q}^*\Big\};$$

$$\mathcal{G}_2 = \Big\{\Big(-\frac{v^{1-\alpha}}{\theta},\frac{v^{-\alpha}}{\theta}\Big)\,|\,v^{3-2\alpha}-\theta^3v^{2+\alpha}-\theta^2=0,\ \theta\in\mathbb{Q}^*,\ \alpha\in\{0,1\}\Big\};$$

$$\mathcal{G}_3 = \Big\{\Big(\frac{v^2}{1-\alpha-\alpha v^3},-\frac{v}{1-\alpha-\alpha v^3}\Big)\,|\,v^3+v^2-1=0,\ \alpha\in\{0,1\}\Big\};$$

$$\mathcal{G}_4 = \Big\{\Big(\frac{v^3}{1-\alpha+v^2-\alpha v^3},-\frac{v^2}{1-\alpha+v^2-\alpha v^3}\Big)\,|\,v^3+(-1)^\alpha(2v^2+v)+1=0,\ \alpha\in\{0,1\}\Big\}.$$

□

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