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PROPERTIES OF DIVISOR PRIME GRAPH

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Abstract. Number theory is a mathematical discipline that uses concepts from graph theory. Recently, various graphs have been defined in relation to various number theoretic functions. One such graph is the divisor prime graph, which is associated with the positive divisors of a positive integer. Let n be a positive integer and $D(n)$ be the set of all positive divisors of n. The *divisor prime graph* $PG_D(n)$ is defined as a graph whose vertex set is $D(n)$ and any two vertices x and y are adjacent in $PG_D(n)$ iff $\gcd(x, y) = 1$. In this study, families of divisor prime graphs for different positive integers are investigated, along with their graph theoretic characteristics such as adjacency, diameter, radius, clique number, chromatic number, planarity, connectivity, independence number and density.

Keywords: Divisor, Prime factor, Greatest common divisor, Connectedness, Diameter, Girth, Radius, Isomorphism, Planar graph.

2010 Mathematics Subject Classification: 05C38, 05C90, 05C4.

1. Introduction

In 2000, Singh and Santhosh [15] introduced the idea of divisor graphs. A divisor graph G is an ordered pair (V, E) where V is a subset of $\mathbb{Z}$ and $uv \in E$ if and only if either $u|v$ or $v|u$ for all $u \neq v$. Many authors had studied an alternative construction of graphs by associating with algorithmic approach on MV-algebras[9], Zero divisor graphs[1, 2], total graphs, prime graphs[14]. Any graph isomorphic to a divisor graph is also called a divisor graph. Additionally, they have pointed out some of the graphs those which are divisor

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graphs and also those which are not. Chartrand et al. [6] studied many more additional properties of divisor graph. Le Anh Vinh [17] also established the existence of a divisor graph of order n and size m for every pair of $m, n \in \mathbb{Z}$ with $0 \leq m \leq n$. Christopher Frayer [8] conducted research on the necessary conditions for a Cartesian product graph to be a divisor graph. Yu-ping Tsao [16] has examined several properties of $D([n])$ and its complement, including the vertex-chromatic number, the clique number, the cover number, and independence number, where $[n] = \{i : 1 \leq i \leq n,\ n \in N\}$. Nathanson [13] introduced the concepts and the notion of congruences from number theory in Graph Theory. He initiated the new way for the emergence of a new class of graphs, namely, arithmetic graphs. An arithmetic graph is one in which any two vertices a and b are adjacent if and only if $a + b \equiv c(mod\ n)$ where $c \in S$, a pre-assigned subset of V. Its vertex set V is the set of the first n positive integers $1, 2, 3, \ldots, n$. Let $(G,\ \cdot)$ be a finite group and $S \subseteq G$ such that $s^{-1} \in S$ for all $s \in S$. S is called symmetric subset of G. A Cayley graph $C(G, S)$ is the graph in which the vertex set $V = G$ and the edge set $E = \{(a, b) : a^{-1}b \in S\ or\ b^{-1}a \in S,\ \forall a,\ b \in G\}$. If $(G,\ \cdot) = (\mathbb{Z}_n, +)$ and the symmetric set S is associated with some arithmetic function, then such Cayley graphs are called arithmetic Cayley graphs. Dejter and Giudici [7], Berrizabeitia and Giudici [3] and others have studied the cycle structure of Cayley graphs associated with certain arithmetic functions. The circumference and girth of the arithmetic Cayley graphs are investigated by Madhavi and Maheswari [11], associated with the Euler totient function $\phi(n)$, and divisor function $d(n)$. The cycle structure of these graphs has many applications in engineering and communication networks. Chalapati, Madhavi and Venkataramana [5] studied the enumeration of triangles in these graphs. The Divisor Prime graph was a novel idea developed by S. M. Nair and J. S. Kumar[12], who also looked into its structural characteristics. They integrated the concepts of prime graphs and divisor function graphs in this follow-up. In that study, maximum and minimum degrees, a null graph, an Euler graph, a cycle, a complete graph, and a bipartite graph are examined for a divisor prime graph.

