International Journal of Maps in Mathematics

*Volume **8**, Issue **1**, 2025, Pages:**2-34***

E-ISSN: 2636-7467

www.simadp.com/journalmim

SHAPE STABILITY OF A QUADRATURE SURFACE PROBLEM IN INFINITE RIEMANNIAN MANIFOLDS

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Abstract. In this paper, we revisit a quadradure surface problem in shape optimization. With tools from infinite-dimensional Riemannian geometry, we give simple control over how an optimal shape can be characterized. The framework of the infinite-dimensional Riemannian manifold is essential in the control of optimal geometric shape. The covariant derivative plays a key role in calculating and analyzing the qualitative properties of the shape hessian. Control only depends on the mean curvature of the domain, which is a minimum or a critical point. In the two-dimensional case, Gauss-Bonnet's theorem gives a control within the framework of the algorithm for the minimum.

Keywords: Stability, quadrature surface, shape optimization, Riemannian manifold, Gauss-Bonnet theorem

2010 Mathematics Subject Classification: Primary 49Q10, Secondary 53B20.

1. Introduction

The search for the notion of quadratures made a prodigious leap forward (1669-1704) thanks to Leibniz and Newton who, with the infinitesimal calculus, made the link between quadrature and derivative. An interesting reminder could be to explain the link with shape optimization. Regarding, a bounded domain $\Omega \subset \mathbb{R}^N$ with regular boundary, for instance $\mathcal{C}^2$, μ a signed measure compactly supported in Ω, it is well known there is a measure σ called a balayage measure carried by the surface $\partial\Omega$ and having the same potential as μ outside $\bar{\Omega}$,

Received:2024.02.20 *Revised:2024.04.23* *Accepted:2024.08.15*

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see for instance [19], [22] for more details about this topic. And in this case, by a classical approximation technique, one has the following relation:

$$\int_{\partial\Omega} h d\sigma = \langle h, \mu \rangle \qquad \forall h \in \mathcal{H}(\bar{\Omega}) \tag{1.1}$$

where $\mathcal{H}(\bar{\Omega})$ denotes the set of harmonic functions in a neighborhood of $\bar{\Omega}$. And we say that $\partial\Omega$ is a quadrature surface with respect to μ if (1.1) is satisfied.

This notion is closely linked with the overdetermined Cauchy elliptic problem. And one can claim that $\partial\Omega$ is a quadrature surface if and only if there is a solution to the following overdetermined Cauchy problem

$$\begin{cases} -\Delta u_\Omega & = & \mu & \text{in} & \Omega, \\ u_\Omega & = & 0 & \text{on} & \partial\Omega, \\ -\frac{\partial u_\Omega}{\partial \vec{\nu}} & = & 1 & \text{on} & \partial\Omega. \end{cases} \tag{1.2}$$

The above quadrature surface free boundary problem has some physical motivations and can be related to many areas such as free streamlines, jets, Hele-show flows, electromagnetic shaping, gravitational problems etc. It has been intensively studied at least during the last forty years, see for example [33], [15] and the references therein for more details. Among these works, some authors have established an intimate link between the existence of quadrature surfaces and the solution of free boundary problems governed by overdetermined partial differential equations, see for instance [17], [32], [31], [13] and references therein.

The quadrature surface problem (1.2) can be tackled by a shape optimization approach when μ is regular enough, for instance by taking it in $L^2(\Omega), supp(\mu) \subset \Omega$. Fore more details see for instance [3] and [13].

Before proceeding further, let us remind that in optimisation or in the study of minimal action, one of the essential questions is the characterization of an optimum if it exists. When one is in a differentiable environment, i.e. if the objective function is differentiable as well as its constraints, if any, the first derivative and second one (hessian) play a fundamental role. In finite dimensions, the characterization results are very well known even when we are in Banach spaces.

On the other hand, when we have to deal with admissible sets of regular openings of $\mathbb{R}^N, N \geq 2$ containing the optimum to be characterized, the question is to be treated in a more delicate way. Indeed, if we consider a shape optimization problem where the variable is a regular open subset of class $\mathcal{C}^2$ and in which a boundary value problem of partial differential equations is posed, there is the computation of the second derivative. Added to this, the equivalence of norms is to be handled if any exist. In this paper, we aim at studying these issues of characterization of critical or optimal domains in the case where the minimum of the considered shape functional exists, in infinite dimensional Riemannian structures. To do so, it is crucial to find a space of forms and associated metrics.

Finding a shape space and an associated metric is a challenging task and different approaches lead to various models. One possible approach is to do as in [25], [24]. These authors proposed, a survey of various suitable inner products is given, e.g., the curvature weighted metric and the Sobolev metric. There are various types of metrics on shape spaces, e.g., inner metrics [4], [24] like the Sobolev metrics, outer metrics [6], [20], [24], metamorphosis metrics [35], the Wasserstein or Monge-Kantorovic metric on the shape space of probability measures [2], [7], the Weil-Petersson metric [21], current metrics [14], and metrics based on elastic deformations [16], [26]. However, it is a challenging task to model both, the shape space and the associated metric. There does not exist a common shape space or shape metric suitable for all applications. The suitability of an approach depends on the requirements in a given situation. In recent works, it has been shown that PDE constrained shape optimization problems can be embedded in the framework of optimization on shape spaces. E.g., in [28], shape optimization is considered as optimization on a Riemannian shape manifold, the manifold of smooth shapes. Moreover, an inner product, which is called Steklov- Poincaré metric, for the application of finite element (FE) methods, is proposed in [29].

As pointed out in [27], shape optimization can be viewed as optimization on Riemannian shape manifolds and the resulting optimization methods can be constructed and analyzed within this framework. This combines algorithmic ideas from [1] with the Riemannian geometrical point of view established in [4]
In [25], [24], a geometric structure of two-dimensional $\mathcal{C}^\infty$ shapes was introduced and subsequently generalized to shapes in higher dimensions in [23], [4], [5]. Essentially, closed curves (and closed higher-dimensional surfaces) are identified with mappings of the unit sphere to any shape under consideration. In two dimensions, this can be naturally motivated by Riemannian mapping theorem. In this work, we focus on two-dimensional shapes as subsets. And considering [3], [13], we think that it is possible to write our work in high dimensions and even if Ω is an open set with boundary of a compact $N-$dimensional Riemannian manifolds noted $\mathcal{M}$.

One of our main question is the following:

Is it possible to express the Hessian of a shape functional to get sufficient conditions so that the critical domain of the functional J assumes its minimum? To answer this question, we study the positiveness of the quadratic form of the functional J which is related to the quadrature surface that is nothing but the following free boundary problem

$$\begin{cases} -\Delta u_\Omega = f \ \text{ in } \Omega \\ u_\Omega = 0 \ \text{ on } \partial\Omega \\ -\frac{\partial u_\Omega}{\partial \vec{\nu}} = k \ \text{ on } \partial\Omega \end{cases}$$

k is a positive constant, and $f \in L^2(\Omega), supp f \subset \Omega$, $\vec{\nu}$ is the exterior unit normal vector. The above quadrature surface can be formulated as the following shape optimization problem:

$$\min_{\Omega \subset \mathbb{R}^2} J(\Omega)$$

under the following partial differential equations contraints

$$\begin{cases} -\Delta u_\Omega = f \ \text{ in } \Omega \\ u_\Omega = 0 \ \text{ on } \partial\Omega \end{cases}$$

where

$$J(\Omega) = -\frac{1}{2}\int_\Omega |\nabla u_\Omega|^2 dx + \frac{k^2}{2}|\Omega| \tag{1.3}$$

is a real valued shape differentiable objective function, with $|\Omega| = \displaystyle\int_\Omega dx$.

In [3], [13], there are all details on existence results of quadrature surface by using shape optimization tools.

And the second question is the computation problem of the Hessian in the infinite Riemannian framework and how it can be related to the second shape derivative to deduce qualitative properties when the minimum of a regular enough shape functional exists or when Ω is a critical point the latter property means, that the first derivative of $J(\Omega)$ is equal to zero.

The paper is organized as follows:

In section 2, we give a brief survey, based on works in [25], [24], about the characterization of the tangent space in a framework of Riemannian manifolds of infinite dimensions.

Section 3 deals with the optimality condition of first order for the shape optimization and the computation of the covariant derivative. The latter plays a key role in our final result. We shall give a direct way to compute it which appears as a simplified expression.

In section 4, we shall recall some technical but classical computations of shape second derivative and establish a result (stated as a proposition) giving the expression of the quadratic form associated to the quadrature surface problem.

Section 5 which contains our main contributions, is devoted to the positiveness of the shape hessian in a Riemannian point of view of infinite dimensions. And, we shall propose a simple control which allows to get key information on the optimal shape domains when the latter are strict local minimum or critical points of the considered shape functional.

2. Characterization of tangent space at a point of B_e

The aim is to analyze the correlation of the Riemannian geometry on infinite dimensional maniolds B_e with shape optimization.
The authors would like to stress, what follows has been already done in pioneering works, see [25], [24], [23]. We only reproduce some fundamental steps related to our work.

Let Ω be a simply connected and compact subset of $\mathbb{R}^2$ with $\Omega \neq \emptyset$ and $\mathcal{C}^\infty$ boundary $\partial\Omega$. As is always the case in shape optimization, the boundary of the shape is all that matters. Thus we can identify the set of all shapes with the set of all those boundaries.

Let $Emb(\mathbb{S}^1, \mathbb{R}^2)$ be the set of all smooth embeddings on $\mathbb{S}^1$ in the plan $\mathbb{R}^2$, its elements are the injective mappings $c : \mathbb{S}^1 \longrightarrow \mathbb{R}^2$. Let $Diff(\mathbb{S}^1)$ stands for the set of all $\mathcal{C}^\infty$ diffeomorphism on $\mathbb{S}^1$ which acts diferentiably on $Emb(\mathbb{S}^1, \mathbb{R}^2)$. Let us consider B_e as the quotient

$$Emb(\mathbb{S}^1, \mathbb{R}^2)/Diff(\mathbb{S}^1).$$

In terms of sets, we have

$$B_e(\mathbb{S}^1, \mathbb{R}^2) := \{ \ [c] \ / \ c \in Emb\} \text{ where } [c] := \{c' \in Emb \ / \ c' \sim c\}. \tag{2.4}$$

To characterize the tangent space to B_e we start with the characterization of the tangent space to Emb denoted T_cEmb and the tangent space to the orbit of c by $Diff(\mathbb{S}^1)$ at c denoted by $T_c(Diff(\mathbb{S}^1).c)$. Thus the tangent space to B_e is then identified with a suplementary subspace of $T_c(Diff(\mathbb{S}^1).c)$ in T_cEmb.

