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ON GENERALIZED SASAKIAN SPACE FORMS WITH CONCIRCULAR AND PROJECTIVE CURVATURE TENSOR

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Abstract. In this paper we study the Concircular pseudosymmetric, $\tilde{C}(\xi, X) \cdot R = 0$, $\tilde{C} \cdot Q = 0$, $Q \cdot \tilde{C} = 0$, Projective pseudosymmetric, $P(\xi, X) \cdot R = 0$, $P \cdot Q = 0$ and $Q \cdot P = 0$ in generalized Sasakian space forms.

1. Introduction

Alegre P, Blair DE, Carriazo A. [1] introduced and studied the concept of generalized Sasakian space forms. An almost contact metric manifold (M, ϕ, ξ, η, g) is said to be a generalized Sasakian space form if there exist differentiable functions f_1, f_2, f_3 such that curvature tensor R of M is given by

$$\begin{aligned} R(X,Y)Z &= f_1\{g(Y,Z)X - g(X,Z)Y\} \\ &+ f_2\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\} \\ &+ f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\}, \end{aligned}$$

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for any vector fields X, Y, Z on M. Throughout the paper we denote generalized Sasakian space form as $M(f_1, f_2, f_3)$, which appears as a natural generalization of the Sasakian space form $M(c)$, which can be obtained as a particular case of generalized Sasakian space form by taking $f_1 = \frac{c+3}{4}$, $f_2 = \frac{c-1}{4}$ and $f_3 = \frac{c-1}{4}$, where c denotes constant ϕ-sectional curvature. The notion of generalized Sasakian space forms have been weakened by many geometers such as [2, 3, 4, 5, 8, 9, 14, 15, 17, 19] with different curvature tensors.

A Riemannian manifold is called locally symmetric if $\nabla R = 0$, where R is the Riemannian curvature tensor of (M, g). As a proper generalization of locally symmetric manifold, the notion of semi-symmetric manifold was defined by $(R(X,Y) \cdot R)(U,V)W = 0$.

For a $(0,k)$-tensor field T on $M, k \geq 1$, and a symmetric $(0,2)$-tensor field g on M, we define the tensor fields $R \cdot T$ and $Q(g,T)$ by

$$(R \cdot T)(X_1, \ldots, X_k; X, Y) = -T(R(X,Y)X_1, X_2, \ldots, X_k) - \ldots - T(X_1, \ldots X_{k-1}, R(X,Y)X_k)$$

and

$$Q(g,T)(X_1, \ldots, X_k; X, Y) = -T((X \wedge_g Y)X_1, X_2, \ldots, X_k) - \ldots - T(X_1, \ldots X_{k-1}, (X \wedge_g Y)X_k).$$

Where $X \wedge_g Y$ is the endomorphism given by

$$(X \wedge_g Y)Z = g(Y,Z)X - g(X,Z)Y. \tag{1.1}$$

A Riemannian manifold M is said to be pseudosymmetric [11] if

$$R \cdot R = L_R Q(g,R) \tag{1.2}$$

holds on $U_R = \{x \in M | R - \frac{r}{n(n-1)} G \neq 0 \ at \ x\}$, where G is the $(0,4)$-tensor defined by $G(X_1, X_2, X_3, X_4) = g((X_1 \wedge X_2)X_3, X_4)$ and L_R is some smooth function on M. A Riemannian manifold M is said to be Concircular pseudosymmetric if

$$R \cdot \tilde{C} = L_{\tilde{C}} Q(g, \tilde{C}) \tag{1.3}$$

holds on the set $U_{\tilde{C}} = \{x \in M : \tilde{C} \neq 0\}$ at x, where $L_{\tilde{C}}$ is some function on $U_{\tilde{C}}$ and $\tilde{C}$ is the Concircular curvature tensor. It is known that every pseudosymmetric manifold is Concircular pseudosymmertic, but the converse is not true. If $L_{\tilde{C}} = 0$ on $U_{\tilde{C}}$, then a Concircular pseudosymmetric manifold is Concircular semisymmetric. But $L_{\tilde{C}}$ need not be zero, in general and hence there exists Concircular pseudosymmetric manifolds which are not Concircular semisymmetric. Thus the class of Concircular pseudosymmetric manifolds is a natural extension of the class of Concircular semisymmetric manifolds.

