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FINITE TIME BLOW-UP FOR FRACTIONAL TEMPORAL SCHRÖDINGER EQUATIONS AND SYSTEMS ON THE HEISENBERG GROUP

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Abstract. The aim of this research paper is to establish sufficient conditions for the nonexistence of global weak solution to the nonlinear Schrödinger equation on the Heisenberg group. The results are shown by the use of test function theory and extended to systems of the same type.

1. Introduction

The main purpose of this paper is to present results concerning the local nonexistence of solutions for the following nonlinear time fractional Schrödinger equation posed in Heisenberg group

$$i^{\alpha}\ {}_{0}^{C}D_t^{\alpha}u + \Delta_{\mathbb{H}}u = \lambda|u|^p + \mu a(\eta).\nabla_{\mathbb{H}}|u|^q, \tag{1.1}$$

equipped with the initial data

$$u(\eta, 0) = g(\eta),$$

where $u(\eta, t)$ is a complex-valued function, $\Delta_{\mathbb{H}}$ is the Kohn-Laplace operator on the $(2N+1)$-dimensional Heisenberg group, $0 < \alpha < 1$, i^{α} is the principal value of i^{α}, ${}_{0}^{C}D_t^{\alpha}$ is the Caputo fractional derivative of order α, $\lambda = \lambda_1 + i\lambda_2$, $(\lambda_1, \lambda_2) \in \mathbb{R}^2 - \{(0;0)\}$,

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$\mu = \mu_1 + i\mu_2$, $(\mu_1, \mu_2) \in \mathbb{R}^2$ and $p > q > 1$. The symbol $\nabla_{\mathbb{H}}$ denotes the gradient over $\mathbb{H}$ and $a(\eta) = (A_1(\eta); A_2(\eta); ...; A_N(\eta)) \in \mathbb{R}^N$ is a given vector function, assumed to satisfy

$$|a(T^{\frac{Q\alpha}{2}}\tilde{\eta})| \simeq T^{\nu}, \qquad |Div_{\mathbb{H}}(a(T^{\frac{Q\alpha}{2}}\tilde{\eta}))| \simeq T^{\tau}. \tag{1.2}$$

Therefore $a(\eta).\nabla_{\mathbb{H}}|u|^q$ is the scalar product of $a(\eta)$ and $\nabla_{\mathbb{H}}|u|^q$ and $g(\eta) = g_1(\eta) + ig_2(\eta)$, $(g_1(\eta); g_2(\eta)) \in \mathbb{R}^2$, $g \in L^1(\mathbb{H})$. Then we extend our analysis to the 2×2 system:

$$\begin{cases} i^{\alpha} \, {}_{0}^{c} D_t^{\alpha} u + \Delta_{\mathbb{H}} u = \lambda |v|^p + \mu a(\eta).\nabla_{\mathbb{H}}|v|^q \\ i^{\beta} \, {}_{0}^{C} D_t^{\beta} v + \Delta_{\mathbb{H}} v = \lambda |u|^k + \mu b(\eta).\nabla_{\mathbb{H}}|u|^{\sigma} \\ u(\eta, 0) = g(\eta); \qquad v(\eta, 0) = h(\eta), \end{cases} \tag{1.3}$$

where $0 < \beta \leq \alpha < 1$; $k > \sigma > 1$. The vector functions $a(\eta) = (A_1(\eta); A_2(\eta); ...; A_N(\eta))$ and $b(\eta) = (B_1(\eta); B_2(\eta); ...; B_N(\eta))$ are assumed to satisfy

$$\begin{aligned} |a(T^{\frac{Q(\alpha+\beta)}{2}}\tilde{\eta})| \simeq T^{\nu_1}, \qquad |Div_{\mathbb{H}}(a(T^{\frac{Q(\alpha+\beta)}{2}}\tilde{\eta}))| \simeq T^{\tau_1}, \\ |b(T^{\frac{Q(\alpha+\beta)}{2}}\tilde{\eta})| \simeq T^{\nu_2}, \qquad |Div_{\mathbb{H}}(b(T^{\frac{Q(\alpha+\beta)}{2}}\tilde{\eta}))| \simeq T^{\tau_2}. \end{aligned} \tag{1.4}$$

Our method of proof relies on a method due to Baras and Pierre [4]. It had been remained dormant until Zhang ([16], [17], [18]) revived it. Later, this method has been successfully applied in a great number of situations by Mitidieri and Pohozaev [12] and Hakem et al [8]. This work is organized as follows. In Section 2, we present some fundamental and basic results. In section 3, we prove our main results.