Proposition 2.1. *Let $c \in Emb$, then the tangent space at c to Emb is given by:*

$$T_c Emb = \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{R}^2). \tag{2.5}$$

Proof. Let $h \in T_c Emb$, then h is obtained by looking at a path of embeddings which passes through c. Let $c : I \times \mathbb{S}^1 \longrightarrow \mathbb{R}^2$ be an embedding path such that $c(t,\theta) = c(\theta) + th(\theta)$ where $h : \mathbb{S}^1 \longrightarrow \mathbb{R}^2$ is $\mathcal{C}^\infty$, we have : $\frac{d}{dt}_{|t=0} c(t,\theta) = h(\theta)$. Since $c(t,\theta)$ is an embedding path then $c(t,\theta)$ is an immersion, thus

$$T_c Emb = Im(T_0 c(t,\theta)) = \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{R}^2). \tag{2.6}$$

□

Proposition 2.2. *The tangent space to the orbit of c by $Diff(\mathbb{S}^1)$, is the subspace of $T_c Emb$ formed by vectors $m(\theta)$ of the type $c_\theta(\theta) = c'(\theta)$ times a function.*

Proof. We have $Diff(\mathbb{S}^1).c \subset Emb$ because these are all the bijective reparametrizations of the same curve $c(\theta)$ therefore $T_c(Diff(\mathbb{S}^1).c) \subset T_c Emb$. Let $m \in T_c(Diff(\mathbb{S}^1).c)$ then m is obtained by looking at a family of parametrizations $c(t,\theta) := c(\phi(t,\theta))$ of the curve $c(\theta)$ where

$$\phi(t,.) : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$$

is a diffeomorphism of $\mathbb{S}^1$ and t is the parameter of the variation of the reparametrization $\phi(t,s)$ of $\mathbb{S}^1$. We have $\frac{d}{dt}_{|t=0} c(t,\theta) = c'(\theta) \frac{d}{dt}_{|t=0} \phi(0,\theta)$ since, $c(t,\theta)$ is a parametrization of the curve $c(\theta)$ so it is an immersion. Thus we have

$$T_c(Diff(\mathbb{S}^1).c) = Im(T_0 c(t,\theta)) = c'(\theta) \frac{d}{dt}_{|t=0} \phi(0,\theta). \tag{2.7}$$

□

Remark 2.1. *The choice of the supplementary must abide by the action of $Diff(\mathbb{S}^1)$ i.e we choose a supplementary of $T_c(Diff(\mathbb{S}^1).c)$ in $T_c Emb$ stable by the action of $Diff(\mathbb{S}^1)$. For that it suffices to define a metric on Emb for which $Diff(\mathbb{S}^1)$ acts isometrically and define the supplementary of $T_c(Diff(\mathbb{S}^1).c)$ as its orthogonal with respect to this metric.*

Definition 2.1. *Let G^0 be metric invariant by the action of $Diff(\mathbb{S}^1)$ on the manifold $Emb(\mathbb{S}^1, \mathbb{R}^2)$, defined by the application:*

$$\begin{array}{rcccl} G^0 & : & T_c Emb \times T_c Emb & \to & \mathbb{R} \\ & & (h,m) & \mapsto & \displaystyle\int_{\mathbb{S}^1} \langle h(\theta), m(\theta) \rangle |c'(\theta)| d\theta \end{array}$$

where $\langle h(\theta), m(\theta) \rangle$ is the ordinary scalar product of $h(\theta)$ and $m(\theta)$ in $\mathbb{R}^2$.

Proposition 2.3. *Let $c \in B_e$ then T_cB_e is colinear to the outer unit normal of Ω where Ω is a simply connected and compact subset of $\mathbb{R}^2$ and $\vec{\nu}$ is the outer unit normal of the domain Ω . In other words*

$$T_cB_e \simeq \{h \mid h = \alpha\vec{\nu}, \alpha \in \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{R})\}.$$

Proof. From the results shown above the orthogonal of $T_c(Diff(\mathbb{S}^1).c)$ in T_cEmb is the set of $h(\theta)$ in T_cEmb which are orthogonal for the metric G^0 to all $m(\theta) = \frac{d}{dt}_{|t=0}\phi(0,\theta)c'(\theta)$ this means that $h(\theta)$ must be perpendicular to $c'(\theta)$. So $h(\theta) = \alpha(\theta)\vec{\nu}(\theta)$ where $\alpha(\theta) \in \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{R})$. Therefore we have

$$T_cB_e \simeq \{h|h = \alpha\vec{\nu},\ \alpha \in \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{R})\}$$

where $\vec{\nu}$ is the outer unit normal of the form Ω defined at the boundary by $\partial\Omega = c$ such that $\vec{\nu}(\theta) \perp c'(\theta)$ for all $\theta \in S^1$ and c' defines the circumferential derivative. □

Now let us consider the following terminology:

$$ds = |c_\theta|d\theta \qquad \text{arc length.}$$

Definition 2.2. *A Sobolev-type metric on the manifold $B_e(\mathbb{S}^1, \mathbb{R}^2)$ is map:*

$$\begin{array}{rcccl} G^A & : & T_cB_e \times T_cB_e & \to & \mathbb{R} \\ & & (h,m) & \mapsto & \displaystyle\int_{\mathbb{S}^1} (1 + AK_c^2(\theta))\langle h(\theta), m(\theta)\rangle |c'(\theta)|d\theta \end{array}$$

where K_c is the curvature of c and A a positive real.

Remark 2.2. (1) *By setting $h = \alpha\vec{\nu}$, $m = \beta\vec{\nu}$ and by parametrizing $c(s)$ by arc length we have*

$$G^A(h,m) = \int_{\partial\Omega} (1 + AK_c^2(\theta))\alpha\beta ds.$$

(2) *If $A > 0$, G^A is a Riemannian metric.*

3. Optimality condition of first order and covariant derivative

The shape optimization problem that we have, consists in finding the solution of the following optimization problem:

$$\min_\Omega J(\Omega),$$

where

$$J(\Omega) = -\frac{1}{2}\int_\Omega |\nabla u_\Omega|^2 dx + \frac{k^2}{2}|\Omega|$$

is a shape functional. We seek the shape derivative associated with the functional $J(\Omega)$ following the direction of the vector field $V:\mathbb{R}^2\to\mathbb{R}^2$, $\mathcal{C}^\infty$ class :

$$dJ(\Omega)[V]=\int_{\partial\Omega}\left(k^2-\left(\frac{\partial u_\Omega}{\partial\vec{\nu}}\right)^2\right)\langle V,\vec{\nu}\rangle d\sigma.$$

If $V_{|\partial\Omega}=\alpha\vec{\nu}$ we can still write

$$dJ(\Omega)[V]=\int_{\partial\Omega}\left(k^2-\left(\frac{\partial u_\Omega}{\partial\vec{\nu}}\right)^2\right)\alpha d\sigma. \tag{3.8}$$

It should noted that there is a link between the shape derivative of J and the gradient in Riemannian structures see [27] and [37]. To illustrate our claim, let us consider the Sobolev metric G^A to ease the understanding of the computations. We think that it is quite possible to generalize this study in higher dimensions and even with other metrics.
Our purpose is to calculate the gradient of $J:B_e\to\mathbb{R}$ then we have :

$$dJ(\Omega)[V]=G^A(gradJ(\Omega),V) \tag{3.9}$$

if $V_{|\partial\Omega}=h$ we have

$$\begin{aligned} dJ_c(h) &= G^A(gradJ(\Omega),h)\\ dJ_c(h) &= \int_{\partial\Omega}\left(1+AK_c^2\right)gradJ\alpha. \end{aligned}$$

But from (3.9),

$$dJ_c(h)=\int_{\partial\Omega}\left(k^2-\left(\frac{\partial u_\Omega}{\partial\vec{\nu}}\right)^2\right)\alpha d\sigma$$

and thus

$$\int_{\partial\Omega}\left(k^2-\left(\frac{\partial u_\Omega}{\partial\vec{\nu}}\right)^2\right)\alpha d\sigma=\int_{\partial\Omega}\left(1+AK_c^2\right)gradJ\alpha d\sigma$$

so that

$$gradJ=\frac{1}{1+AK_c^2}\left(k^2-\left(\frac{\partial u_\Omega}{\partial\vec{\nu}}\right)^2\right).$$

The next step is to compute the explicit form of the covariant derivative $\nabla_h m\in T_cB_e$ with $h,m\in T_cB_e$.

Definition 3.1. *Let $\mathcal{M}$ be a set. A chart of $\mathcal{M}$ is a triplet $(\mathcal{U},\psi,\mathcal{E})$ where $\mathcal{U}$ is a subset of $\mathcal{M}$, $\mathcal{E}$ is a Banach space, and ψ is a bijection from $\mathcal{U}$ to an open set $\mathcal{E}$. We say that two charts $(\mathcal{U}_1,\psi_1,\mathcal{E}_1)$ and $(\mathcal{U}_2,\psi_2,\mathcal{E}_2)$ are $\mathcal{C}^r$-compatible if:*

- *$\psi_1(\mathcal{U}_1\cap\mathcal{U}_2)$(respectively $\psi_2(\mathcal{U}_1\cap\mathcal{U}_2)$) is an open set in $\mathcal{E}_1$ (respectively $\mathcal{E}_2$).*

- *The map* $\psi_1 \circ \psi_2^{-1}$ *(respectively* $\psi_2 \circ \psi_1^{-1}$*) of* $\psi_2(\mathcal{U}_1 \cap \mathcal{U}_2)$ *in* $\psi_1(\mathcal{U}_1 \cap \mathcal{U}_2)$ *(respectively* $\psi_1(\mathcal{U}_1 \cap \mathcal{U}_2)$ *in* $\psi_2(\mathcal{U}_1 \cap \mathcal{U}_2)$ *is of class* $\mathcal{C}^r$.

A $\mathcal{C}^r$*-atlas of* $\mathcal{M}$ *is a set of charts that are two by two* $\mathcal{C}^r$*-compatible and whose domains cover* $\mathcal{M}$*. Two atlases are* $\mathcal{C}^r$*-equivalent if their union is still a* $\mathcal{C}^r$*-atlas. A Banach manifold of class* $\mathcal{C}^r$ *is a set* $\mathcal{M}$ *equipped with a class of equivalence of* $\mathcal{C}^r$*-atlases.*

Definition 3.2. *Let* $\mathcal{M}$ *be a real (smooth) Banach manifold and* g *be a section of the bundle* $T\mathcal{M}^* \times T\mathcal{M}^*$ *of symmetric bilinear forms on* $T\mathcal{M}$*. We say that* g *is a weak Riemannian metric on* $\mathcal{M}$ *if and only if, for every* $p \in \mathcal{M}$, g_p *is a positive definite bilinear map on* $T_p\mathcal{M}$*, ie. if and only if:*

- $g_p(X, X) \geq 0$ *, for all* $X \in T_p\mathcal{M}$*,*
- $g_p(X, X) = 0$ *if and only if* $X = 0$*.*

Definition 3.3. *Let* $\mathcal{M}$ *be a real (smooth) Banach manifold and* g *be a section of the bundle* $T\mathcal{M}^* \times T\mathcal{M}^*$ *of symmetric bilinear forms on* $T\mathcal{M}$*. We say that* g *is a strongly Riemannian metric on* $\mathcal{M}$ *if, for every* $p \in \mathcal{M}$*, the injection from* $T_p\mathcal{M}$ *to* $T_p\mathcal{M}^*$ *defined by*

$$\tilde{g}_p : \left| \begin{array}{ccc} T_p\mathcal{M} & \longrightarrow & T_p\mathcal{M}^* \\ X & \mapsto & \{i_X g_p : Y \mapsto g_p(X, Y)\} \end{array} \right.$$

induces an isomorphism between $T_p\mathcal{M}$ *and* $T_p\mathcal{M}^*$*.*

Proposition 3.1. *Given a Banach manifold* $\mathcal{M}$ *equipped with a strongly Riemannian metric* g*, there exists a unique linear connection* ∇ *on the tangent bundle* $T\mathcal{M}^*$ *preserving* g *and having zero torsion. It is called the Levi-Civita connection of* g*.*

For the proof of this proposition, see Proposition A.2.6 in [36].

The following results (Propositions 3.2; 3.3 and Theorem 3.1) have been already established in a pioneering work, see [27]. We only bring a new proof and additional details in the computations of the covariant derivative. In the last part of the paper containing our main contributions, the covariant derivative plays a key role in the study of the positiveness of the quadratic form. We shall come back to this fact.