Motivated by the above work in this paper we study the Concircular pseudosymmetric, $\tilde{C}(\xi, X) \cdot R = 0$, $\tilde{C} \cdot Q = 0$, $Q \cdot \tilde{C} = 0$, Projective pseudosymmetric, $P(\xi, X) \cdot R = 0$, $P \cdot Q = 0$ and $Q \cdot P = 0$ in generalized Sasakian space forms.

2. Preliminaries

An n-dimensional Riemannian manifold M is called an almost contact metric manifold [7] if there exist a $(1,1)$ tensor field ϕ, a vector field ξ and a 1-form η such that

$$\phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta(\phi X) = 0, \tag{2.4}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad g(X, \xi) = \eta(X), \tag{2.5}$$

$$g(\phi X, Y) = -g(X, \phi Y). \tag{2.6}$$

For an n-dimensional generalized Sasakian space form [1], we have

$$\begin{aligned} R(X,Y)Z &= f_1\{g(Y,Z)X - g(X,Z)Y\} + f_2\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X \\ &+ 2g(X,\phi Y)\phi Z\} + f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi \\ &- g(Y,Z)\eta(X)\xi\}, \end{aligned} \tag{2.7}$$

$$S(X,Y) = [(n-1)f_1 + 3f_2 - f_3]g(X,Y) + [-3f_2 - (n-2)f_3]\eta(X)\eta(Y), \tag{2.8}$$

$$r = (n-1)\{nf_1 + 3f_2 - 2f_3\}. \tag{2.9}$$

From (2.7) and (2.8), we get

$$\eta(R(X,Y)Z) = (f_1 - f_3)\{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\}, \tag{2.10}$$

$$R(X,Y)\xi = (f_1 - f_3)\{\eta(Y)X - \eta(X)Y\}, \tag{2.11}$$

$$R(\xi,X)Y = (f_1 - f_3)\{g(X,Y)\xi - \eta(Y)X\}, \tag{2.12}$$

$$S(X,\xi) = (n-1)(f_1 - f_3)\eta(X), \tag{2.13}$$

where R is the Riemannian curvature tensor, S is the Ricci tensor and r is the Scalar curvature.

3. **Concircular Pseudosymmetric Generalized Sasakian Space Forms**

This section deals with the study of Concircular pseudosymmetric generalized Sasakian space forms. A transformation of an n-dimensional Riemannian manifold M, which transforms every geodesic circle of M into a geodesic circle is called a concircular transformation ([13], [21]). A concircular transformation is always a conformal transformation ([13]). Here geodesic circle means a curve in M whose first curvature is constant and whose second curvature is identically zero. Thus the geometry of concircular transformations, that is, the concircular geometry is a generalization of inversive geometry in the sense that the change of metric is more general than that induced by a circle preserving diffeomorphism. The interesting invariant of a concircular transformation is the concircular curvature tensor $\tilde{C}$, which is defined by ([21])

$$\tilde{C}(X,Y)Z = R(X,Y)Z - \frac{r}{n(n-1)}[g(Y,Z)X - g(X,Z)Y], \tag{3.14}$$

where R is the curvature tensor and r is the scalar curvature of the manifold.

Let $M(f_1, f_2, f_3)$ be an n-dimensional Concircular pseudosymmetric generalized Sasakian space form. Then from (1.3), we have

$$(R(\xi,Y)\cdot\tilde{C})(U,V)W = L_{\tilde{C}}[(\xi\wedge Y)\cdot\tilde{C}(U,V)W]. \tag{3.15}$$

By (3.15), we get

$$\begin{aligned}
&R(\xi,Y)\tilde{C}(U,V)W - \tilde{C}(R(\xi,Y)U,V)W - \tilde{C}(U,R(\xi,Y)V)W - \tilde{C}(U,V)R(\xi,Y)W\\
&= L_{\tilde{C}}[(\xi\wedge Y)\tilde{C}(U,V)W - \tilde{C}((\xi\wedge Y)U,V)W - \tilde{C}(U,(\xi\wedge Y)V)W\\
&-\tilde{C}(U,V)(\xi\wedge Y)W].
\end{aligned} \tag{3.16}$$