2. Preliminaries

For the reader convenience, some background facts used in the sequel are recalled.

The Heisenberg group $\mathbb{H}$ whose points will be denoted by $\eta = (x, y, \tau)$, is the Lie group $(\mathbb{R}^{2N+1}, \circ)$ with the non-commutative group operation $\circ$ defined by

$$\eta \circ \eta' = (x + x', y + y', \tau + \tau' + 2(x \cdot y' - x' \cdot y))$$

for all $\eta = (x, y, \tau), \eta' = (x', y', \tau') \in \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}$, where $\cdot$ denotes the standard scalar product in $\mathbb{R}^N$.

This group operation endows $\mathbb{H}$ with the structure of a Lie group.

The Laplacian $\Delta_{\mathbb{H}}$ over $\mathbb{H}$ is obtained from the vector fields $X_i = \frac{\partial}{\partial x_i} + 2y_i \frac{\partial}{\partial \tau}$ and $Y_i = \frac{\partial}{\partial y_i} - 2x_i \frac{\partial}{\partial \tau}$, by

$$\Delta_{\mathbb{H}} = \sum_{i=1}^{N} \left(X_i^2 + Y_i^2 \right).$$

Observe that the vector field $T = \frac{\partial}{\partial\tau}$ does not appear in the equality above. This fact makes us presume a "loss of derivative" in the variable τ. The compensation comes from the relation

$$[X_i, Y_j] = -4T, \quad i, j \in 1, 2, 3, ..., N.$$

The relation above proves that $\mathbb{H}$ is a nilpotent Lie group of order 2. Explicit computation gives the expression

$$\Delta_{\mathbb{H}} = \sum_{i=1}^{N} \left(\frac{\partial^2}{\partial x_i^2} + \frac{\partial^2}{\partial y_i^2} + 4y_i \frac{\partial^2}{\partial x_i \partial\tau} - 4x_i \frac{\partial^2}{\partial y_i \partial\tau} + 4(x_i^2 + y_i^2) \frac{\partial^2}{\partial\tau^2} \right).$$

A natural group of dilatations on $\mathbb{H}$ is given by

$$\delta_\lambda(\eta) = (\lambda x, \lambda y, \lambda^2\tau), \qquad \lambda > 0,$$

whose Jacobian determinant is λ^Q, where $Q = 2N + 2$ is the homogeneous dimension of $\mathbb{H}$. The operator $\Delta_{\mathbb{H}}$ is a degenerate elliptic operator. It is invariant with respect to the left translation of $\mathbb{H}$ and homogeneous with respect to the dilations δ_λ. More precisely, we have

$$\Delta_{\mathbb{H}}(u(\eta \circ \eta')) = (\Delta_{\mathbb{H}}u)(\eta \circ \eta'), \Delta_{\mathbb{H}}(u \circ \delta_\lambda) = \lambda^2(\Delta_{\mathbb{H}}u) \circ \delta_\lambda, \eta, \eta' \in \mathbb{H}.$$

The natural distance from η to the origin is introduced by Folland and Stein, see [**?**]

$$|\eta|_{\mathbb{H}} = \left(\tau^2 + \left(\sum_{i=1}^{N} (x_i^2 + y_i^2) \right)^2 \right)^{\frac{1}{4}}.$$

The gradient $\nabla_{\mathbb{H}}$ over $\mathbb{H}$ is defined by

$$\nabla_{\mathbb{H}} = (X_1; X_2; ...X_N; Y_1; Y_2; ...; Y_N).$$

Let

$$M = \begin{pmatrix} I_N & 0 & 2y \\ 0 & I_N & -2x \end{pmatrix}$$

where I_N is the identity matrix of size N. Then

$$\nabla_{\mathbb{H}} = M \nabla_{\mathbb{R}^{2N+1}}.$$

A simple computation gives the expression

$$|\nabla_{\mathbb{H}}|^2 = 4(|x|^2 + |y|^2) \left(\frac{\partial u}{\partial\tau} \right)^2 + \sum_{i=1}^{N} \left(\left(\frac{\partial u}{\partial x_i} \right)^2 + \left(\frac{\partial u}{\partial y_i} \right)^2 + 4 \frac{\partial u}{\partial\tau} \left(y_i \frac{\partial u}{\partial x_i} - x_i \frac{\partial u}{\partial y_i} \right) \right).$$

The divergence operator in $\mathbb{H}$ is defined by

$$Div_{\mathbb{H}}(u) = Div_{\mathbb{R}^{2N+1}}(Mu).$$

To derive the nonexistence of results for the problem (1.1), we shall state some results about fractional derivative and fractional integral which will be used in the proof of our main results (see for instance [10], [14]).