Proposition 3.2. *Let* $\Omega \subset \mathbb{R}^2$ *where* Ω *is a simply connected and compact subset of* $\mathbb{R}^2$ *and be at least of class* $\mathcal{C}^2$*, and* $\vec{\nu}$ *is the outer unit normal of the domain* Ω *and* $V, W, Z \in \mathcal{C}^\infty(\mathbb{R}^2, \mathbb{R}^2)$ *vector fields which are orthogonal to the boundaries i.e*

$$V_{|\partial\Omega} = \alpha\vec{\nu}$$

with $\alpha := \langle V_{|\partial\Omega}, \vec{\nu} \rangle$,

$$W_{|\partial\Omega} = \beta\vec{\nu}$$

with $\beta := \langle W_{|\partial\Omega}, \vec{\nu} \rangle$ *and*

$$Z_{|\partial\Omega} = \gamma\vec{\nu}$$

with $\gamma := \langle Z_{|\partial\Omega}, \vec{\nu} \rangle$ *such that* $V_{|\partial\Omega} = h := \alpha\vec{\nu}$, $W_{|\partial\Omega} = m =: \beta\vec{\nu}$ *and* $Z_{|\partial\Omega} = l := \gamma\vec{\nu}$ *belongs to the tangent space of* B_e. *Then the shape derivative* $h(G^A(m,l))$ *associated with the Riemannian metric* G^A *can be expressed as follows:*

$$\begin{aligned} h(G^A(m,l)) &= \int_{\partial\Omega} \Big(2AK_c^3\alpha\beta\gamma + (1+AK_c^2)\tfrac{\partial\beta}{\partial\vec{\nu}}\gamma\alpha \Big)\, ds \\ &+ \int_{\partial\Omega} \Big((1+AK_c^2)\tfrac{\partial\gamma}{\partial\vec{\nu}}\beta\alpha + K_c(1+AK_c^2)\alpha\beta\gamma \Big)\, ds. \end{aligned}$$

Proof. We set

$$F(c_t(\theta)) = (1+AK_c^2(\theta))\langle m(\theta), l(\theta)\rangle,$$

so that $G^A(m,l) = \int_{\mathbb{S}^1} F(c_t(\theta))|c_t'(\theta)|d\theta$. Then we calculate the following expression

$$h(G^A(m,l)) = \frac{d}{dt}_{|t=0} \left(\int_{\mathbb{S}^1} F(c_t(\theta))|c_t'(\theta)|d\theta \right)[V],$$

where $c_t(\theta)$ denotes a family of (parameterized) curves with $c_0(\theta) = c(\theta)$ and $c_t'(\theta)$ denotes the derivative with respect to θ of the curve $c_t : \theta \longrightarrow c_t(\theta)$. We have

$$\begin{aligned} h(G^A(m,l)) &= \int_{\mathbb{S}^1} \Big(\tfrac{\partial[(1+AK_c^2)\beta\gamma]}{\partial\vec{\nu}}\alpha|c_t'(\theta)| + \tfrac{\partial(|c_t'(\theta)|)}{\partial\vec{\nu}}(1+AK_c^2)\beta\gamma\alpha) \Big)\, d\theta \\ &= \int_{\mathbb{S}^1} \Big(2AK_c(\tfrac{\partial K_c}{\partial\vec{\nu}})\alpha\beta\gamma + (1+AK_c^2)\tfrac{\partial\beta}{\partial\vec{\nu}}\gamma\alpha + (1+AK_c^2)\tfrac{\partial\gamma}{\partial\vec{\nu}}\beta\alpha \Big)\, d\theta \\ &+ \int_{\mathbb{S}^1} \frac{\partial|c_t'(\theta)|}{\partial\vec{\nu}}(1+AK_c^2)\beta\gamma\alpha d\theta. \end{aligned}$$

Now let us calculate $\frac{\partial K_c}{\partial\vec{\nu}}$. We have

$$\frac{\partial K_c}{\partial\vec{\nu}} = \frac{\langle\vec{\nu}, c_\theta\rangle}{|c_\theta|^2}K_\theta + \frac{\langle\vec{\nu}, ic_\theta\rangle}{|c_\theta|}K^2 + \frac{1}{|c_\theta|}\left(\frac{1}{|c_\theta|}\left(\frac{\langle\vec{\nu}, ic_\theta\rangle}{|c_\theta|}\right)_\theta\right)_\theta.$$

Then we have $\langle\vec{\nu}, c_\theta\rangle = 0$ because $\vec{\nu} \perp c_\theta$ and moreover,

$$\begin{aligned} \frac{\langle\vec{\nu}, ic_\theta\rangle}{|c_\theta|} &= \langle\vec{\nu}, \frac{ic_\theta}{|c_\theta|}\rangle \\ \frac{\langle\vec{\nu}, ic_\theta\rangle}{|c_\theta|} &= \langle\vec{\nu}, \vec{\nu}\rangle \\ \frac{\langle\vec{\nu}, ic_\theta\rangle}{|c_\theta|} &= \|\vec{\nu}\|^2 = 1. \end{aligned}$$

Hence, we obtain that:

$$\frac{\partial K_c}{\partial \vec{\nu}} = K_c^2 + \frac{1}{|c_\theta|}\left(\frac{1}{|c_\theta|}\left(\frac{\langle \vec{\nu}, ic_\theta\rangle}{|c_\theta|}\right)_\theta\right)_\theta.$$

Let us compute step by step the above last term in the right hand side.

First, we have

$$\begin{aligned}
\left(\frac{\langle \vec{\nu}, ic_\theta\rangle}{|c_\theta|}\right)_\theta &= \frac{\partial}{\partial\theta}\left(\frac{\langle \vec{\nu}, ic_\theta\rangle}{|c_\theta|}\right)\\
&= \frac{\frac{\partial}{\partial\theta}\langle \vec{\nu}, ic_\theta\rangle|c_\theta| - \frac{\partial|c_\theta|}{\partial\theta}\langle \vec{\nu}, ic_\theta\rangle}{|c_\theta|^2}\\
&= \frac{\frac{\partial}{\partial\theta}\langle \vec{\nu}, ic_\theta\rangle|c_\theta|}{|c_\theta|^2}\\
&= \frac{\frac{\partial}{\partial\theta}\langle \vec{\nu}, ic_\theta\rangle}{|c_\theta|}
\end{aligned}$$

$$\begin{aligned}
\left(\frac{\langle \vec{\nu}, ic_\theta\rangle}{|c_\theta|}\right)_\theta &= \frac{\langle \frac{\partial\vec{\nu}}{\partial\theta}, ic_\theta\rangle + \langle \vec{\nu}, i\frac{\partial c_\theta}{\partial\theta}\rangle}{|c_\theta|}\\
&= \frac{\langle \vec{\nu}'(\theta), ic_\theta\rangle + \langle \vec{\nu}(\theta), ic_{\theta\theta}\rangle}{|c_\theta|}\\
&= \frac{\langle \vec{\nu}'(\theta), ic_\theta\rangle}{|c_\theta|} + \frac{\langle \vec{\nu}(\theta), ic_{\theta\theta}\rangle}{|c_\theta|}\\
&= \langle \vec{\nu}'(\theta), \frac{ic_\theta}{|c_\theta|}\rangle + \frac{\langle \vec{\nu}(\theta), ic_{\theta\theta}\rangle}{|c_\theta|}\\
&= \langle \vec{\nu}'(\theta), \vec{\nu}(\theta)\rangle + \frac{\langle \vec{\nu}(\theta), ic_{\theta\theta}\rangle}{|c_\theta|}.
\end{aligned}$$

Note that $\|\vec{\nu}\|^2 = 1$ which is nothing $\langle \vec{\nu}, \vec{\nu}\rangle = 1$. Therefore, by differentiation, we get

$$\begin{aligned}
\langle \vec{\nu}'(\theta), \vec{\nu}(\theta)\rangle + \langle \vec{\nu}(\theta), \vec{\nu}'(\theta)\rangle &= 0\\
2\langle \vec{\nu}'(\theta), \vec{\nu}(\theta)\rangle &= 0\\
\langle \vec{\nu}'(\theta), \vec{\nu}(\theta)\rangle &= 0.
\end{aligned}$$

Indeed, proceeding further the computation, we have

$$\begin{aligned}\left(\frac{\langle\vec{\nu},ic_\theta\rangle}{|c_\theta|}\right)_\theta &= \frac{\langle\vec{\nu}(\theta),ic_{\theta\theta}\rangle}{|c_\theta|}\\ &= \langle\vec{\nu}(\theta),\frac{ic_{\theta\theta}}{|c_\theta|}\rangle\\ &= \langle\vec{\nu}(\theta),\frac{-K_c|c_\theta|c_\theta}{|c_\theta|}\rangle\\ &= \langle\vec{\nu}(\theta),-K_c c_\theta\rangle\\ &= -K_c\langle\vec{\nu}(\theta),c_\theta\rangle = 0.\end{aligned}$$

Finally, from all the above steps, we have $\frac{1}{|c_\theta|}\left(\frac{1}{|c_\theta|}\left(\frac{\langle\vec{\nu},ic_\theta\rangle}{|c_\theta|}\right)_\theta\right)_\theta = 0$ and we get

$$\frac{\partial K_c}{\partial\vec{\nu}} = K_c^2.$$

Therefore, we have

$$\begin{aligned}h(G^A(m,l)) &= \int_{\partial\Omega}\left(2AK_c(\tfrac{\partial K_c}{\partial\vec{\nu}})\alpha\beta\gamma+(1+AK_c^2)\tfrac{\partial\beta}{\partial\vec{\nu}}\gamma\alpha+(1+AK_c^2)\tfrac{\partial\gamma}{\partial\vec{\nu}}\beta\alpha\right)d\theta\\ &\quad+\int_{\partial\Omega}\frac{\partial|c_t'(\theta)|}{\partial\vec{\nu}}(1+AK_c^2)\beta\gamma\alpha d\theta\\ &= \int_{\partial\Omega}\left(2AK_c\times K_c^2\alpha\beta\gamma+(1+AK_c^2)\tfrac{\partial\beta}{\partial\vec{\nu}}\gamma\alpha+(1+AK_c^2)\tfrac{\partial\gamma}{\partial\vec{\nu}}\beta\alpha\right)d\theta\\ &\quad+\int_{\partial\Omega}\frac{\partial|c_t'(\theta)|}{\partial\vec{\nu}}(1+AK_c^2)\beta\gamma\alpha d\theta. \qquad (3.10)\end{aligned}$$

Let us calculate now the following expression

$$\frac{\partial(|c_t'(\theta)|)}{\partial\vec{\nu}}.$$

To do this we parametrize $c(\theta)$ by arc length i.e $|c'(\theta)| = 1$. Since

$$\langle c'(\theta),c'(\theta)\rangle = 1$$

and differentiating it, we have

$$\langle c''(\theta),c'(\theta)\rangle = 0.$$

Then $c''(\theta) = c_{\theta\theta}(\theta)$ is proportional to $\vec{\nu}(s)$ so $c''(\theta) = K_c(\theta)\vec{\nu}(\theta)$ (this is the definition of the curvature of the curve c).

Let us compute now $\frac{d}{dt}(|c_t'(\theta)|)$ at $t=0$, where

$$\begin{aligned} |c_t'(\theta)| &= |\frac{d}{d\theta}(c(\theta)+t\vec{\nu}(\theta))| \\ &= |c'(\theta)+t\vec{\nu}'(\theta)| \\ &= (|c'(\theta)|^2+t^2|\vec{\nu}'(\theta)|^2+2t\langle c'(\theta),\vec{\nu}'(\theta)\rangle)^{\frac{1}{2}} \end{aligned} \tag{3.11}$$

From the Taylor's expansion of the previous expression in t, we see that

$$\frac{d}{dt}_{|t=0}|c_t'(\theta)| = \langle c'(\theta),\vec{\nu}'(\theta)\rangle$$

and since

$$\langle c'(\theta),\vec{\nu}(\theta)\rangle = 0$$

by differentiating we have

$$\langle c'(\theta),\vec{\nu}'(\theta)\rangle = -\langle c''(\theta),\vec{\nu}(\theta)\rangle = K_c,$$

and hence $\frac{d}{dt}(|c_t'(\theta)|) = K_c$.