In this paper, we have studied the properties of the divisor prime graph, its diameter, girth, colorability, planarity, density, etc. Any number theory or graph theory terms can be looked up in [4, 10], or any other standard literature.

2. Preliminaries

A *graph* is a set of objects represented graphically, with links connecting some pairs of objects. The points that represent the connected objects are referred to as *vertices or points*, and the links that join the vertices are called *edges or lines*. The majority of the definitions we have included here come from scholarly articles and standard literature.

A *graph* G is a pair of set $G = (V, E)$, where V is a set of all vertices or points and E is a set of all edges or lines, connecting the vertices.

A graph is called *connected* if there is a path between every pair of vertices, otherwise, it is a *disconnected* graph.

In a graph, two vertices are said to be *adjacent* if they are connected by a common edge and two edges are said to be adjacent, if there is a common vertex between the two edges.

The *degree* of vertex in a graph is defined as the number of edges incident to the vertex, say v, or the number of vertices that is adjacent to the vertex v. It is denoted by $deg(v)$. The minimum and maximum degrees of a graph G is denoted by $\delta(G)$ and $\Delta(G)$.

If all the vertices in the graph have the same degree, then the graph is called a *regular* graph. If k is the degree of the vertex, then the graph is called a k-regular graph. A connected 2-regular graph is also called a cycle graph.

A graph is said to be *complete* if each and every vertex is connected to each other. A complete graph of n vertices (i.e K_n) is a $(n-1)$- regular graph. A graph $G = (V, E)$ whose vertices can be partitioned into two disjoint and independent sets $V = V_1 \cup V_2$ such that every edge of E connects a vertex in V_1 to a vertex in V_2 is called a *bipartite* graph. A bipartite graph in which each vertex of the first set is connected to every vertex of the second set is called a *complete bipartite* graph. A *star* graph is a complete bipartite graph of the form $K_{1,n-1}$ with n-vertices, i.e., one set will have only one vertex and all the remaining vertices belong to the other set, and all these vertices are adjacent to that single vertex and not to each other. A star graph with n vertices is denoted by S_n. A graph where the degree of all its vertices is 0 is called a *null* graph and a graph where there is only one point (thus degree=0) is called a *trivial* graph.

A *walk* of a graph is an alternating sequence of points and lines beginning and ending with points, where each line is incident with the two points immediately preceding and following it. If all the lines of a walk are distinct, then it is called a *trail* and if all the points are distinct, then it is called a *path*.

The *distance* between the two vertices is the length of a geodesic between that pair of vertices. Distance between a pair of vertices u and v is denoted by $d(u, v)$. The maximum distance of a vertex, say v, from all the other vertex is called the *eccentricity* of a vertex. It is denoted by $e(v)$. The minimum eccentricity of all the vertices in the graph is considered the *radius* of the graph. It is denoted by $r(G)$. The maximum eccentricity of all the vertices in the graph is considered the *diameter* of the graph. It is denoted by $d(G)$.

Simple graphs G and H are called *isomorphic* if there is a bijection f from the vertices of G to the vertices of H such that (v, w) is an edge in G if and only if $(f(v), f(w))$ is an edge of H.

A simple, connected graph is called *planar* if there is a way to draw it on a plane so that no edges cross. Such a drawing is called an *embedding* of the graph in the plane.

The *Girth* of a simple graph is the shortest cycle contained in the graph and if there is no cycle in the graph then its girth is undefined. A complete subgraph in a graph is often called a *clique*. A clique having n number of vertices is called $n - clique$. The size of the largest clique of a graph G is called the *clique number* of G. It is denoted by $cl(G)$.

A subset I of V is an independent set of a graph $G = (V, E)$ if the vertices in I are not adjacent to each other. The independence number $\beta_0(G)$ is the size of a largest independent set in G.