By using the expression (2.12) in (3.16), we have

$$\begin{aligned}
&(L_{\tilde{C}} - (f_1 - f_3))[g(Y,\tilde{C}(U,V)W)\xi - \eta(\tilde{C}(U,V)W)Y - g(Y,U)\tilde{C}(\xi,V)W\\
&+\eta(U)\tilde{C}(Y,V)W - g(Y,V)\tilde{C}(U,\xi)W + \eta(V)\tilde{C}(U,Y)W\\
&-g(Y,W)\tilde{C}(U,V)\xi + \eta(W)\tilde{C}(U,V)Y] = 0.
\end{aligned} \tag{3.17}$$

By taking the inner product with ξ in (3.17), we obtain

$$\begin{aligned}
&(L_{\tilde{C}} - (f_1 - f_3))[g(Y,\tilde{C}(U,V)W) - \eta(\tilde{C}(U,V)W)\eta(Y) - g(Y,U)\eta(\tilde{C}(\xi,V)W)\\
&+\eta(U)\eta(\tilde{C}(Y,V)W) - g(Y,V)\eta(\tilde{C}(U,\xi)W) + \eta(V)\eta(\tilde{C}(U,Y)W)\\
&-g(Y,W)\eta(\tilde{C}(U,V)\xi) + \eta(W)\eta(\tilde{C}(U,V)Y)] = 0.
\end{aligned} \tag{3.18}$$

By (3.18), we get either $L_{\tilde{C}} = (f_1 - f_3)$ or

$$\begin{aligned}
&[g(Y,\tilde{C}(U,V)W) - \eta(\tilde{C}(U,V)W)\eta(Y) - g(Y,U)\eta(\tilde{C}(\xi,V)W)\\
&+\eta(U)\eta(\tilde{C}(Y,V)W) - g(Y,V)\eta(\tilde{C}(U,\xi)W) + \eta(V)\eta(\tilde{C}(U,Y)W)\\
&-g(Y,W)\eta(\tilde{C}(U,V)\xi) + \eta(W)\eta(\tilde{C}(U,V)Y)] = 0.
\end{aligned} \tag{3.19}$$

Let $\{e_1, e_2, \ldots, e_n\}$ is an orthonormal basis of the tangent space at each point of the manifold. Putting $U = Y = e_i$ in (3.19) and taking summation over $i, (1 \leq i \leq n)$ and by virtue of (3.14), we have

$$S(V, W) = (n-1)(f_1 - f_3)g(V, W). \tag{3.20}$$

On contracting (3.20), we get

$$r = n(n-1)(f_1 - f_3). \tag{3.21}$$

Therefore, $M(f_1, f_2, f_3)$ is an Einstein manifold. Hence we state the following theorem.

Theorem 3.1. *Let $M(f_1, f_2, f_3)$ be an n-dimensional generalized Sasakian space form. If $M(f_1, f_2, f_3)$ is Concircular pseudosymmetric then $M(f_1, f_2, f_3)$ is an Einstein manifold or $L_{\tilde{C}} = (f_1 - f_3)$ holds on $M(f_1, f_2, f_3)$.*

Now, by using (3.21) in (3.14) then we get

$$\eta(\tilde{C}(X, Y)Z) = 0 \tag{3.22}$$

and

$$\eta(\tilde{C}(\xi, Y)Z) = 0. \tag{3.23}$$

By virtue of (3.22) and (3.23) in (3.19), we obtain

$$g(Y, \tilde{C}(U, V)W) = \tilde{C}(U, V, W, Y) = 0. \tag{3.24}$$

This implies that $M(f_1, f_2, f_3)$ is Concircularly flat. Hence we conclude the following theorem.

Theorem 3.2. *Let $M(f_1, f_2, f_3)$ be an n-dimensional generalized Sasakian-space form. If $M(f_1, f_2, f_3)$ is Concircular pseudosymmetric then $M(f_1, f_2, f_3)$ is either Concircularly flat or $L_{\tilde{C}} = (f_1 - f_3)$ holds on $M(f_1, f_2, f_3)$.*

If we assume that $M(f_1, f_2, f_3)$ is not Concircularly semi symmtric, a Concircular pseudosymmetric generalized Sasakian space form. Then we get $R \cdot \tilde{C} = (f_1 - f_3)Q(g, \tilde{C})$, which implies that the pseudosymmetry function $L_{\tilde{C}} = (f_1 - f_3)$. Therefore we have the following:

Corollary 3.1. *Every generalized Sasakian space form $M(f_1, f_2, f_3)$ is Concircular pseudosymmetric of the form $R \cdot \tilde{C} = (f_1 - f_3)Q(g, \tilde{C})$.*