One can conclude that

$$\begin{aligned} h(G^A(m,l)) &= \int_{\partial\Omega}\Big(2AK_c^3\alpha\beta\gamma+(1+AK_c^2)\tfrac{\partial\beta}{\partial\vec{\nu}}\gamma\alpha\Big)\,ds \\ &+ \int_{\partial\Omega}\Big((1+AK_c^2)\tfrac{\partial\gamma}{\partial\vec{\nu}}\beta\alpha+K_c(1+AK_c^2)\alpha\beta\gamma\Big)\,ds. \end{aligned}$$

□

Proposition 3.3. *Let $\Omega\subset\mathbb{R}^2$ where Ω is a simply connected and compact subset of $\mathbb{R}^2$ and be at least of class $\mathcal{C}^2$, and $\vec{\nu}$ is the outer unit normal of the domain Ω and $V,W,Z\in\mathcal{C}^\infty(\mathbb{R}^2,\mathbb{R}^2)$ vector fields which are orthogonal to the boundaries i.e*

$$V_{|\partial\Omega} = \alpha\vec{\nu}$$

with $\alpha := \langle V_{|\partial\Omega},\vec{\nu}\rangle$,

$$W_{|\partial\Omega} = \beta\vec{\nu}$$

with $\beta := \langle W_{|\partial\Omega},\vec{\nu}\rangle$ and

$$Z_{|\partial\Omega} = \gamma\vec{\nu}$$

with $\gamma := \langle Z_{|\partial\Omega},\vec{\nu}\rangle$ such that $V_{|\partial\Omega} = h := \alpha\vec{\nu}$, $W_{|\partial\Omega} = m =: \beta\vec{\nu}$ and $Z_{|\partial\Omega} = l := \gamma\vec{\nu}$ belongs to the tangent space of B_e. Then the expression $G^A(\nabla_h m,l)+G^A(m,\nabla_h l)$ associated with the Riemannian metric G^A for all $l\in T_C B_e$, can be expressed as follows:

$$G^A(\nabla_h m,l)+G^A(m,\nabla_h l) = \int_{\partial\Omega}(1+AK_c^2)\,(\nabla_V W\gamma+\beta\nabla_V Z)\,ds.$$

Proof. By using the definition of the Riemannian metric G^A we have

$$\begin{aligned} G^A(\nabla_h m, l) &= \int_{\partial\Omega} (1 + AK_c^2)\nabla_h m\gamma \\ &= \int_{\partial\Omega} (1 + AK_c^2)\nabla_V W\gamma \end{aligned} \tag{3.12}$$

and

$$\begin{aligned} G^A(m, \nabla_h l) &= \int_{\partial\Omega} (1 + AK_c^2)\beta\nabla_h l \\ &= \int_{\partial\Omega} (1 + AK_c^2)\beta\nabla_V Z. \end{aligned} \tag{3.13}$$

Therefore, by adding up the equations (3.12) and (3.13) we have

$$\begin{aligned} G^A(\nabla_h m, l) + G^A(m, \nabla_h l) &= \int_{\partial\Omega} (1 + AK_c^2)\nabla_V W\gamma + \int_{\partial\Omega} (1 + AK_c^2)\beta\nabla_V Z \\ &= \int_{\partial\Omega} (1 + AK_c^2)\left(\nabla_V W\gamma + \beta\nabla_V Z\right) ds. \end{aligned}$$

□

Remark 3.1. *We would like to find a linear connection that preserves the Riemannian metric G^A and if such a connection exists, it is such that $hG^A(m, l) = G^A(\nabla_h m, l) + G^A(m, \nabla_h l)$. By using the Propositions 3.2 and 3.3 we have*

$$\begin{aligned} \int_{\partial\Omega} (1 + AK_c^2)\left(\nabla_V W\gamma + \beta\nabla_V Z\right) ds &= \int_{\partial\Omega} \left(2AK_c^3\alpha\beta\gamma + (1 + AK_c^2)\tfrac{\partial\beta}{\partial\vec{\nu}}\gamma\alpha\right) ds \\ &+ \int_{\partial\Omega} \left((1 + AK_c^2)\tfrac{\partial\gamma}{\partial\vec{\nu}}\beta\alpha + K_c\alpha\beta\gamma + AK_c^3\alpha\beta\gamma\right) ds. \\ &= \int_{\partial\Omega} \left(3AK_c^3 + K_c\right)\alpha\beta\gamma + (1 + AK_c^2)\frac{\partial\beta}{\partial\vec{\nu}}\gamma\alpha \\ &+ (1 + AK_c^2)\frac{\partial\gamma}{\partial\vec{\nu}}\beta\alpha ds. \end{aligned}$$

From which, we get a simplified expression:

$$\begin{aligned} \nabla_V W\gamma + \beta\nabla_V Z &= \left(\frac{3AK_c^3 + K_c}{1 + AK_c^2}\right)\alpha\beta\gamma + \frac{\partial\beta}{\partial\nu}\gamma\alpha + \frac{\partial\gamma}{\partial\nu}\beta\alpha \\ \nabla_V W\gamma &= \left(\frac{3AK_c^3 + K_c}{1 + AK_c^2}\right)\alpha\beta\gamma + \frac{\partial\beta}{\partial\vec{\nu}}\gamma\alpha + \frac{\partial\gamma}{\partial\vec{\nu}}\beta\alpha - \beta\nabla_V Z. \end{aligned}$$

By pointing out that

$$\nabla_V Z = \frac{\partial\gamma}{\partial\vec{\nu}}\alpha,$$

we have

$$\begin{aligned}\nabla_V W\gamma &= \left(\frac{3AK_c^3+K_c}{1+AK_c^2}\right)\alpha\beta\gamma + \frac{\partial\beta}{\partial\vec{\nu}}\gamma\alpha + \beta\frac{\partial\gamma}{\partial\vec{\nu}}\alpha - \beta\frac{\partial\gamma}{\partial\vec{\nu}}\alpha \\ &= \left(\frac{3AK_c^3+K_c}{1+AK_c^2}\right)\alpha\beta\gamma + \frac{\partial\beta}{\partial\vec{\nu}}\gamma\alpha.\end{aligned} \tag{3.14}$$

Finally, we have

$$\nabla_V W = \left(\frac{3AK_c^3+K_c}{1+AK_c^2}\right)\alpha\beta + \frac{\partial\beta}{\partial\vec{\nu}}\alpha. \tag{3.15}$$

Now let us verify that the connection ∇ is linear

- Let $V, W, Z \in \mathcal{C}^\infty(\mathbb{R}^2, \mathbb{R}^2)$ be vector fields which are orthogonal to the boundary of Ω, we have

$$\begin{aligned}\nabla_V(W+Z) := \nabla_h(m+l) &= \frac{\partial(\beta+\gamma)}{\partial\vec{\nu}}\alpha + \left(\frac{3AK_c^3+K_c}{1+AK_c^2}\right)\alpha(\beta+\gamma) \\ &= \langle D_V(W+Z), \vec{\nu}\rangle + \left(\frac{3AK_c^3+K_c}{1+AK_c^2}\right)\langle V, \vec{\nu}\rangle\langle W+Z, \vec{\nu}\rangle \\ &= \langle D_V W, \vec{\nu}\rangle + \langle D_V Z, \vec{\nu}\rangle + \left(\frac{3AK_c^3+K_c}{1+AK_c^2}\right)\langle V, \vec{\nu}\rangle\langle W, \vec{\nu}\rangle \\ &+ \left(\frac{3AK_c^3+K_c}{1+AK_c^2}\right)\langle V, \vec{\nu}\rangle\langle Z, \vec{\nu}\rangle \\ &= \left(\frac{3AK_c^3+K_c}{1+AK_c^2}\right)\alpha\beta + \frac{\partial\beta}{\partial\vec{\nu}}\alpha + \left(\frac{3AK_c^3+K_c}{1+AK_c^2}\right)\alpha\gamma + \frac{\partial\gamma}{\partial\vec{\nu}}\alpha \\ \nabla_V(W+Z) &= \nabla_V W + \nabla_V Z.\end{aligned} \tag{3.16}$$

- Let f be a scalar field and $V, W \in \mathcal{C}^\infty(\mathbb{R}^2, \mathbb{R}^2)$ be vector fields which are orthogonal to the boundary of Ω, we have

$$\begin{aligned}\nabla_{fV} W &= \nabla_{fh} m \\ \nabla_{fV} W &= f\frac{\partial\beta}{\partial\vec{\nu}}\alpha \\ \nabla_{fV} W &= f\nabla_V W.\end{aligned} \tag{3.17}$$

From the equations (3.16) and (3.17) one can conclude that ∇ is linear.

Remark 3.2. *The map*

$$\xi : \left| \begin{array}{ccc} T_c B_e & \longrightarrow & T_c B_e^* \\ h & \mapsto & \{i_h G^A : m \mapsto G^A(h,m)\} \end{array} \right.$$

is linear, injective and also surjective. Indeed given G^A as defined above, for any $y \in T_cB_e^*$, we want to find a vector $h \in T_cB_e$ such that

$$\begin{aligned} \xi(h) &= y \\ &\iff i_hG^A = y \\ &\iff i_hG^A(m) = y(m), \qquad \forall m \in T_cB_e \\ &\iff G^A(h,m) = y(m), \qquad \forall m \in T_cB_e. \end{aligned} \tag{3.18}$$

From the equation (3.18) we can write

$$\begin{aligned} G^A(h,m) &= y(m) = \int_{\partial\Omega} (1+AK_c^2).\frac{y(m)}{|\partial\Omega|(1+AK_c^2)}ds \\ G^A(h,m) &= \int_{\partial\Omega} (1+AK_c^2).\frac{1}{1+AK_c^2}.\frac{1}{|\partial\Omega|}yds \end{aligned} \tag{3.19}$$

Using the definition above, we have

$$G^A(h,m) = \int_{\partial\Omega} (1+AK_c^2)\langle h,m\rangle ds. \tag{3.20}$$

Then from the equations (3.19) and (3.20) we can show that

$$\langle h,m\rangle = \frac{y(m)}{|\partial\Omega|(1+AK_c^2)} \qquad \forall m \in T_cB_e. \tag{3.21}$$

Does there exist a $h = \alpha\vec{\nu}$ such that

$$\langle h,m\rangle = \frac{y(m)}{|\partial\Omega|(1+AK_c^2)} \qquad \forall m \in T_cB_e \text{ with } m := \beta\vec{\nu}\ ? \tag{3.22}$$

One can show that $y(m)$ is linear with respect to m. Indeed $y(m) = G^A(h,m)$ and G^A is linear with m. Since $y(m) \in \mathbb{R}$ then we can write $y(m) = k.m$ for $k \in T_cB_e$ with $k := \gamma\vec{\nu}$ and $\gamma \in \mathcal{C}^\infty(\mathbb{S}^1,\mathbb{R})$. Therefore from (3.22) we have

$$\alpha\beta = \frac{y(\beta\vec{\nu})}{|\partial\Omega|(1+AK_c^2)} \qquad \forall \beta \in \mathcal{C}^\infty(\mathbb{S}^1,\mathbb{R}) \tag{3.23}$$

$$\alpha\beta = \frac{k.\beta\vec{\nu}}{|\partial\Omega|(1+AK_c^2)} \tag{3.24}$$

$$\alpha\beta = \frac{\gamma\beta}{|\partial\Omega|(1+AK_c^2)} \tag{3.25}$$

$$\tag{3.26}$$

therefore

$$\alpha = \frac{\gamma}{|\partial\Omega|(1+AK_c^2)}$$

then

$$h = \frac{\gamma}{|\partial\Omega|(1 + AK_c^2)}$$

$$\int_{\partial\Omega} ds = |\partial\Omega|.$$

Now we can conclude that for

$$h = \frac{\gamma}{|\partial\Omega|(1 + AK_c^2)}$$

then

$$\forall y \in T_c B_e^* \qquad \exists h \in T_c B_e : \xi(h) = y.$$

Therefore the map ξ induces an isomorphism between $T_c B_e$ and $T_c B_e^*$. Consequently, from the Definition 3.3 we can check that G^A is a strongly Riemannian metric and then by using the Proposition 3.1 one can deduce that ∇ exists, it is unique and coincides with the Levi-Civita connection.

And then we are now able to claim the following theorem.