The *divisor function* or *Tau function*, is a number-theoretic function that counts the positive divisors of an integer n. It is represented by the symbol $\tau(n)$. In the prime factorization of n, it can be written as the product of one and the exponent of each prime factor. The Tau function can be found mathematically for a positive integer n with prime factorization $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$, where $p_1, p_2, \ldots, p_k$ are distinct prime numbers and $a_1, a_2, \ldots, a_k$ are positive integers representing the exponents. Then $\tau(n) = (a_1 + 1)(a_2 + 1) \cdots (a_k + 1)$.

Numerous branches of number theory, such as the study of perfect numbers, integer sequences, and cryptography, employ the Tau function. In conclusion, the Tau function counts the number of prime factors, while prime factorization is the process of breaking down a positive integer into its prime factors.

3. Properties of Divisor Prime Graph

The divisior prime graph presents a pictorial view of the relation between the positive divisors of a natural number n. We expect that the investigation of the theoretical properties of these graphs can help to determine some number theoretic properties of these numbers.

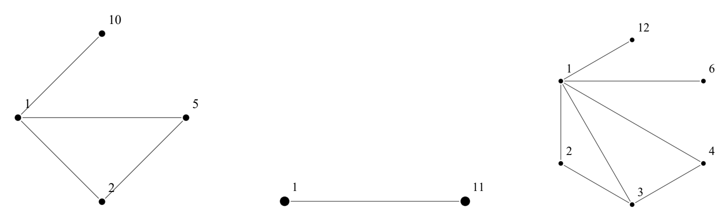
In this section, we discuss some properties of the divisor prime graph $PG_D(n)$ for $n \in \mathbb{Z}^+$ like diameter, girth, radius, clique number, planarity, etc.

Let us start the with the formal definition of the *divisor prime graph* PG_{D_n}.

Definition 3.1. (Divisor Prime Graph)[12]

Let $n \in \mathbb{Z}^+$ and $D(n) = \{m \in \mathbb{Z}^+ : m|n\}$. The divisor prime graph $PG_D(n)$ is defined as a graph with the vertex set $D(n)$ and any two vertices x and y are adjacent in $PG_D(n)$ if $\gcd(x, y) = 1$.

Example 3.1. *The divisor prime graphs for $n = 10, 11, 12$ are shown in figure* **1**.



(a) (b) (c)

FIGURE 1. (a) $PG_D(10)$ (b) $PG_D(11)$ (c) $PG_D(12)$

Theorem 3.1. [12] *For all $n \in \mathbb{Z}^+$, $PG_D(n)$ is connected.*

Proof. Since for every $n \in \mathbb{Z}^+$, $1 \in D(n)$ and $\gcd(1, d) = 1$ for all $d \in D(n)$, so the vertex 1 is adjacent to every vertex d in $D(n)$. Hence $PG_D(n)$ is connected for all n. □

Theorem 3.2. [12] *For all $n \in \mathbb{Z}^+$,*

$$\Delta(PG_D(n)) = \tau(n) - 1 \quad \textit{and} \quad \delta(PG_D(n)) = 1.$$

Moreover, $\Delta(PG_D(n)) = \delta(PG_D(n))$ if $n = 1$ or n is a prime.

Proof. For every $n \in \mathbb{Z}^+$, let $\tau(n)$ is the number of divisor of n. The vertex 1 is adjacent to all the vertices $d \neq 1$ of $PG_D(n)$ and we have $deg(1) = \tau(n) - 1$, which is maximum possible degree for any graph with $\tau(n)$ vertices. Thus $\Delta(PG_D(n)) = \tau(n) - 1$.

Also $\gcd(n, d) \neq 1$ for all divisors $d \neq 1$ of n and $\gcd(n, 1) = 1$, so n is adjacent only to 1 and since $PG_D(n)$ is connected so there is no isolated vertex. Thus $\delta(PG_D(n)) = 1$.