4. Generalized Sasakian space form satisfying $\tilde{C}(\xi, X) \cdot R = 0$

In this section we study generalized Sasakian space form satisfying $\tilde{C}(\xi, X) \cdot R = 0$. Let $M(f_1, f_2, f_3)$ be an n-dimensional generalized Sasakian space form satisfying $\tilde{C}(\xi, X) \cdot R = 0$. Then, we have

$$\begin{aligned}(\tilde{C}(\xi, X) \cdot R)(U, V)W &= \tilde{C}(\xi, X)R(U, V)W - R(\tilde{C}(\xi, X)U, V)W \\ &- R(U, \tilde{C}(\xi, X)V)W - R(U, V)\tilde{C}(\xi, X)W = 0. \qquad (4.25)\end{aligned}$$

Putting $W = \xi$ in (4.25) and by virtue of (2.11), we obtain

$$\begin{aligned}&(f_1 - f_3)\eta(\tilde{C}(\xi, X)U)V - (f_1 - f_3)\eta(\tilde{C}(\xi, X)V)U \\ &-[(f_1 - f_3) - \frac{r}{n(n-1)}]\{\eta(X)R(U, V)\xi - R(U, V)X\} = 0. \qquad (4.26)\end{aligned}$$

Let $\{e_1, e_2, \ldots, e_n\}$ is an orthonormal basis of the tangent space at each point of the manifold and taking inner product with e_i in (4.26) and on simplification, we get

$$S(X, V) = (n-1)(f_1 - f_3)g(X, V). \qquad (4.27)$$

On Contracting (4.27), we have

$$r = n(n-1)(f_1 - f_3). \qquad (4.28)$$

Conversely, if $f_1 = f_3$ then from (2.12) and (3.14) trivially we get $\tilde{C}(\xi, X) \cdot R = 0$. If $S(X, V) = (n-1)(f_1 - f_3)g(X, V)$ with scalar curvature $r = n(n-1)(f_1 - f_3)$, we obtain $\tilde{C}(\xi, X) \cdot R = 0$. And then comparing r with (2.9) we have $3f_2 + (n-2)f_3 = 0$. Hence we conclude the following theorem.

Theorem 4.1. *An n-dimensional generalized Sasakian space form M satisfying the condition $\tilde{C}(\xi, X) \cdot R = 0$ if and only if either $f_1 = f_3$ or the manifold is an Einstein manifold with scalar curvature $r = n(n-1)(f_1 - f_3)$.*

Remark 4.1. *In* [4], *author obtained necessary and sufficient condition for a generalized Sasakian space form M^{2n+1} satisfying $\tilde{C}(\xi, X) \cdot R = 0$ if and only if the functions f_2 and f_3 either satisfy the conditions $(2n-1)f_3 + 3f_2 = 0$ or it has the sectional curvature $(f_1 - f_3)$.*

5. Generalized Sasakian space form satisfying $\tilde{C} \cdot Q = 0$

In this section we study the generalized Sasakian space form satisfying $\tilde{C} \cdot Q = 0$. Let $M(f_1, f_2, f_3)$ be an n-dimensional generalized Sasakian space form satisfying $\tilde{C} \cdot Q = 0$. Then, we have

$$\tilde{C}(X,Y)QZ - Q(\tilde{C}(X,Y)Z) = 0, \tag{5.29}$$

for all smooth vector fields X, Y and Z. Putting $Y = \xi$ in (5.29), we have

$$\tilde{C}(X,\xi)QZ - Q(\tilde{C}(X,\xi)Z) = 0. \tag{5.30}$$

By using (3.14) in (5.30) and on simplification, we obtain

$$\begin{aligned}\left[(f_1 - f_3) - \frac{r}{n(n-1)}\right] \quad & [(n-1)(f_1 - f_3)\eta(Z)X - S(X,Z)\xi \\ & -\eta(Z)QX + (n-1)(f_1 - f_3)g(X,Z)\xi] = 0. \end{aligned}\tag{5.31}$$

Taking inner product with ξ in (5.31), we have

$$\left[(f_1 - f_3) - \frac{r}{n(n-1)}\right] [(n-1)(f_1 - f_3)g(X,Z) - S(X,Z)] = 0. \tag{5.32}$$

From (5.32), either $[(f_1 - f_3) - \frac{r}{n(n-1)}] = 0$ or

$$S(X,Z) = (n-1)(f_1 - f_3)g(X,Z). \tag{5.33}$$

Hence, we state the following theorem.