Let $f \in L^1(0;T),\ T > 0$, be a given function. The Riemann-Liouville left-sided fractional integral ${}_0I_t^\alpha f$ of order $\alpha > 0$ is defined by

$$\left({}_0I_t^\alpha f\right)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds; \quad for \quad a.e. \quad t \in [0;T],$$

where Γ is the Gamma function.

The Riemann-Liouville right-sided fractional integral ${}_tI_T^\alpha f$ of order $\alpha > 0$ is defined by

$$\left({}_tI_T^\alpha f\right)(t) = \frac{1}{\Gamma(\alpha)} \int_t^T (s-t)^{\alpha-1} f(s)ds; \quad for \quad a.e. \quad t \in [0;T].$$

Let $0 < \alpha < 1$ and $f \in AC^1[0;T],\ T > 0$. The Caputo left-sided and right-sided fractional derivatives of order α of f are defined, respectively, by

$$\left({}_0^C D_t^\alpha f\right)(t) =_0 I_t^{1-\alpha} f'(t) \quad for \quad a.e. \quad t \in [0;T],$$

and

$$\left({}_t^C D_T^\alpha f\right)(t) =_{-t} I_T^{1-\alpha} f'(t) \quad for \quad a.e. \quad t \in [0;T].$$

The following fractional integration by parts will be used later to define the weak solutions to (1.1) and (1.3).

Lemma 2.1. *Let* $0 < \alpha < 1$. *If* $f \in C[0;T]$, ${}_0^C D_t^\alpha f \in L^1(0;T)$, $g \in C^1[0;T]$ *and* $g(T) = 0$, *then*

$$\int_0^T \left({}_0^C D_t^\alpha f\right)(t) g(t)dt = \int_0^T (f(t) - f(0))(t) \left({}_t^C D_T^\alpha g\right)(t)dt.$$

The following results will be used several times.

Lemma 2.2. *Let* $T > 0$, $r \geq 1$ *and* $f : [0;T] \longrightarrow \mathbb{R}$ *be the function given by*

$$f(t) = \left(1 - \frac{t}{T}\right)^r, \qquad 0 \leq t \leq T.$$

Then, for any $0 < \alpha < 1$, *we have*

$$\left({}_t^C D_T^\alpha f\right)(t) = \frac{\Gamma(r+1)}{\Gamma(r+1-\alpha)} T^{-r}(T-t)^{r-\alpha}, \qquad 0 \leq t \leq T.$$

Given a complex number $z \in \mathbb{C}$. We denote by Rez its real part and by Imz its imaginary part.

Lemma 2.3. [1] *Let $f \in \mathcal{L}^1\left(\mathbb{R}^{2N+1}\right)$ and $\int_{\mathbb{R}^{2N+1}} f d\eta > 0$. Then there exists a test function $0 \leq \varphi \leq 1$ such that*

$$\int_{\mathbb{R}^{2N+1}} f\varphi d\eta \geq 0.$$

Let us set $\mathcal{H}_T = \mathbb{H} \times (0,T)$ and $\mathcal{H} = \mathbb{H} \times (0,\infty)$ for $T > 0$. We put

$$G_1(\alpha; g(\eta)) = cos\left(\frac{\alpha\pi}{2}\right) g_1(\eta) - sin\left(\frac{\alpha\pi}{2}\right) g_2(\eta);$$
$$G_2(\alpha; g(\eta)) = cos\left(\frac{\alpha\pi}{2}\right) g_2(\eta) + sin\left(\frac{\alpha\pi}{2}\right) g_1(\eta).$$