Since for n prime $PG_D(n) \cong K_2$, thus $\Delta(PG_D(n)) = \delta(PG_D(n))$. □

Theorem 3.3. [12] *$PG_D(n)$ is non-eulerian for all n.*

Theorem 3.4. *For all $n \in \mathbb{Z}^+$, $Diam(PG_D(n)) \leq 2$.*

Proof. If $n = 1$ then $PG_D(n) \cong K_1$ and $Diam(PG_D(n)) = 0$.

If n is prime then $PG_D(n) \cong K_2$ and $Diam(PG_D(n)) = 1$.

It is clear from theorem **3.1** that for composite n, any two non adjacent vertices u and v in $PG_D(n)$, $u - 1 - v$ is always the shortest $u - v$-path. So $Diam(PG_D(n)) = 2$.

So we can conclude that $Diam(PG_D(n)) \leq 2$. □

Theorem 3.5. *For all $n \in \mathbb{Z}^+$, $rad(PG_D(n)) = 1$.*

Proof. Since $PG_D(n)$ is connected and $Diam(PG_D(n)) \leq 2$ by theorems **3.1** and **3.4**, the ecentricity $1 \leq e(v) \leq 2$, $\forall\ v \in V(PG_D(n))$, therefore $rad(PG_D(n)) = min\{e(v) : v \in V(PG_D(n))\} = 1$. □

Theorem 3.6. *For all $n \in \mathbb{Z}^+$, let k be the number of distinct prime divisors of n, then $cl(PG_D(n)) = k + 1$.*

Proof. Let $p_i,\ i = 1, 2, \ldots, k$ be the distinct prime divisors of n. Since for distinct i and j, $\gcd(p_i, p_j) = 1$, so p_i *adjacent* p_j . Thus the vertices $1, p_1, p_2, \ldots, p_k$ induced a complete subgraph of order $k + 1$.

Let $v = p_i p_j$ be a vertex of $PG_D(n)$, then $\gcd(v, p_i) \neq 1$, $\gcd(v, p_j) \neq 1$ but $\gcd(v, p_r) = 1$, $\forall r \neq i, j$ which implies that the vertex v adjacent to all p_r, $r \neq i, j$. So the vertices $1, p_1, p_2, \ldots, p_{i-1}, p_{i+1}, \ldots, p_{j-1}, p_{j+1}, \ldots, p_k, v$ will induce a complete subgraph of order k. That is any set of vertices more than $k + 1$ cannot induce a complete graph. Thus the complete subgraph induced by the vertices $1, p_1, p_2, \ldots, p_k$ is the maximal clique in $PG_D(n)$. Hence $cl(PG_D(n)) = k + 1$. □

Theorem 3.7. *For all $n \in \mathbb{Z}^+$, $girth(PG_D(n)) = 3$ or ∞.*

Proof. If $n = 1$ then $girth(PG_D(n)) = \infty$.

If n is prime then $PG_D(n) \cong K_2$ and $girth(PG_D(n)) = \infty$.

If $n = p^k$ where $k \in \mathbb{Z}^+$, then $PG_D(n) \cong K_{1,\tau(n)-1}$ and so $girth(PG_D(n)) = \infty$.

If n has more than one prime divisors, it is clear from theorem **3.6** that $PG_D(n)$ always contain a cycle of length 3. So $girth(PG_D(n)) = 3$.

So we can conclude that $girth(PG_D(n)) = 3$ *or* ∞. □

We obtained some counter examples to the statement in *Theorem 2.5* given by Nair and Kapur in [12]. So we modified the theorem and provided a proof and an example supporting our result.