Theorem 5.1. *An n-dimensional generalized Sasakian space form $M(f_1, f_2, f_3)$ satisfies the curvature condition $\tilde{C} \cdot Q = 0$, then the manifold is an Einstein manifold or the scalar curvature $r = n(n-1)(f_1 - f_3)$.*

6. Generalized Sasakian space form satisfying $Q \cdot \tilde{C} = 0$

In this section we study generalized Sasakian space form satisfying $Q \cdot \tilde{C} = 0$. Let $M(f_1, f_2, f_3)$ be an n-dimensional generalized Sasakian space form satisfying $Q \cdot \tilde{C} = 0$. Then, we have

$$Q(\tilde{C}(X,Y)Z) - \tilde{C}(QX,Y)Z - \tilde{C}(X,QY)Z - \tilde{C}(X,Y)QZ = 0, \tag{6.34}$$

for all smooth vector fields X, Y and Z. Putting $Y = \xi$ in (6.34), we have

$$Q(\tilde{C}(X,\xi)Z) - \tilde{C}(QX,\xi)Z - \tilde{C}(X,Q\xi)Z - \tilde{C}(X,\xi)QZ = 0. \tag{6.35}$$

By using (3.14) in (6.35) and on simplification, we obtain

$$\left[(f_1 - f_3) - \frac{r}{n(n-1)}\right][2S(X,Z)\xi - 2(n-1)(f_1 - f_3)\eta(Z)X] = 0. \tag{6.36}$$

Taking inner product with ξ in (6.36), we have

$$\left[(f_1 - f_3) - \frac{r}{n(n-1)}\right][2S(X,Z) - 2(n-1)(f_1 - f_3)\eta(Z)\eta(X)] = 0. \tag{6.37}$$

Putting $Z = \xi$ in (6.37), then from (6.37) either $[(f_1 - f_3) - \frac{r}{n(n-1)}] = 0$ or

$$S(X,\xi) = (n-1)(f_1 - f_3)\eta(X), \tag{6.38}$$

which implies

$$Q\xi = (n-1)(f_1 - f_3)\xi. \tag{6.39}$$

Hence, we state the following theorem.

Theorem 6.1. *An n-dimensional generalized Sasakian space form $M(f_1, f_2, f_3)$ satisfies the curvature condition $Q \cdot \tilde{C} = 0$, then the Ricci operator of ξ of a generalized Sasakian space form is equal to $(n-1)$ times of $(f_1 - f_3)\xi$ or the scalar curvature $r = n(n-1)(f_1 - f_3)$.*

7. Projective Pseudosymmetric Generalized Sasakian Space Forms

This section deals with the study of Projective pseudosymmetric generalized Sasakian space forms. The projective curvature tensor is an important concept of Riemannian geometry, which one uses to calculate the basic geometric measurements on a manifold. The projective transformation on a manifold is a transformation under which geodesic transforms into geodesic. The projective curvature tensor is given by ([4])

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{(n-1)}[S(Y,Z)X - S(X,Z)Y]. \tag{7.40}$$

Let $M(f_1, f_2, f_3)$ be an n-dimensional Projective pseudosymmetric generalized Sasakian space form. Then we have

$$(R(\xi,Y)\cdot P)(U,V)W = L_P[(\xi \wedge Y)\cdot P(U,V)W]. \tag{7.41}$$

By (7.41), we get

$$\begin{aligned} &R(\xi,Y)P(U,V)W - P(R(\xi,Y)U,V)W - P(U,R(\xi,Y)V)W - P(U,V)R(\xi,Y)W \\ &= L_P[(\xi\wedge Y)P(U,V)W - P((\xi\wedge Y)U,V)W - P(U,(\xi\wedge Y)V)W \\ &-P(U,V)(\xi\wedge Y)W]. \end{aligned} \tag{7.42}$$

By using the expression (2.12) in (7.42), we have

$$(L_P - (f_1 - f_3))[g(Y, P(U,V)W)\xi - \eta(P(U,V)W)Y - g(Y,U)P(\xi,V)W$$
$$+\eta(U)P(Y,V)W - g(Y,V)P(U,\xi)W + \eta(V)P(U,Y)W$$
$$-g(Y,W)P(U,V)\xi + \eta(W)P(U,V)Y] = 0. \tag{7.43}$$