3. Main results

3.1. Case of a single equation.

Definition 3.1. *A locally integrable function $u \in L^{max\{p;q\}}_{loc}(\mathcal{H}_\mathcal{T})$ is called a local weak solution to (1.1) in $\mathcal{H}_\mathcal{T}$ subject to the initial data $g \in L^1(\mathbb{H})$ if the equality*

$$\begin{aligned} &\lambda \int_{\mathcal{H}_\mathcal{T}} |u|^p \varphi d\eta dt + i^\alpha \int_{\mathcal{H}_\mathcal{T}} g(\eta) \, {}^C_t D^\alpha_T \varphi d\eta dt = \int_{\mathcal{H}_\mathcal{T}} u \left(i^\alpha \, {}^C_t D^\alpha_T \varphi + \Delta_\mathbb{H}\varphi\right) d\eta dt \\ &+\mu \int_{\mathcal{H}_\mathcal{T}} |u|^q \varphi div_\mathbb{H}(a(\eta)) d\eta dt + \mu \int_{\mathcal{H}_\mathcal{T}} a(\eta).\nabla_\mathbb{H}\varphi |u|^q d\eta dt \end{aligned} \tag{3.5}$$

is satisfied for every test function $\varphi \in C^{1,2}_{t,\eta}(\mathcal{H}_\mathcal{T})$ with $\varphi(.,T) = 0$.
Moreover, if $T > 0$ can be arbitrarily chosen, then u is said to be a global weak solution to (1.1).

Now, we are in position to announce our first result:

Theorem 3.1. *Let $p > q > 1$ and $g \in L^1(\mathbb{H})$. Suppose that one of the following cases holds:*
(I)

$$\lambda_1 \int_\mathbb{H} G_1(\alpha; g(\eta)) d\eta > 0, \tag{3.6}$$

and

$$\mu_1 = 0, \qquad 1 < p < 1 + \frac{1}{N+1},$$

or

$$\mu_1 \neq 0, \qquad N < -2 + p \quad min\left\{\frac{1}{p-1}; \frac{\tau}{\alpha(p-q)}; \frac{\alpha - 2\nu}{2\alpha(p-q)}\right\}.$$

(II)

$$\lambda_2 \int_\mathbb{H} G_2(\alpha; g(\eta)) d\eta > 0, \tag{3.7}$$

and

$$\mu_2 = 0, \qquad 1 < p < 1 + \frac{1}{N+1},$$

or

$$\mu_2 \neq 0, \qquad N < -2 + p \quad min\left\{\frac{1}{p-1}; \frac{\tau}{\alpha(p-q)}; \frac{\alpha - 2\nu}{2\alpha(p-q)}\right\}.$$

Then the problem (1.1) admits no global weak solution.

For simplicity, we use C to denote a positive constant which may vary from line to line.

Proof. The proof is by contradiction. For that, let u be a solution and φ be a smooth nonnegative test function such that:

$$\varphi(\eta; t) = \varphi_1(t)\varphi_2(\eta), \tag{3.8}$$

where for $T > 0$, we take

$$\varphi_1(t) = \left(1 - \frac{t}{T}\right)^m, \qquad \varphi_2(\eta) = \Phi^\omega\left(\frac{\tau^2 + |x|^4 + |y|^4}{T^{2\alpha}}\right),$$

where $\omega >> 1$, $m > max\left\{1; \frac{\alpha p}{p-1}\right\}$ and $\phi \in C_0^\infty(\mathbb{R}^{\mathbb{N}})$ be a cut-off nonincreasing function such that

$$\Phi\left(r\right) = \begin{cases} 1\ ,\ 0 \leq r \leq 1 \\ \searrow\ ,\ 1 \leq r \leq 2 \\ 0\ ,\ r \geq 2. \end{cases} \tag{3.9}$$

Let us set

$$\rho = \frac{\tau^2 + |x|^4 + |y|^4}{R^{2\alpha}}.$$

Then we have

$$\begin{aligned} \Delta_{\mathbb{H}}\Phi^\omega &= \tfrac{4\omega(N+4)}{T^{2\alpha}}\left(|x|^2 + |y|^2\right)\Phi'\Phi^{\omega-1} \\ &+ \tfrac{16\omega}{T^{4\alpha}}\left(\left(|x|^6 + |y|^6\right) + 2\tau\left(|x|^2 - |y|^2\right)x.y + \tau^2\left(|x|^2 + |y|^2\right)\right)\Phi''\Phi^{\omega-1} \\ &+ \tfrac{16\omega(\omega-1)}{T^{4\alpha}}\left(\left(|x|^6 + |y|^6\right) + 2\tau\left(|x|^2 - |y|^2\right)x.y + \tau^2\left(|x|^2 + |y|^2\right)\right)\Phi'^2\Phi^{\omega-2}. \end{aligned} \tag{3.10}$$