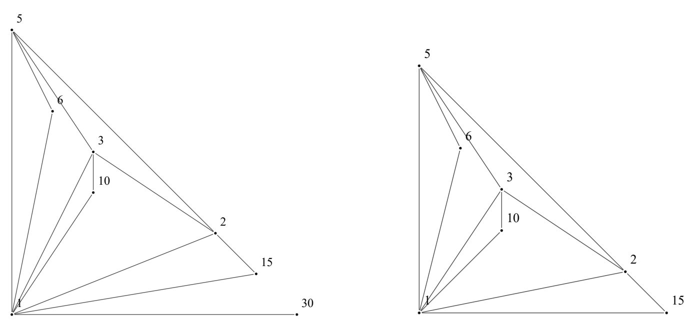
Theorem 3.8. *For any $k \in \mathbb{Z}^+$, if $n = p_1p_2 \cdots p_k$, then $PG_D(n) - n$ can not be a complete graph.*

Proof. For $n = p_1p_2 \cdots p_k$,

$D(n) = \{1, p_1, p_2, \cdots, p_k, p_1p_2, \cdots p_1p_k, p_1p_2 \ldots, p_{k-1}, p_1p_2, p_1p_2 \ldots p_k\}$.

So $V(PG_D(n)) - n = \{1, p_1, p_2, \cdots, p_k, p_1p_2, \cdots p_1p_k, p_1p_2 \ldots, p_{k-1}, \cdots p_2, p_2 \ldots p_k\}$.

The vertices $1, p_1, p_2, \cdots, p_k$ will induce a complete subgraph. But the vertex set of the graph contains more vertices in the form of product of primes. These vertices are adjacent to vertex 1 as well as to some of the vertices of $p_1, p_2, \cdots, p_k$ but not to all. Hence $PG_D(n) - n$ is not a complete graph. □



(a) (b)

FIGURE 2. (a) $PG_D(30)$ (b) $PG_D(30) - 30$

Following the theorems **3.6 and 3.8** finally we can give the following result.

Theorem 3.9. *Let $n_1 = p_1^{\alpha_1}p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ and $n_2 = q_1^{\alpha_1}q_2^{\alpha_2} \cdots q_k^{\alpha_k}$, then $PG_D(n_1) \cong PG_D(n_2)$.*

Proof. Since $n_1 = p_1^{\alpha_1}p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ and $n_2 = q_1^{\alpha_1}q_2^{\alpha_2} \cdots q_k^{\alpha_k}$, so

$$\tau(n_1) = \tau(n_2) = (\alpha_1 + 1)(\alpha_2 + 1) \cdots (\alpha_k + 1).$$

That is $|V(PG_D(n_1))| = |V(PG_D(n_2))|$.

Let us consider the mapping $f : D(n_1) \to D(n_2)$ defined by

$$f(p_1^{r_1}p_2^{r_2} \cdots p_k^{r_k}) = q_1^{r_1}q_2^{r_2} \cdots q_k^{r_k}$$

where $0 \leq r_i \leq \alpha_i$ for each $i = 1, 2, \ldots, k$.

So f is a one-one correspondence from $D(n_1)$ onto $D(n_2)$, *i.e.* f is a one-one correspondence from $V(PG_D(n_1))$ onto $V(PG_D(n_2))$.

It is now sufficient to prove that f preserves the adjacency from $PG_D(n_1)$ to $PG_D(n_2)$ *i.e.* $(a,b) \in E(PG_D(n_1)) \Leftrightarrow (f(a), f(b)) \in E(PG_D(n_2))$.

If possible, let $(a,b) \in E(PG_D(n_1))$ but $(f(a), f(b)) \notin E(PG_D(n_2))$.

Then there exists at least one q_i such that $q_i | \gcd(f(a), f(b))$

$\Rightarrow q_i|f(a) \textit{ and } q_i|f(b)$

$\Rightarrow \exists\ p_i \textit{ such that } f(p_i) = q_i \textit{ and } p_i|a \textit{ and } p_i|b$

$\Rightarrow\ p_i| \gcd(a,b)$

which is a contradiction to the fact that $(a,b) \in E(PG_D(n_1))$.