By taking the inner product with ξ in (7.43), we obtain

$$(L_P - (f_1 - f_3))[g(Y, P(U,V)W) - \eta(P(U,V)W)\eta(Y) - g(Y,U)\eta(P(\xi,V)W)$$
$$+\eta(U)\eta(P(Y,V)W) - g(Y,V)\eta(P(U,\xi)W) + \eta(V)\eta(P(U,Y)W)$$
$$-g(Y,W)\eta(P(U,V)\xi) + \eta(W)\eta(P(U,V)Y)] = 0. \tag{7.44}$$

By (7.44), we get either $L_P = (f_1 - f_3)$ or

$$[g(Y, P(U,V)W) - \eta(P(U,V)W)\eta(Y) - g(Y,U)\eta(P(\xi,V)W)$$
$$+\eta(U)\eta(P(Y,V)W) - g(Y,V)\eta(P(U,\xi)W) + \eta(V)\eta(P(U,Y)W)$$
$$-g(Y,W)\eta(P(U,V)\xi) + \eta(W)\eta(P(U,V)Y)] = 0. \tag{7.45}$$

Let $\{e_1, e_2, \ldots, e_n\}$ is an orthonormal basis of the tangent space at each point of the manifold. Putting $U = Y = e_i$ in (7.45) and taking summation over $i, (1 \leq i \leq n)$ and by virtue of (7.40), we have

$$S(V,W) = (n-1)(f_1 - f_3)g(V,W) - \left[\frac{r}{n-1} - n(f_1 - f_3)\right]\eta(V)\eta(W). \tag{7.46}$$

On contracting (7.46), we get

$$r = n(n-1)(f_1 - f_3). \tag{7.47}$$

By using (7.47) in (7.46), we obtain

$$S(V,W) = (n-1)(f_1 - f_3)g(V,W). \tag{7.48}$$

Therefore, $M(f_1, f_2, f_3)$ is an Einstein manifold. Hence, we state the following theorem.

Theorem 7.1. *Let $M(f_1, f_2, f_3)$ be an n-dimensional generalized Sasakian space form. If $M(f_1, f_2, f_3)$ is Projective pseudosymmetric then $M(f_1, f_2, f_3)$ is an Einstein manifold or $L_P = (f_1 - f_3)$ holds on $M(f_1, f_2, f_3)$.*

Now, by using (2.10) and (7.48) in (7.40) then we get

$$\eta(P(X,Y)Z) = 0 \tag{7.49}$$

and

$$\eta(P(\xi,Y)Z) = 0. \tag{7.50}$$

By virtue of (7.49) and (7.50) in (7.45), we obtain

$$g(Y, P(U,V)W) = P(U,V,W,Y) = 0. \tag{7.51}$$

This implies that $M(f_1, f_2, f_3)$ is Projectively flat. Hence, we conclude the following theorem.

Theorem 7.2. *Let $M(f_1, f_2, f_3)$ be an n-dimensional generalized Sasakian space form. If $M(f_1, f_2, f_3)$ is Projective pseudosymmetric then $M(f_1, f_2, f_3)$ is either Projectively flat or $L_P = (f_1 - f_3)$ holds on $M(f_1, f_2, f_3)$.*

If we assume that $M(f_1, f_2, f_3)$ is not Projectively semisymmtric, a Projective pseudosymmetric generalized Sasakian space form. Then we get $R \cdot P = (f_1 - f_3)Q(g, P)$, which implies that the pseudosymmetry function $L_P = (f_1 - f_3)$. Therefore we have the following:

Corollary 7.1. *Every generalized Sasakian space form $M(f_1, f_2, f_3)$ is Projective pseudosymmetric of the form $R \cdot P = (f_1 - f_3)Q(g, P)$.*

8. Generalized Sasakian space form satisfying $P(\xi, X) \cdot R = 0$

In this section we study generalized Sasakian space form satisfying $P(\xi, X) \cdot R = 0$. Let $M(f_1, f_2, f_3)$ be an n-dimensional generalized Sasakian-space form satisfying $P(\xi, X) \cdot R = 0$. Then, we have

$$\begin{aligned} &P(\xi,X)R(U,V)W - R(P(\xi,X)U,V)W \\ &-R(U,P(\xi,X)V)W - R(U,V)P(\xi,X)W = 0. \end{aligned} \tag{8.52}$$