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^2$ where Ω is a simply connected and compact subset of $\mathbb{R}^2$ and be at least of class $\mathcal{C}^2$, and $\vec{\nu}$ is the outer unit normal of the domain Ω and $V, W \in \mathcal{C}^\infty(\mathbb{R}^2, \mathbb{R}^2)$ vector fields which are orthogonal to the boundaries i.e*

$$V_{|\partial\Omega} = \alpha\vec{\nu}$$

with $\alpha := \langle V_{|\partial\Omega}, \vec{\nu} \rangle$ and

$$W_{|\partial\Omega} = \beta\vec{\nu}$$

with $\beta := \langle W_{|\partial\Omega}, \vec{\nu} \rangle$ such that $V_{|\partial\Omega} = h := \alpha\vec{\nu}$, $W_{|\partial\Omega} = m =: \beta\vec{\nu}$ belongs to the tangent space of B_e. Then the covariant derivative associated with the Riemannian metric G^A can be expressed as follows:

$$\begin{aligned} \nabla_V W : &= \nabla_h m = \frac{\partial \beta}{\partial \vec{\nu}}\alpha + \left(\frac{3AK_c^3 + K_c}{1 + AK_c^2} \right) \alpha\beta \\ &= \langle D_V W, \vec{\nu} \rangle + \left(\frac{3AK_c^3 + K_c}{1 + AK_c^2} \right) \langle V, \vec{\nu} \rangle \langle W, \vec{\nu} \rangle \end{aligned}$$

where $D_V W$ is the directional derivative of the vector field W in the direction V.

Proof. It is a straight consequence of the above propositions and remarks. □

Remark 3.3. *It is quite possible to begin the proof of the above theorem by the application of shape calculus rules for volume and boundary functionals as in* [12], [34], [18] *on the following functional*

$$\int_{\partial\Omega}(1+AK_c^2)\alpha\beta d\sigma.$$

The remaining computations are mostly similar. We only underline that, at the end, it is necessary to see that the local covariant derivative is $\nabla_{X_x}Y = \frac{d}{dt_{|t=0}}(Y(x+tX(x)))$ *where* $Y = X = \vec{\nu}$ *and* $D_{\vec{\nu}}\vec{\nu} = 0$ *since* $|\vec{\nu}|^2 = 1, D_{\vec{\nu}}$ *being the Jacobian matrix.*

4. Sufficient condition for the minimality of a shape functional

In this section, assuming at first that there is at least one critical point, we shall first present the sufficient condition on the existence of a local minimum for a functional $J(\Omega)$ given as follows:

$$J(\Omega) = \int_{\Omega} f_0(u_\Omega, grad(u_\Omega)) \tag{4.27}$$

where f_0 is a function of $\mathbb{R}\times\mathbb{R}^n$ that we suppose to be smooth and u_Ω denotes a smooth solution of a boundary value problem.
And in the second part, in the case where $J(\Omega) = -\frac{1}{2}\displaystyle\int_{\Omega}|grad(u_\Omega)|^2dx + \frac{k^2}{2}|\Omega|$, we compute the second shape derivative.

The fundamental question is then to study the existence of the local strict minima of this functional under possible constraints that Ω is a critical point. That means that the first order derivative with respect to the domain is equal to zero at the domain Ω. We shall examine, for that, how this solution u_Ω varies when its domain of definition Ω moves.

Let us recall the classical method of studying a critical point. Let $(B, \|\ .\ \|_1)$ be a Banach space and let $E : (B, \|\ .\ \|_1) \longrightarrow \mathbb{R}$ be a function of class C^2 whose differential Df vanishes at 0. The Taylor-Young formula is then written as

$$E(u) = E(0) + D^2\ E(0)\ .\ (u,u) + o(||u||_1^2). \tag{4.28}$$

In particular, if the Hessian form $D^2E(0)$ is coercive in the norm $\|\ .\ \|_1$, then the critical point 0 is a strict local minimum of E. The fundamental difficulty in the study of critical forms is caused by the appearance of a second norm $\|\ .\ \|_2$ finer than $\|\ .\ \|_1$ $(i.e\ \ \|\ .\ \|_2 \leq C\|\ .\ \|_1)$. The Hessian form, is not in general, coercive for the norm $\|\ .\ \|_1$ but it is for the standard norm $\| \,.\, \|_2$. If these norms are not equivalent, which is generally the case, concluding that the minimum is strict is impossible, even locally for the strong norm. It is quite possible to give several examples. But let us reproduce a simple example of such a situation on the space $H_0^1(0,1)$ that was presented in the thesis [8]. Let us consider the functional E defined by

$$E(u) = \|u\|^2_{L^2(0,1)} - \|u\|^4_{H_0^1(0,1)}.$$

We can check that E is twice differentiable on $H_0^1(0,1)$. Further more, one has at 0:

$$\begin{cases} E'(0) = 0 \\ E''(0).(h,h) = 2\|h\|^2_{L^2(0,1).} \end{cases}$$

For each direction, we find that 0 is a strictly local minimum. Indeed, for all nonzero $u_0 \in H_0^1(0,1)$ and for all $t \in \mathbb{R}$, we have

$$E(tu_0) = t^2\|u_0\|^2_{L^2(0,1)} - t^4\|u_0\|^4_{H_0^1(0,1))} > 0 \ if \ t^2 < \frac{\|u_0\|^2_{L^2(0,1)}}{\|u_0\|^4_{H_0^1(0,1)}}.$$

However, 0 is not a local minimum even for the H_0^1 norm. Indeed, there is no $r > 0$ such that

$$\|u\|_{H_0^1(0,1)} < r \Longrightarrow E(u) > E(0) = 0 \ \text{ i.e } \ \|u\|^2_{L^2(0,1)} > \|u\|^4_{H_0^1(0,1)},$$

since we can always build a sequence in $H_0^1(0,1)$ such that

$$\begin{cases} \|u_n\|_{H_0^1(0,1)} = r/2, \\ \|u_n\|_{L^2(0,1)} \longrightarrow 0 \text{ when } n \longrightarrow +\infty. \end{cases}$$

To solve this problem, we will use the Taylor's formula with an integral remainder, instead of (4.28) i.e

$$E(u) - E(0) = \int_0^1 (1-t) \ E''(tu)(u,u) \ dt. \tag{4.29}$$

This formula allows to express exactly the difference in energy between a critical form Ω_0 and a neighboring form Ω via an integral term that we can carefully estimate thanks to the study of the variations of the Hessian.

Theorem 4.1. *Let $f_0 : \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{R}$, $(s,v) \longmapsto f_0(s,v)$ be a function of class $\mathcal{C}^3$ and f a function in $C^{0,\gamma}(\mathbb{R}^N, \mathbb{R}), \gamma \in (0,1)$. Let $L_0 = div(Agrad(.))$ be strictly and uniformly elliptical operator with A in $\mathcal{C}^2(\mathbb{R}^N, \ M_N(\mathbb{R}^N))$. Let E be the defined shape functional on the class $\mathcal{O}$ of open class $\mathcal{C}^{2,\gamma}$ as*

$$J(\Omega) = \int_\Omega f_0(u_\Omega, \ grad(u_\Omega)),$$

where $M_N(\mathbb{R}^N)$ stands for the space of square matrix of order N and u_Ω is the solution of the homogeneous Dirichlet problem

$$\begin{cases} L_0 u = f \quad in \quad \Omega, \\ u = 0 \quad on \quad \partial\Omega. \end{cases}$$

Let $\Omega_0 \in \mathcal{O}$, then, there exist a real $\eta_0 > 0$ and an increasing function $\omega : (0, \eta_0] \longrightarrow (0, +\infty)$ with $\lim_{r \searrow 0} \omega(r) = 0$, which depend only on Ω_0, L_0, f_0 and f, such that for all $\eta \in (0, \eta_0]$ and for all $\theta \in \mathcal{C}^{2,\alpha}(\mathbb{R}^N, \ \mathbb{R}^N)$ satisfying

$$\|\theta - Id_{\mathbb{R}^N}\|_{2,\alpha} \leq \eta,$$

we have the following estimate valid for all t in $[0, 1]$,

$$\left| \frac{d^2}{dt^2} \ J(\Omega_t) - \frac{d^2}{dt^2}_{|t=0} \ J(\Omega_t) \right| \leq \omega(\eta) \| < V, \vec{\nu} > \|^2_{H^{1/2}(\partial\Omega_0)} \tag{4.30}$$

where $\Omega_t = \Phi_t(\Omega_0), t \in [0, 1]$ stands for the flow related to the vector field V.

This is exactly the Theorem 1 in [9] and for its proof the reader is invited to see this paper.

In the case where Ω_0 is a critical point for the functional J, to show that it is a strict local minimum, we have to study the positiveness of a quadratic form which we are going to denote by Q . This quadratic form is obtained by computing the second derivative of J with respect to the domain. So before proceeding, we need some hypothesis ;
let us suppose that:

- (i) - Ω is a $\mathcal{C}^2$- regular open domain.
- (ii) - $V(x; t) = \alpha(x)\vec{\nu}(x), \ \alpha \in H^{\frac{1}{2}}(\partial\Omega), \ \forall \ t \in [0, \epsilon[$.

In [10], (see also [9], [8]), the authors showed that it is not sufficient to prove that the quadratic form is positive to claim that: a critical shape is a minimum. In fact most of the time people use the Taylor Young formula to study the positiveness of the quadratic form.
For $t \in [0, \epsilon[$, $j(t) := J(\Omega_t) = J(\Omega) + tdJ(\Omega, V) + \frac{1}{2}t^2 d^2 J(\Omega, V, V) + o(t^2)$.
The quantity $o(t^2)$ is expressed with the norm of $\mathcal{C}^2$. The $H^{\frac{1}{2}}(\partial\Omega)$ norm appears in the expression of $d^2 J(\Omega, V, V)$. And these two norms are not equivalent. The quantity $o(t^2)$ is not smaller than $||V||_{H^{\frac{1}{2}}(\partial\Omega)}$, see the example in [10]. Then such an argument does not insure that the critical point is a local strict minimum.

In our study, we shall see that the main result in [10] can be satisfied in a simple way thanks to the hessian obtained via the Sobolev metric G^A in which the norm of $H^{1/2}(\partial\Omega)$ appears directly. And this overcomes the clasical issue. In fact the study of the sign of $\int_{\partial\Omega} H d\sigma$ becomes the only control from which one can get information on the optimal domain and then on the optimal shape.

Proposition 4.1.

Let Ω be a critical point for the functional J, then

$$\begin{aligned} Q(\alpha) &= d^2J(\Omega; V; V) \\ &= -(N-1)\int_{\partial\Omega} H\alpha^2 d\sigma + k^2\int_{\Omega} |grad(\Lambda)|^2 dx \\ &= -(N-1)k^2\int_{\partial\Omega} H\alpha^2 d\sigma + k^2\int_{\partial\Omega} \alpha L\alpha d\sigma, \end{aligned}$$

where Λ is a solution of the following boundary value problem

$$\begin{cases} -\Delta\Lambda = 0 & in \quad \Omega \\ \quad \Lambda = \alpha & on \quad \partial\Omega. \end{cases} \tag{4.31}$$

H is the mean curvature of $\partial\Omega$ and L is a pseudo differential operator which is known as the Steklov-Poincaré or capacity or Dirichlet to Neumann(see e.g [11]*) operator, defined by* $L\alpha = \dfrac{\partial\Lambda}{\partial\vec{\nu}}$.

Proof. We use the definition of the derivative with respect to the domain and we apply it to $dJ(\Omega, V)$.