Thus $(a,b) \in E(PG_D(n_1)) \Leftrightarrow (f(a), f(b)) \in E(PG_D(n_2))$ for all $a, b \in V(PG_D(n_1))$. Hence $PG_D(n_1) \cong PG_D(n_2)$. □

Theorem 3.10. *Let $n \in \mathbb{Z}^+$ then $PG_D(n)$ is planar if n is any one of the following form $p^k,\ p^k q,\ p^2q^2$ or pqr, where $p,\ q,\ r$ are primes and k is nonzero positive integer.*

Proof. If $n = 1$ then $PG_D(n)$ is trivial and so is planar.

If n is prime then $PG_D(n) \cong K_2$ and $PG_D(n)$ is planar.

If $n = p^k$ where $k \in \mathbb{Z}^+$, then $PG_D(n) \cong K_{1,\tau(n)-1}$ and so $PG_D(n)$ is planar as shown in figure **3(a)**.

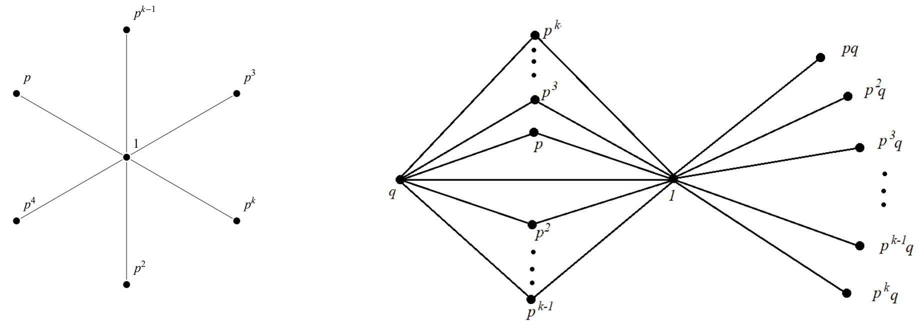
If $n = p^k q$, the vertices are $1, p, p^2, p^3, \cdots p^{k-1}, p^k, q, pq, p^2q, p^3q, \cdots p^{k-1}q, p^kq$ and the graph $PG_D(n)$ which is clearly a planar graph as shown in figure **3(b)**.

If $n = p^2q^2$, $1, p, p^2, q, q^2, pq, p^2q, pq^2, p^2q^2$ are the only vertices and from the figure **4(a)** it is clear that the graph $PG_D(n)$ is planar graph.

If $n = pqr$, the vertices are $1,\ p,\ q,\ r,\ pq,\ qr,\ pr,\ \textit{and}\ pqr$ and the graph $PG_D(n)$ which is clearly a planar graph as shown in figure **4(b)**.

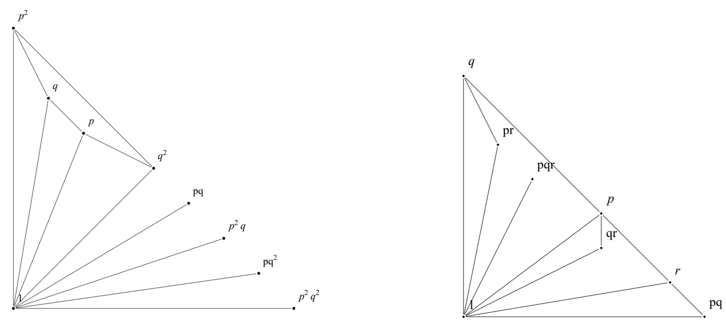
If $n = p^3q^2$, then the vertices $1, p, p^2, p^3, q, q^2$ together give $K_{3,3}$ as a induced subgraph of $PG_D(p^3q^2)$ because of which we can conclude that the graph $PG_D(p^3q^2)$ is not a planar graph. Thus for all $n = p^iq^j\ \forall i,\ j > 2$ the graph $PG_D(p^iq^j)$ is not a planar graph.

If $n = p^2qr$, then the vertices $1, p, p^2, q, r$ together give the induce subgraph K_5 of $PG_D(p^2qr)$ and so the graph $PG_D(p^2qr)$ is not a planar graph. Thus for all $n = p^iq^jr^k\ \forall i,\ j,\ k > 1$ the graph $PG_D(p^iq^jr^k)$ is not a planar graph.