Putting $W = \xi$ in (8.52) and by virtue of (2.11), we obtain

$$\begin{aligned} &(f_1 - f_3)\eta(P(\xi,X)U)V - (f_1 - f_3)\eta(P(\xi,X)V)U \\ &-(n-2)(f_1 - f_3)\{\eta(X)R(U,V)\xi - R(U,V)X\} = 0. \end{aligned} \tag{8.53}$$

Let $\{e_1, e_2, \ldots, e_n\}$ is an orthonormal basis of the tangent space at each point of the manifold and taking inner product with e_i in (8.53) and on simplification, we get

$$S(X,V) = (n-1)(f_1 - f_3)g(X,V) + (n-2)(f_1 - f_3)\eta(X)\eta(V). \tag{8.54}$$

Therefore, $M(f_1, f_2, f_3)$ is an η-Einstein manifold. Hence, we state the following theorem.

Theorem 8.1. *An n-dimensional generalized Sasakian space form $M(f_1, f_2, f_3)$ satisfying the condition $P(\xi, X) \cdot R = 0$ is an η-Einstein manifold.*

9. Generalized Sasakian space form satisfying $P \cdot Q = 0$

Let $M(f_1, f_2, f_3)$ be an n-dimensional generalized Sasakian space form satisfying $P \cdot Q = 0$. Then, we have

$$P(X, Y)QZ - Q(P(X, Y)Z) = 0, \tag{9.55}$$

for all smooth vector fields X, Y and Z. Putting $Y = \xi$ in (9.55), we have

$$P(X, \xi)QZ - Q(P(X, \xi)Z) = 0. \tag{9.56}$$

By using (7.40), (2.11) in (9.56) and on simplification, we obtain

$$\frac{1}{(n-1)}S(X, QZ)\xi - 2(f_1 - f_3)S(X, Z)\xi + (n-1)(f_1 - f_3)^2 g(X, Z)\xi = 0. \tag{9.57}$$

Taking inner product with ξ in (9.57), we have

$$S^2(X, Z) = 2(n-1)(f_1 - f_3)S(X, Z) - (n-1)^2 g(f_1 - f_3)^2 g(X, Z). \tag{9.58}$$

Hence, we state the following theorem.

Theorem 9.1. *An n-dimensional generalized Sasakian space form satisfies the curvature condition $P \cdot Q = 0$, then the square of the Ricci tensor S^2 is the linear combination of the Ricci tensor S and the metric tensor g.*

10. Generalized Sasakian space form satisfying $Q \cdot P = 0$

Let $M(f_1, f_2, f_3)$ be an n-dimensional generalized Sasakian space form satisfying $Q \cdot P = 0$. Then, we have

$$Q(P(X, Y)Z) - P(QX, Y)Z - P(X, QY)Z - P(X, Y)QZ = 0, \tag{10.59}$$

for all smooth vector fields X, Y and Z. Putting $Y = \xi$ in (10.59), we have

$$Q(P(X, \xi)Z) - P(QX, \xi)Z - P(X, Q\xi)Z - P(X, \xi)QZ = 0. \tag{10.60}$$

By virtue of (7.40) in (10.60) and on simplification, we obtain

$$2(f_1 - f_3)S(X, Z)\xi - \frac{2}{(n-1)}S(X, QZ)\xi = 0. \tag{10.61}$$

Taking inner product with ξ in (10.61), we have

$$S(X, QZ) = (n-1)(f_1 - f_3)S(X, Z), \tag{10.62}$$

which implies

$$g(Q^2X, Z) = (n-1)(f_1 - f_3)g(QX, Z), \tag{10.63}$$

Let $\{e_1, e_2, e_3, \ldots, e_n\}$ be a local orthonormal basis of the tangent space at a point of the manifold M. Then by putting $X = Z = e_i$ in (10.63) and taking summation over i, we have

$$trace(Q^2) = (n-1)(f_1 - f_3)trace(Q). \tag{10.64}$$

Hence, we state the following theorem.

Theorem 10.1. *An n-dimensional generalized Sasakian space form satisfies the curvature condition $Q \cdot P = 0$, then the trace of the square Ricci operator of a generalized Sasakian space form is equal to $(n-1)(f_1 - f_3)$ times of trace of the Ricci operator.*

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