Using the formula 3.5 we get

$$\begin{aligned} &Re\left\{\lambda\int_{\mathcal{H}_\mathcal{T}}|u|^p\varphi d\eta dt + i^\alpha\int_{\mathcal{H}_\mathcal{T}} g(\eta)\ {}_t^C D_T^\alpha\varphi d\eta dt\right\} = Re\left\{\mu\int_{\mathcal{H}_\mathcal{T}} a(\eta).\nabla_{\mathbb{H}}\varphi|u|^q d\eta dt\right\} \\ &+ Re\left\{\int_{\mathcal{H}_\mathcal{T}} u\left(i^\alpha\ {}_t^C D_T^\alpha\varphi + \Delta_{\mathbb{H}}\varphi\right)d\eta dt + \mu\int_{\mathcal{H}_\mathcal{T}}|u|^q\varphi Div_{\mathbb{H}}(a(\eta))d\eta dt\right\}, \end{aligned} \tag{3.11}$$

which implies

$$\int_{\mathcal{H}_\mathcal{T}}|u|^p\varphi d\eta dt+\frac{1}{\lambda_1}\int_{\mathcal{H}_\mathcal{T}}G_1(\alpha;g(\eta))\,{}_t^C D_T^\alpha\varphi d\eta dt=$$
$$\frac{1}{\lambda_1}\int_{\mathcal{H}_\mathcal{T}}\left(Re(u)cos(\frac{\alpha\pi}{2})-Im(u)sin(\frac{\alpha\pi}{2})\right){}_t^C D_T^\alpha\varphi d\eta dt+\frac{1}{\lambda_1}\int_{\mathcal{H}_\mathcal{T}}Re(u)\Delta_{\mathbb{H}}\varphi d\eta dt \tag{3.12}$$
$$+\frac{\mu_1}{\lambda_1}\int_{\mathcal{H}_\mathcal{T}}|u|^q\left(\varphi Div_{\mathbb{H}}(a(\eta))+a(\eta).\nabla_{\mathbb{H}}\varphi\right)d\eta dt.$$

Then we find

$$\int_{\mathcal{H}_\mathcal{T}}|u|^p\varphi d\eta dt+\frac{1}{\lambda_1}\int_{\mathcal{H}_\mathcal{T}}G_1(\alpha;g(\eta))\,{}_t^C D_T^\alpha\varphi d\eta dt\le\frac{2}{|\lambda_1|}\int_{\mathcal{H}_\mathcal{T}}|u||\,{}_t^C D_T^\alpha\varphi|d\eta dt$$
$$+\frac{1}{|\lambda_1|}\int_{\mathcal{H}_\mathcal{T}}|u||\Delta_{\mathbb{H}}\varphi|d\eta dt+\frac{|\mu_1|}{|\lambda_1|}\int_{\mathcal{H}_\mathcal{T}}|u|^q\left(|\varphi||Div_{\mathbb{H}}(a(\eta))|+|a(\eta)||\nabla_{\mathbb{H}}\varphi|\right)d\eta dt. \tag{3.13}$$

By applying $\varepsilon-$ Young's inequality

$$ab\le\varepsilon a^p+C_\varepsilon b^{p'},p+p'=pp',a,b,\varepsilon,C_\varepsilon\ge 0,$$

to the right-hand side of the above inequality, we obtain

$$\left(1-\frac{\varepsilon(3-2|\mu_1|)}{|\lambda_1|}\right)\int_{\mathcal{H}_\mathcal{T}}|u|^p\varphi d\eta dt+\frac{1}{\lambda_1}\int_{\mathcal{H}_\mathcal{T}}G_1(\alpha;g(\eta))\,{}_t^C D_T^\alpha\varphi d\eta dt\le$$
$$\frac{C_\varepsilon}{|\lambda_1|}\left(2\int_{\mathcal{H}_\mathcal{T}}\varphi^{-\frac{1}{p-1}}|\,{}_t^C D_T^\alpha\varphi|^{\frac{p}{p-1}}d\eta dt+\int_{\mathcal{H}_\mathcal{T}}\varphi^{-\frac{1}{p-1}}|\Delta_{\mathbb{H}}\varphi|^{\frac{p}{p-1}}d\eta dt\right)$$
$$+\frac{C_\varepsilon}{|\lambda_1|}|\mu_1|\int_{\mathcal{H}_\mathcal{T}}\varphi^{-\frac{q}{p-q}}|\varphi|^{\frac{p}{p-q}}|Div_{\mathbb{H}}(a(\eta))|^{\frac{p}{p-q}}d\eta dt \tag{3.14}$$
$$+\frac{C_\varepsilon}{|\lambda_1|}|\mu_1|\int_{\mathcal{H}_\mathcal{T}}\varphi^{-\frac{q}{p-q}}|a(\eta)|^{\frac{p}{p-q}}|\nabla_{\mathbb{H}}\varphi|^{\frac{p}{p-q}}d\eta dt.$$