(a) (b)

FIGURE 3. (a) $PG_D(p^k)$ (b) $PG_D(p^kq)$



(a) (b)

FIGURE 4. (a) $PG_D(p^2q^2)$ (b) $PG_D(pqr)$

It is clear from theorem **3.6** that if n has more than 3 distinct prime factors than $PG_D(n)$ has a clique of order greater than 4. Which implies that $PG_D(n)$ for n with more than 3 prime factors is not planar.

Hence we can conclude that $PG_D(n)$ is planar only for the values $n = 1,\ p^k,\ p^kq,\ p^2q^2$ and pqr, where $p,\ q,\ r$ are distinct primes and k is a nonzero positive integer. □

Theorem 3.11. *For any* $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, $\chi(PG_D(n)) = k + 1$ *where* $\alpha_i \geq 0$.

Proof. We know that for any graph G, $cl(G) \leq \chi(G)$. From theorem **3.6** it is clear that $k + 1 = cl(PG_D(n)) \leq \chi(PG_D(n))$.
Let us assign $k+1$ colors to the vertices $1, p_1, p_2, \cdots p_k$ of the maximal clique. Any of the remaining vertices, say d, will have at least one of the primes $p_1, p_2, \cdots p_k$ as a factor, say p_i. Then $\gcd(d, p_i) = p_i$ and so $(d, p_i) \notin E(PG_D(n))$. Thus d can be assigned with the same color assigned to p_i. Let d_i and d_j be any two vertices such that $p_i|d_i$ and $p_j|d_j$,then they can

be assigned the colors of p_i and p_j respectively. Now let d be a vertex such that $p_ip_j|d$ then d is not adjacent to any of p_i, p_j, d_i and d_j, so we can assign any one color from these vertices to the vertex d. Proceeding with the same argument we can have a proper $(k+1)$- coloring of $PG_D(n)$. Thus $\chi(PG_D(n)) \leq k+1$. Hence we can conclude that $\chi(PG_D(n)) = k+1$. □

Theorem 3.12. *Let* $n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}$ *and* $\alpha_r = max\{\alpha_i : 1 \leq i \leq k\}$. *Then*

$$\beta_0(PG_D(n)) = (\tau(p_r^{\alpha_r}) - 1)(\tau(n/p_r^{\alpha_r})).$$

Proof. Here $n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}$ and $\alpha_r = max\{\alpha_i : 1 \leq i \leq k\}$.
It is easy to see that each of the sets $I_i = \{p_i^s d : 1 \leq s \leq \alpha_i, d|\Pi_{j\neq i}p_j\}$ is an independent set of $PG_D(n)$. The number of elements in I_i is given by

$$|I_i| = \tau(n) - (\tau(n/p_i^{\alpha_i})) = (\tau(p_i^{\alpha_i}) - 1)\tau(n/p_i^{\alpha_i}).$$

To prove the result it is sufficient to show that

$$|I_r| = max\{|I_i| : 1 \leq i \leq k\} = (\tau(p_r^{\alpha_r}) - 1)\tau(n/p_r^{\alpha_r}).$$

Now for $1 \leq i \leq k$ and $i \neq r$, $\tau(p_i^{\alpha_i}) < \tau(p_r^{\alpha_r})$

$$\Rightarrow \tau(p_i^{\alpha_i}n') < \tau(p_r^{\alpha_r}n'), \text{ whhere } n' = \Pi_{j\neq i,r}p_j$$
$$\Rightarrow \tau(n) - \tau(p_i^{\alpha_i}n') > \tau(n) - \tau(p_r^{\alpha_r}n')$$
$$\Rightarrow \tau(n) - \tau(n/p_r^{\alpha_r}) > \tau(n) - \tau(n/p_i^{\alpha_i})$$
$$\Rightarrow (\tau(p_r^{\alpha_r}) - 1)\tau(n/p_r^{\alpha_r}) > (\tau(p_i^{\alpha_i}) - 1)\tau(n/p_i^{\alpha_i})$$
$$\Rightarrow |I_r| > |I_i| \text{ for all } 1 \leq i \leq k \text{ and } i \neq r$$
$$\Rightarrow |I_r| = max\{|I_i| : 1 \leq i \leq k\}$$

Hence $\beta_0(PG_D(n)) = (\tau(p_r^{\alpha_r}) - 1)\tau(n/p_r^{\alpha_r})$. □

Theorem 3.13. $PG_D(n)$ *is non-hamiltonian for all* n.