Then we get

$$\begin{aligned} 2Q(\alpha) &= 2d^2J(\Omega, V, V) \\ &= \int_{\Omega} (\mathrm{div}((k^2 - |grad(u)|^2)V(x,0)))' dx + \int_{\Omega} \mathrm{div}(V(x,0)\mathrm{div}((k^2 - |grad(u)|^2)V(x,0)))dx \\ 2Q(\alpha) &= \left[\int_{\partial\Omega} -2grad(u)grad(u')V(x,0).\vec{\nu} + \mathrm{div}((k^2 - |grad(u)|^2)V(x,0))V(x,0).\vec{\nu}\right] d\sigma. \end{aligned}$$

Since Ω is solution of the quadradure surface problem then $-\dfrac{\partial u}{\partial\vec{\nu}} = k$ on $\partial\Omega$.
By assumption, $\partial\Omega$ is of $\mathcal{C}^2$ class and since $u = 0$ on $\partial\Omega$,
we have
$grad(u) = \dfrac{\partial u}{\partial\vec{\nu}}\vec{\nu} = -k\vec{\nu}$. Hence

$$2Q(\alpha) = \left[\int_{\partial\Omega} 2kgrad(u').\vec{\nu}V(x,0).\vec{\nu} + \mathrm{div}((k^2 - |grad(u)|^2)V(x,0))V(x,0).\vec{\nu}\right] d\sigma.$$

A classical calculus in shape optimization leads to $u' = -\dfrac{\partial u}{\partial \vec{\nu}} V.\vec{\nu}$ on $\partial\Omega$.
Let us recall again that $-\dfrac{\partial u}{\partial \vec{\nu}} = k$ on $\partial\Omega$ and $V.\vec{\nu} = \alpha$. Then, we have $u' = k\alpha$ on $\partial\Omega$ and

$$\vec{\nu} grad(u') = \frac{\partial u'}{\partial \vec{\nu}} = k\frac{\partial \alpha}{\partial \vec{\nu}} = kL\alpha,$$

where L is a pseudo differential operator, defined by $L\alpha = \dfrac{\partial \Lambda}{\partial \vec{\nu}}$ and such that

$$\begin{cases} -\Delta\Lambda & = & 0 & \text{in} & \Omega \\ \Lambda & = & \alpha & \text{on} & \partial\Omega, \end{cases} \tag{4.32}$$

Λ is the extension of α in Ω.

Hence

$$2Q(\alpha) = \int_{\partial\Omega} (2k^2 \alpha L\alpha - \operatorname{div}((|grad(u)|^2 - k^2)\alpha.\vec{\nu})\alpha) d\sigma. \tag{4.33}$$

Let us compute now $\operatorname{div}((|grad(u)|^2 - k^2)\alpha.\vec{\nu})$ on $\partial\Omega$. Since $|grad(u)| = k$ on $\partial\Omega$, we have

$$\operatorname{div}((|grad(u)|^2 - k^2)\alpha.\vec{\nu}) = \alpha grad(|grad(u)|^2 - k^2).\vec{\nu} = \alpha grad(|grad(u)|^2).\vec{\nu}.$$

Since we have supposed that Ω of class $\mathcal{C}^2$, locally, $\partial\Omega$ can be described by a curve φ such that $x_N = \varphi(x'), x' \in \mathbb{R}^{N-1}$ and $D\varphi(x') = 0$. Here, $D\varphi(x')$ is the Jacobian matrix of φ.
Let us set $x_0 = (x', x_N) = (x', \varphi(x')) \in \partial\Omega$ then we have $u(x_0) = 0$.
By differentiating with respect to s_j for all $j \in \{1, \cdots, N-1\}$, we have

$$\frac{\partial u(x_0)}{\partial s_j} + \frac{\partial \varphi\ (x')}{\partial s_j}\ \frac{\partial u\ (x_0)}{\partial \vec{\nu}} = 0,$$

since $\dfrac{\partial \varphi(x')}{\partial s_j} = 0$, we get $\dfrac{\partial u(x_0)}{\partial s_j} = 0$.
Starting from the following equality

$$\frac{\partial u(x_0)}{\partial s_j} + \frac{\partial \varphi(x')}{\partial s_j}\ \frac{\partial u(x_0)}{\partial \vec{\nu}} = 0, \tag{4.34}$$

and by differentiating it with respect to s_i for all $i \in \{1, \cdots, N-1\}$, we have

$$\frac{\partial^2 u(x_0)}{\partial s_i\, \partial s_j} + \frac{\partial \varphi(x')}{\partial s_i}\ \frac{\partial^2 u(x_0)}{\partial \vec{\nu}\, \partial s_j} + \frac{\partial^2 \varphi(x')}{\partial s_i\, \partial s_j}\ \frac{\partial u(x_0)}{\partial \vec{\nu}}$$

$$+\frac{\partial \varphi(x')}{\partial s_j}\ \frac{\partial^2 u(x_0)}{\partial s_i\, \partial \vec{\nu}} + \frac{\partial \varphi(x')}{\partial s_j}\ \frac{\partial \varphi(x')}{\partial s_i}\ \frac{\partial^2 u(x_0)}{\partial \vec{\nu}^2} = 0.$$

Note that $u(x_0) = 0$ and $\dfrac{\partial u(x_0)}{\partial s_j} = 0\ \forall j \in \{1, \cdots, N-1\}$ and summing over the indices i, j, we have

$$\sum_{j=1}^{N-1} \frac{\partial^2 u(x_0)}{\partial s_j^2} + (N-1)\ H\ \frac{\partial u(x_0)}{\partial \vec{\nu}} = 0. \tag{4.35}$$

Since $\dfrac{\partial u(x_0)}{\partial s_i} = 0\ \forall i \in \{1, \cdots, N-1\}$, we have also

$$\begin{aligned} grad(|grad(u)|^2(x_0)).\vec{\nu} &= \tfrac{\partial}{\partial \vec{\nu}}\left[\sum_{i=1}^{N-1} \left(\frac{\partial u(x_0)}{\partial s_i}\right)^2 + \left(\frac{\partial u(x_0)}{\partial \vec{\nu}}\right)^2\right] \\ &= 2\ \frac{\partial u(x_0)}{\partial \vec{\nu}}\ \ \frac{\partial^2 u(x_0)}{\partial \vec{\nu}^2}. \end{aligned}$$

In addition, we can remark that:
$\dfrac{\partial^2 u(x_0)}{\partial \vec{\nu}^2} = -\displaystyle\sum_{i=1}^{N-1} \frac{\partial^2 u(x_0)}{\partial s_i^2} - f$ on $\partial\Omega$.
Therefore, we have

$$\begin{aligned} grad(|grad(u)|^2(x_0)).\vec{\nu} &= 2\ \frac{\partial u(x_0)}{\partial \vec{\nu}} \left(-\sum_{i=1}^{N-1} \frac{\partial^2 u(x_0)}{\partial s_i^2} - f\right) \\ &= 2\ \frac{\partial u(x_0)}{\partial \vec{\nu}} \left((N-1)\ H\ \frac{\partial u(x_0)}{\partial \vec{\nu}} - f\right). \end{aligned}$$

When the support of the function f is in Ω, then $f = 0$ on $\partial\Omega$.

Finally we have

$$\begin{aligned} 2Q(\alpha) &= \int_{\partial\Omega} 2k^2\alpha L\alpha - 2(N-1)H\alpha^2\ \left(\frac{\partial u(x_0)}{\partial \vec{\nu}}\right)^2 d\sigma \\ &= \int_{\partial\Omega} 2k^2\alpha L\alpha - 2k^2(N-1)H\alpha^2 d\sigma. \end{aligned} \tag{4.36}$$

And by the Green's formula we get

$$\int_{\partial\Omega} \alpha L\alpha d\sigma = \int_{\Omega} |grad(\Lambda)|^2 dx. \tag{4.37}$$

□

5. Positiveness of the quadratic form in the infinite Riemannian point of view

Definition 5.1. *Let $J : \Omega \to \mathbb{R}$ be an functional. One defines the hessian Riemannian shape as follows:*

$$HessJ(\Omega)[V] := \nabla_V gradJ$$

where ∇_V denotes the derivative following the vector field V.

Theorem 5.1. *The hessian Riemannian shape defined by the Riemannian metric G^A verifies the following condition:*

$$G^A(HessJ(\Omega)[V], W) = d(dJ(\Omega)[W])[V] - dJ(\Omega)[\nabla_V W].$$

Proof. Our purpose is to show that

$$G^A(HessJ(\Omega)[V], W) = d(dJ(\Omega)[W])[V] - dJ(\Omega)[\nabla_V W].$$

So let us use the compatibility of the metric G^A with the Levi-Civita connection. We have

$$\begin{aligned} V.G^A(gradJ, W) &= G^A(gradJ, \nabla_V W) + G^A(\nabla_V gradJ, W), \\ G^A(\nabla_V gradJ, W) &= V.G^A(gradJ, W) - G^A(gradJ, \nabla_V W). \end{aligned}$$

Since $G^A(HessJ(\Omega)[V], W) = G^A(\nabla_V gradJ, W)$, we have

$$\begin{aligned} G^A(HessJ(\Omega)[V], W) &= V.G^A(gradJ, W) - G^A(gradJ, \nabla_V W), \\ G^A(HessJ(\Omega)[V], W) &= V.(WJ) - (\nabla_V W).J), \\ G^A(HessJ(\Omega)[V], W) &= d(dJ(\Omega)[W])[V] - dJ(\Omega)[\nabla_V W] \end{aligned}$$

where $V, W \in \mathcal{C}^\infty(\mathbb{R}^2, \mathbb{R}^2)$ are vector fields normal to the boundary $\partial\Omega$ and $d(dJ(\Omega)[W])[V]$ defines the standard Hessian shape. □

Remark 5.1. *In our quadrature surface case, for $W = m\vec{\nu}$ and $V = h\vec{\nu}$, we have*

$$dJ(\Omega)[V] = \int_{\partial\Omega} \left(k^2 - \left(\frac{\partial u_\Omega}{\partial \vec{\nu}}\right)^2\right) \alpha d\sigma$$

and then,

$$d\left(dJ(\Omega)[W]\right)[V] = d\left(\int_{\partial\Omega} \left(k^2 - \left(\frac{\partial u_\Omega}{\partial \vec{\nu}}\right)^2\right) m d\sigma\right)[V].$$

Setting

$$\psi := k^2 - \left(\frac{\partial u_\Omega}{\partial \vec{\nu}}\right)^2,$$

then

$$\psi_t := k^2 - \left(\frac{\partial u_{\Omega_t}}{\partial \vec{\nu}}\right)^2,$$

we have

$$d\left(dJ(\Omega)[W]\right)[V] = d\left(\int_{\partial\Omega_t} \psi_t m d\sigma\right)[h]. \tag{5.38}$$

This is nothing but

$$\begin{aligned} d\left(dJ(\Omega)[W]\right)[V] &= \int_{\partial\Omega} \frac{\partial \psi_t}{\partial t}_{|t=0} m d\sigma + \int_{\partial\Omega} \frac{\partial(\psi m)}{\partial \vec{\nu}} h d\sigma + \int_{\partial\Omega} K_c \psi m h d\sigma \\ &= \int_{\partial\Omega} \left[\frac{\partial \psi_t}{\partial t}_{|t=0} m + \left(\frac{\partial \psi}{\partial \vec{\nu}} + K_c \psi\right) m h + \psi \frac{\partial m}{\partial \vec{\nu}} h\right] d\sigma. \end{aligned}$$

Let us compute

$$
\begin{aligned}
\frac{\partial \psi_t}{\partial t}_{|t=0} m &= \frac{\partial \left[k^2 - \left(\frac{\partial u_{\Omega_t}}{\partial \vec{\nu}_t}\right)^2\right]}{\partial t}_{|t=0} m, \\
\frac{\partial \psi_t}{\partial t}_{|t=0} m &= -2m \left[\frac{\partial (grad(u_{\Omega_t})).\vec{\nu}}{\partial t}_{|t=0} + D^2 u_\Omega V.\vec{\nu} + grad(u_\Omega).\left(\frac{\partial \vec{\nu}_t}{\partial t}_{|t=0} + D_{\vec{\nu}} V\right)\right], \\
\frac{\partial \psi_t}{\partial t}_{|t=0} m &= -2m \left[grad(u'_\Omega).\vec{\nu} + D^2 u_\Omega V.\vec{\nu} + grad(u_\Omega).\left(\frac{\partial \vec{\nu}_t}{\partial t}_{|t=0} + D_{\vec{\nu}} V\right)\right],
\end{aligned} \tag{5.39}
$$

where $D^2 u_\Omega$ *is the hessian matrix and* $D_{\vec{\nu}}$ *the jacobian matrix of* $\vec{\nu}$ *.*

Let us calculate now the following expression: $\frac{\partial \vec{\nu}_t}{\partial t}_{|t=0}$.