Proof. $PG_D(n)$ is non-hamiltonian as $deg(n) = 1$ in $PG_D(n)$ and there can not exists any hamiltonian cycle in $PG_D(n)$. □

A subset D of V is a dominating set for a graph $G = (V, E)$ if every vertex in $V - D$ is adjacent to at least one member of D. The domination number $\gamma(G)$ is the size of a smallest dominating set for G.

Theorem 3.14. *Domination number of* $PG_D(n)$ *is* 1 *for all* n.

Proof. It is clear from the definition that for the set $D = 1 \subset V(PG_D(n))$, every vertex $v \in V(PG_D(n)) - 1$ is adjacent to 1. That is 1 is a dominating set for $PG_D(n)$ and is the smallest dominating set. Hence $\gamma(PG_D(n)) = 1$. □

A subset C of V is a dominating cut vertex set for a graph $G = (V, E)$ if $G - C$ is either a disconnected graph or a trivial graph. The point connectivity $\kappa(G)$ is the size of a smallest cut vertex set for G. A subset C' of E is a cut set for a graph $G = (V, E)$ if $G - C'$ is a disconnected graph. The line connectivity $\lambda(G)$ is the size of a smallest cut set for G.

Theorem 3.15. *Both point and line connectivity of $PG_D(n)$ is 1 for all n.*

Proof. It is clear from the definition of $PG_D(n)$ that $PG_D(n) - 1$ is a disconnected graph, so $\kappa(PG_D(n)) = 1$. Also $PG_D(n) - (1, n)$ is always disconnected, so $\lambda(PG_D(n)) = 1$. □

Theorem 3.16. *For all prime p, $Den(PG_D(p)) = 1$.*

Proof. Since for every prime p, $PG_D(p) \cong K_2$ so $Den(PG_D(p)) = 1$. □

Theorem 3.17. *For all prime p and $k \in \mathbb{Z},\ k \geq 0$, $Den(PG_D(p^k)) = \dfrac{2}{k+1}$.*

Proof. For $n = p^k$, $PG_D(p^k)$ has $k+1$ vertices and k edges. So $Den(PG_D(p^k)) = \dfrac{k}{{}^{k+1}C_2} = \dfrac{k}{\{(k+1)k\}/2} = \dfrac{2}{k+1}$. □

Theorem 3.18. *Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ then $Den(PG_D(n)) = \dfrac{\sum_{d_i|n} \tau(\frac{n}{d_i})}{2 \cdot {}^{\tau(n)}C_2}$.*

Proof. For each divisor d_i of n, $deg(d_i) = \tau(\frac{n}{d_i})$. So $o(E(PG_D(n))) = \dfrac{1}{2} \sum_{d_i|n} \tau\left(\dfrac{n}{d_i}\right)$.

Hence $Den(PG_D(n)) = \dfrac{\sum_{d_i|n} \tau(\frac{n}{d_i})}{2 \cdot {}^{\tau(n)}C_2}$. □

4. Conclusion

In this study, we have investigated the nature and characteristics of the divisor prime graph. The adjacency, diameter, radius, clique number, chromatic number, planarity, connectivity, independence number, and domination number features of the divisor prime graph have also been investigated. Since this is a preliminary investigation of divisor prime graphs, the reader may be thinking about various issues. Studying the energy and distance eigenvalues of the divisor prime graph can reveal some potential problems.

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