We have

$$
\frac{\partial \vec{\nu}_t}{\partial t}_{|t=0} = -grad_\Gamma(V.\vec{\nu}) - (D_{\vec{\nu}_0}.\vec{\nu})V.\vec{\nu} \quad on \quad \Gamma,
$$

where $grad_\Gamma$ *is the tangential gradient,* $\Gamma = \partial\Omega$ *and* $\vec{\nu}_0 = \vec{\nu}$ *hence*

$$
\frac{\partial \vec{\nu}_t}{\partial t}_{|t=0} = -grad_\Gamma(V.\vec{\nu}) - (D_{\vec{\nu}}.\vec{\nu})V.\vec{\nu} \quad on \quad \Gamma.
$$

Since $D_{\vec{\nu}}.\vec{\nu} \equiv 0$, *then*

$$
\frac{\partial \vec{\nu}_t}{\partial t}_{|t=0} = -grad_\Gamma(V.\vec{\nu}) \quad on \quad \Gamma.
$$

So

$$
\frac{\partial \psi_t}{\partial t}_{|t=0} m = -2m \left[grad(u'_\Omega).\vec{\nu} + D^2 u_\Omega V.\vec{\nu} + grad(u_\Omega).\left(-grad_\Gamma(V.\vec{\nu}) + D_{\vec{\nu}}.V\right)\right].
$$

And finally, we get

$$
\begin{aligned}
d\left(dJ(\Omega)[W]\right)[V] = \int_{\partial\Omega} \Big[&-2m\Big(grad(u'_\Omega)\cdot\vec{\nu} + D^2 u_\Omega V\cdot\vec{\nu} \\
&+ grad(u_\Omega)\cdot\big(-grad_\Gamma(V\cdot\vec{\nu}) + D_{\vec{\nu}} V\big)\Big) \\
&+ \left(\frac{\partial \psi}{\partial \vec{\nu}} + K_c \psi\right) mh + \psi \frac{\partial m}{\partial \vec{\nu}} h\Big] d\sigma,
\end{aligned}
$$

$$
\begin{aligned}
d\left(dJ(\Omega)[W]\right)[V] = \int_{\partial\Omega} \Big[&-2\langle W,\vec{\nu}\rangle\Big(grad(u'_\Omega)\cdot\vec{\nu} + D^2 u_\Omega V\cdot\vec{\nu} \\
&+ grad(u_\Omega)\cdot\big(-grad_\Gamma(V\cdot\vec{\nu}) + D_{\vec{\nu}} V\big)\Big) \\
&+ \left(\frac{\partial \psi}{\partial \vec{\nu}} + K_c \psi\right) \langle W,\vec{\nu}\rangle\langle V,\vec{\nu}\rangle + \psi\langle D_V W,\vec{\nu}\rangle\Big] d\sigma.
\end{aligned}
$$

On the one hand, having the following Riemannian hessian formula

$$G^A\left(HessJ(\Omega)[V], W\right) = d\left(dJ(\Omega)[W]\right)[V] - dJ(\Omega)[\nabla_V W] \tag{5.40}$$

it is possible to bring additional details on its computation.

Proposition 5.1. *We have*

$$\begin{aligned} G^A(HessJ(\Omega)[V], W) &= \int_{\partial\Omega} \left[-2\langle W, \vec{\nu}\rangle \left(grad\left(u'_\Omega\right).\vec{\nu} + D^2 u_\Omega V.\vec{\nu} + grad\left(u_\Omega\right).\left(-grad_\Gamma(V.\vec{\nu})\right.\right.\right. \\ &+ \left.\left.\left. D_{\vec{\nu}}V\right)\right)\right] d\sigma \\ &+ \int_{\partial\Omega} \left[\frac{\partial}{\partial\vec{\nu}}\left(k^2 - \left(\frac{\partial u_\Omega}{\partial\vec{\nu}}\right)^2\right) + K_c\left(k^2 - \left(\frac{\partial u_\Omega}{\partial\vec{\nu}}\right)^2\right)\right. \\ &- \left.\frac{3AK_c^3 + K_c}{1 + AK_c^2}\left(k^2 - \left(\frac{\partial u_\Omega}{\partial\vec{\nu}}\right)^2\right)\right]\langle V, \vec{\nu}\rangle\langle W, \vec{\nu}\rangle d\sigma. \end{aligned} \tag{5.41}$$

Proof.

$$\begin{aligned} G^A\left(HessJ(\Omega)[V], W\right) &= \int_{\partial\Omega} \left[-2\langle W, \vec{\nu}\rangle\left(grad(u'_\Omega).\vec{\nu} + D^2 u_\Omega V.\vec{\nu} + grad(u_\Omega).\left(-grad_\Gamma(V.\vec{\nu}) + D_{\vec{\nu}}V\right)\right)\right. \\ &+ \left.\left(\frac{\partial\psi}{\partial\vec{\nu}} + K_c\psi\right)\langle W, \vec{\nu}\rangle\langle V, \vec{\nu}\rangle + \psi\langle D_V W, \vec{\nu}\rangle\right] d\sigma \\ &- \int_{\partial\Omega} \psi\langle\nabla_V W, \vec{\nu}\rangle d\sigma, \\ &= \int_{\partial\Omega} \left[-2\langle W, \vec{\nu}\rangle\left(grad(u'_\Omega).\vec{\nu} + D^2 u_\Omega V.\vec{\nu} + grad(u_\Omega).\left(-grad_\Gamma(V.\vec{\nu}) + D_{\vec{\nu}}V\right)\right)\right. \\ &+ \left.\left(\frac{\partial\psi}{\partial\vec{\nu}} + K_c\psi\right)\langle W, \vec{\nu}\rangle\langle V, \vec{\nu}\rangle + \psi\langle D_V W, \vec{\nu}\rangle\right] d\sigma \\ &- \int_{\partial\Omega} \psi\left[\langle D_V W, \vec{\nu}\rangle + \left(\frac{3AK_c^3 + K_c}{1 + AK_c^2}\right)\langle V, \vec{\nu}\rangle\langle W, \vec{\nu}\rangle\right] d\sigma, \\ &= \int_{\partial\Omega} \left[-2\langle W, \vec{\nu}\rangle\left(grad\left(u'_\Omega\right).\vec{\nu} + D^2 u_\Omega V.\vec{\nu} + grad\left(u_\Omega\right).\left(-grad_\Gamma\left(V.\vec{\nu}\right)\right.\right.\right. \\ &+ \left.\left.\left. D_{\vec{\nu}}V\right)\right)\right] d\sigma \\ &+ \int_{\partial\Omega} \left[\frac{\partial\psi}{\partial\vec{\nu}} + K_c\psi - \psi K_c\left(\frac{3AK_c^2 + 1}{1 + AK_c^2}\right)\right]\langle V, \vec{\nu}\rangle\langle W, \vec{\nu}\rangle d\sigma. \end{aligned}$$

Replacing ψ by its expression, we have

$$\begin{aligned} G^A(HessJ(\Omega)[V], W) &= \int_{\partial\Omega} \left[-2\langle W, \vec{\nu}\rangle \left(grad\left(u'_\Omega\right).\vec{\nu} + D^2 u_\Omega V.\vec{\nu} + grad\left(u_\Omega\right).\left(-grad_\Gamma\left(V.\vec{\nu}\right)\right.\right.\right. \\ &+ \left.\left.\left. D_{\vec{\nu}}V\right)\right)\right] d\sigma \\ &+ \int_{\partial\Omega} \left[\frac{\partial}{\partial\vec{\nu}}\left(k^2 - \left(\frac{\partial u_\Omega}{\partial\vec{\nu}}\right)^2\right) + K_c\left(k^2 - \left(\frac{\partial u_\Omega}{\partial\vec{\nu}}\right)^2\right)\right. \\ &- \left.\frac{3AK_c^3 + K_c}{1 + AK_c^2}\left(k^2 - \left(\frac{\partial u_\Omega}{\partial\vec{\nu}}\right)^2\right)\right]\langle V, \vec{\nu}\rangle\langle W, \vec{\nu}\rangle d\sigma. \end{aligned} \tag{5.42}$$

□

On the other hand, let us compute $G^A(HessJ(\Omega)[V],W)$ by using directly the Sobolev-type metric G^A. Then we have the following proposition.

Proposition 5.2.

$$\begin{aligned} G^A(HessJ(\Omega)[V],W) &= \int_{\partial\Omega}\left[\frac{\partial}{\partial\vec{\nu}}\left(k^2-\left(\frac{\partial u_\Omega}{\partial\vec{\nu}}\right)^2\right)\right. \\ &+ \left. K_c\left(k^2-\left(\frac{\partial u_\Omega}{\partial\vec{\nu}}\right)^2\right)\right]\langle V,\vec{\nu}\rangle\langle W,\vec{\nu}\rangle d\sigma. \end{aligned} \tag{5.43}$$

Proof.

$$\begin{aligned} G^A(HessJ(\Omega)[V],W) &= \int_{\partial\Omega}\left(1+AK_c^2\right)HessJ(\Omega)[V]W, \\ &= \int_{\partial\Omega}\left(1+AK_c^2\right)\nabla_V gradJ(\Omega)W, \\ &= \int_{\partial\Omega}\left(1+AK_c^2\right)\nabla_h gradJ(\Omega)m. \end{aligned}$$

Since $gradJ(\Omega)=\frac{1}{1+AK_c^2}\psi$, we have

$$\begin{aligned} \nabla_h gradJ(\Omega) &= \frac{\partial}{\partial\vec{\nu}}\left(gradJ(\Omega)\right)\alpha+\left(\frac{3AK_c^3+K_c}{1+AK_c^2}\right)gradJ(\Omega)\alpha, \\ &= \frac{\partial}{\partial\vec{\nu}}\left(\frac{1}{1+AK_c^2}\psi\right)\alpha+\frac{1}{1+AK_c^2}\psi\left(\frac{3AK_c^3+K_c}{1+AK_c^2}\right)\alpha, \\ &= \frac{\partial}{\partial\vec{\nu}}\left[(1+AK_c^2)^{-1}\right]\psi\alpha+\frac{\partial\psi}{\partial\vec{\nu}}\left(\frac{1}{1+AK_c^2}\right)\alpha+\frac{1}{1+AK_c^2}\psi\left(\frac{3AK_c^3+K_c}{1+AK_c^2}\right)\alpha, \\ &= -2AK_c\frac{\partial K_c}{\partial\vec{\nu}}\left(1+AK_c^2\right)^{-2}\psi\alpha+\frac{\partial\psi}{\partial\vec{\nu}}\left(\frac{1}{1+AK_c^2}\right)\alpha \\ &+ \frac{1}{1+AK_c^2}\psi\left(\frac{3AK_c^3+K_c}{1+AK_c^2}\right)\alpha. \end{aligned}$$

Note that $\frac{\partial K_c}{\partial\vec{\nu}}=K_c^2$, which implies that:

$$\begin{aligned} \nabla_h gradJ(\Omega) &= \frac{-2AK_c^3}{(1+AK_c^2)^2}\psi\alpha+\frac{\partial\psi}{\partial\vec{\nu}}\left(\frac{1}{1+AK_c^2}\right)\alpha \\ &+ \frac{1}{1+AK_c^2}\psi\left(\frac{3AK_c^3+K_c}{1+AK_c^2}\right)\alpha. \end{aligned} \tag{5.44}$$

Then, coming back to our hessian computation, we have

$$\begin{aligned}
G^A(HessJ(\Omega)[V],W) &= \int_{\partial\Omega}\left(1+AK_c^2\right)\left[\frac{-2AK_c^3}{(1+AK_c^2)^2}\psi\alpha+\frac{\partial\psi}{\partial\vec{\nu}}\left(\frac{1}{1+AK_c^2}\right)\alpha\right.\\
&+ \left.\frac{1}{1+AK_c^2}\psi\left(\frac{3AK_c^3+K_c}{1+AK_c^2}\right)\alpha\right]\beta d\sigma,\\
&= \int_{\partial\Omega}\left[\frac{-2AK_c^3}{1+AK_c^2}\psi\alpha+\frac{\partial\psi}{\partial\vec{\nu}}\alpha+\psi\left(\frac{3AK_c^3+K_c}{1+AK_c^2}\right)\alpha\right]\beta d\sigma,\\
&= \int_{\partial\Omega}\left[\frac{\partial\psi}{\partial\vec{\nu}}+\psi\left(\frac{AK_c^3+K_c}{1+AK_c^2}\right)\right]\alpha\beta d\sigma,\\
&= \int_{\partial\Omega}\left[\frac{\partial\psi}{\partial\vec{\nu}}+\psi K_c\left(\frac{1+AK_c^2}{1+AK_c^2}\right)\right]\alpha\beta d\sigma. \qquad (5.45)
\end{aligned}$$

Replacing ψ by its expression, we have

$$\begin{aligned}
G^A(HessJ(\Omega)[V],W) &= \int_{\partial\Omega}\left[\frac{\partial}{\partial\vec{\nu}}\left(k^2-\left(\frac{\partial u_\Omega}{\partial\vec{\nu}}\right)^2\right)\right.\\
&+ \left.K_c\left(k^2-\left(\frac{\partial u_\Omega}{\partial\vec{\nu}}\right)^2\right)\right]\langle V,\vec{\nu}\rangle\langle W,\vec{\nu}\rangle d\sigma. \qquad (5.46)
\end{aligned}$$

□

Remark 5.2. *Let us note first that there is a symmetry relation with respect to the hessian which is in the case of our considered Riemannian structure a self adjoint operator with respect to the metric G^A.*

And the second fact is that it is important to underline that the formulas (5.42) *obtained from the formula in Theorem 5.1 and* (5.46) *computed by a direct method with the metric G^A in two different ways, have to give the same expression even if Ω is not a critical point. And then from these computations, one deduces that*

$$\begin{aligned}
&\int_{\partial\Omega}\left[-2\langle W,\vec{\nu}\rangle\left(grad(u'_\Omega).\vec{\nu}+D^2u_\Omega V.\vec{\nu}+grad(u_\Omega).\left(-grad_\Gamma(V.\vec{\nu})+D_{\vec{\nu}}V\right)\right)\right]d\sigma\\
&\qquad = \int_{\partial\Omega}\frac{3AK_c^3+K_c}{1+AK_c^2}\left(k^2-\left(\frac{\partial u_\Omega}{\partial\vec{\nu}}\right)^2\right)\langle V,\vec{\nu}\rangle\langle W,\vec{\nu}\rangle d\sigma. \qquad (5.47)
\end{aligned}$$

Remark 5.3. *In this remark, we compute $G^A(V,HessJ(\Omega)[W])$ to show the symmetry relation with respect to the hessian with the computation of the direct method with the metric*

G^A.

$$G^A(V, HessJ(\Omega)[W]) = \int_{\partial\Omega} \left(1 + AK_c^2\right) HessJ(\Omega)[W]V,$$
$$= \int_{\partial\Omega} \left(1 + AK_c^2\right) \nabla_W gradJ(\Omega)V,$$
$$= \int_{\partial\Omega} \left(1 + AK_c^2\right) \nabla_m gradJ(\Omega)h \tag{5.48}$$

where $V = h = \alpha\vec{\nu}$ *and* $W = m = \beta\vec{\nu}$. *Since* $gradJ(\Omega) = \frac{1}{1+AK_c^2}\psi$, *we have*

$$\nabla_m gradJ(\Omega) = \frac{\partial}{\partial\vec{\nu}}\left(gradJ(\Omega)\right)\beta + \left(\frac{3Ak_c^3 + K_c}{1 + AK_c^2}\right) gradJ(\Omega)\beta,$$
$$= \frac{\partial}{\partial\vec{\nu}}\left(\frac{1}{1 + AK_c^2}\psi\right)\beta + \frac{1}{1 + AK_c^2}\psi\left(\frac{3Ak_c^3 + K_c}{1 + AK_c^2}\right)\beta.$$

As previously, by the same computations, we get

$$\nabla_m gradJ(\Omega) = \frac{-2AK_c^3}{(1 + AK_c^2)^2}\psi\beta + \frac{\partial\psi}{\partial\vec{\nu}}\left(\frac{1}{1 + AK_c^2}\right)\beta + \frac{1}{1 + AK_c^2}\psi\left(\frac{3Ak_c^3 + K_c}{1 + AK_c^2}\right)\beta.$$

And finally, we have

$$G^A(HessJ(\Omega)[W], V) = \int_{\partial\Omega} \left(1 + AK_c^2\right)\left[\frac{-2AK_c^3}{(1 + AK_c^2)^2}\psi\beta + \frac{\partial\psi}{\partial\vec{\nu}}\left(\frac{1}{1 + AK_c^2}\right)\beta\right.$$
$$\left. + \frac{1}{1 + AK_c^2}\psi\left(\frac{3AK_c^3 + K_c}{1 + AK_c^2}\right)\beta\right]\alpha d\sigma,$$
$$= \int_{\partial\Omega}\left[\frac{\partial}{\partial\vec{\nu}}\left(k^2 - \left(\frac{\partial u_\Omega}{\partial\vec{\nu}}\right)^2\right) + K_c\left(k^2 - \left(\frac{\partial u_\Omega}{\partial\vec{\nu}}\right)^2\right)\right]\langle V, \vec{\nu}\rangle\langle W, \vec{\nu}\rangle d\sigma.$$

Let us have a look at the two formulas of the second derivation when $V = W = \alpha\vec{\nu}$. On the one hand, by Proposition 4.1, we get

$$Q(\alpha) = d^2J(\Omega; V; V),$$
$$= -(N-1)\int_{\partial\Omega} H\alpha^2 d\sigma + k^2\int_{\Omega} |grad(\Lambda)|^2 dx,$$
$$= -(N-1)k^2\int_{\partial\Omega} H\alpha^2 d\sigma + k^2\int_{\partial\Omega} \alpha L\alpha d\sigma. \tag{5.49}$$

On the other hand by Theorem 5.1, we have

$$G^A\left(HessJ(\Omega)[V], W\right) = d\left(dJ(\Omega)[W]\right)[V] - dJ(\Omega)[\nabla_V W]. \tag{5.50}$$

Then for $V = W$ we derive

$$d\left(dJ(\Omega)[V]\right)[V] = d^2J(\Omega; V; V) = G^A\left(HessJ(\Omega)[V], V\right) + dJ(\Omega)[\nabla_V V]. \tag{5.51}$$

- If the quadrature surface problem has a solution Ω, then
  $d\left(dJ(\Omega)[V]\right)[V] = G^A\left(HessJ(\Omega)[V], V\right).$

- In previous works, the second author studied the stability and positiveness of the quadratic form, see [30] for more details. He established a proposition similar to Proposition 4.1 and gave necessary and sufficient qualitative properties in the theoretical point of view.

The one obtained involves the study of a generalized spectral Steklov problem that is reminded in the following corollary.

Corollary 5.1. *Let us consider the following generalized spectral Steklov problem:*

$$\begin{aligned} \Delta\phi_n &= 0 \ \ in \ \ \Omega\backslash K \\ \phi_n &= 0 \ \ on \ \ \partial K \\ (L+(N-1)HI)\phi_n &= (\frac{1}{\mu_n} - \|H^-\|_\infty)\phi_n \ \ on \ \ \partial\Omega, \end{aligned}$$

where I is the identity map, H is the mean curvature of Ω, K is a compact regular enough subset of Ω, $H^- = max\{-H, 0\}$ and μ_n is a decreasing sequence of eigenvalues depending also on H which goes to 0. And one must have the sign of the first eigenvalue

$$\lambda_0 := \frac{1}{\mu_0} - \|H^-\|_\infty = \inf\big\{(N-1)\int Hv^2 d\sigma + \int_{\Omega\backslash K} |grad(\Lambda)|^2 dx, v \in H^{1/2}(\partial\Omega), \int_{\partial\Omega} v^2 d\sigma = 1\big\},$$

where

$$\begin{aligned} \Delta\Lambda &= 0 \ \ in \ \ \Omega\backslash K \\ \Lambda &= 0 \ \ on \ \ \partial K \\ \frac{\partial\Lambda}{\partial\vec{\nu}} &= v \ \ on \ \ \partial\Omega. \end{aligned}$$

And the minimum is reached for ϕ_0 satisfying

$$\begin{aligned} \Delta\phi_0 &= 0 \ \ in \ \ \Omega\backslash K \\ \phi_0 &= 0 \ \ on \ \ \partial K \\ (L+(N-1)HI)\phi_0 &= \lambda_0\phi_0 \ \ on \ \ \partial\Omega. \end{aligned}$$

From our work we can deduce the following conclusions as a corollary.

Corollary 5.2.

- *What is obtained with the Riemannian hessian formula is easier to derive simple control for the characterization of the optimal shape in a number of ways.*

- *In the case of minimum,* $G^A\left(HessJ(\Omega)[V],V\right)\geq 0$. *And this inequality is equivalent to* $\displaystyle\int_{\partial\Omega}\left[\frac{\partial}{\partial\vec{\nu}}\left(k^2-\left(\frac{\partial u_\Omega}{\partial\vec{\nu}}\right)^2\right)\right]\alpha^2 d\sigma\geq 0,\ \forall\alpha\in\mathcal{C}^\infty(\mathbb{R}^2,\mathbb{R})\cap H^{1/2}(\partial\Omega)$.
  This is reduced to $\displaystyle\int_{\partial\Omega}\frac{\partial}{\partial\vec{\nu}}\left(k^2-\left(\frac{\partial u_\Omega}{\partial\vec{\nu}}\right)^2\right)d\sigma\geq 0$.
  One can deduce also another control, since

$$\int_{\partial\Omega}\left[\frac{\partial}{\partial\vec{\nu}}\left(k^2-\left(\frac{\partial u_\Omega}{\partial\vec{\nu}}\right)^2\right)\right]\alpha^2 d\sigma=-2k^2(N-1)\int_{\partial\Omega}H\alpha^2 d\sigma=-2k^2(N-1)\int_{\partial\Omega}K_c\alpha^2 d\sigma.$$

  Before proceeding further, let us underline that in two dimension $H=K_c$. *And knowing that* $\alpha\in\mathcal{C}^\infty(\mathbb{R}^2,\mathbb{R})\cap H^{1/2}(\partial\Omega)$, *the control becomes* $\displaystyle\int_{\partial\Omega}K_c d\sigma=2\pi\chi(\partial\Omega)\leq 0$ *where* $\chi(\partial\Omega)$ *is the Euler- Poincaré characteristic. And from this, we can deduce that by Gauss- Bonnet theorem the control is done on the Euler-Poincaré characteristic. And from this, we have key information to set up algorithm in order to get a good approximation of the optimal shape.*
- *Now, when* Ω *is only a critical point, to get a strict local minimum, we need the following sufficient condition:*

$$\int_{\partial\Omega}\left[\frac{\partial}{\partial\vec{\nu}}\left(k^2-\left(\frac{\partial u_\Omega}{\partial\vec{\nu}}\right)^2\right)\right]\alpha^2 d\sigma=-2k^2(N-1)\int_{\partial\Omega}K_c\alpha^2 d\sigma\geq C_0\|\alpha\|^2, C_0>0.$$

  One can say also that there is $x_0\in\partial\Omega, -2k^2(N-1)K_c(x_0)\displaystyle\int_{\partial\Omega}\alpha^2 d\sigma\geq C_0\|\alpha\|^2$. *And if* $K_c(x_0)<0$, *then* Ω *is a strict local minimum. If the Euler-Poincaré characteristic is positive, then there is not a minimum.*

Conflicts of interests. The authors declare that there is no conflict of interests.

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