Taking $\varepsilon=\frac{|\lambda_1|}{2(3-2|\mu_1|)}$, we get

$$\frac{1}{2}\int_{\mathcal{H}_\mathcal{T}}|u|^p\varphi d\eta dt+\frac{1}{\lambda_1}\int_{\mathcal{H}_\mathcal{T}}G_1(\alpha;g(\eta))\,{}_t^C D_T^\alpha\varphi d\eta dt\le$$
$$\frac{C}{|\lambda_1|}\left(2\int_{\mathcal{H}_\mathcal{T}}\varphi^{-\frac{1}{p-1}}|\,{}_t^C D_T^\alpha\varphi|^{\frac{p}{p-1}}d\eta dt+\int_{\mathcal{H}_\mathcal{T}}\varphi^{-\frac{1}{p-1}}|\Delta_{\mathbb{H}}\varphi|^{\frac{p}{p-1}}d\eta dt\right)$$
$$+\frac{C}{|\lambda_1|}|\mu_1|\int_{\mathcal{H}_\mathcal{T}}\varphi^{-\frac{q}{p-q}}|\varphi|^{\frac{p}{p-q}}|div_{\mathbb{H}}(a(\eta))|^{\frac{p}{p-q}}d\eta dt \tag{3.15}$$
$$+\frac{C}{|\lambda_1|}|\mu_1|\int_{\mathcal{H}_\mathcal{T}}\varphi^{-\frac{q}{p-q}}|a(\eta)|^{\frac{p}{p-q}}|\nabla_{\mathbb{H}}\varphi|^{\frac{p}{p-q}}d\eta dt.$$

Now, we estimate each term of the right hand side of the above equality. By (3.8), we have

$$\int_{\mathcal{H}_\mathcal{T}}\varphi^{-\frac{1}{p-1}}|\,{}_t^C D_T^\alpha\varphi|^{\frac{p}{p-1}}d\eta dt=\left(\int_{\mathbb{H}}\varphi_2(\eta)d\eta\right)\left(\int_0^T\varphi_1^{-\frac{1}{p-1}}|\,{}_t^C D_T^\alpha\varphi_1|^{\frac{p}{p-1}}dt\right). \tag{3.16}$$

On the other hand, from Lemma 2.2, we arrive at

$$\int_0^T\varphi_1^{-\frac{1}{p-1}}|\,{}_t^C D_T^\alpha\varphi_1|^{\frac{p}{p-1}}dt=\left[\frac{\Gamma(m+1)}{\Gamma(m+1-\alpha)}\right]^{\frac{p}{p-1}}\frac{p-1}{p(m+1-\alpha)-(m+1)}T^{1-\frac{\alpha p}{p-1}}. \tag{3.17}$$

Therefore, by using the scaled variable

$$\tilde{\tau} = T^{-\alpha}\tau; \quad \tilde{x} = T^{-\frac{\alpha}{2}}x; \quad \tilde{y} = T^{-\frac{\alpha}{2}}y, \tag{3.18}$$

we obtain

$$\int_{\mathcal{H}_T} \varphi^{-\frac{1}{p-1}} | \, {}^C_t D^\alpha_T \varphi|^{\frac{p}{p-1}} d\eta dt = C(m;p) T^{1-\frac{\alpha p}{p-1}+(N+1)\alpha}, \tag{3.19}$$

where

$$C(m;p) = \left[\frac{\Gamma(m+1)}{\Gamma(m+1-\alpha)}\right]^{\frac{p}{p-1}} \frac{p-1}{p(m+1-\alpha)-(m+1)} \int_{0\leq|\tilde{\eta}|_{\mathbb{H}}\leq 2} \Phi^\omega(\tilde{\rho}) d\tilde{\rho}.$$

Similarly, we get

$$\int_{\mathcal{H}_T} \varphi^{-\frac{1}{p-1}} |\Delta_{\mathbb{H}}\varphi|^{\frac{p}{p-1}} d\eta dt = \left(\int_0^T \varphi_1(t)dt\right)\left(\int_{\mathbb{H}} \varphi_2^{-\frac{1}{p-1}} |\Delta_{\mathbb{H}}\varphi_2|^{\frac{p}{p-1}} d\eta\right). \tag{3.20}$$

A simple computation yields

$$\int_0^T \varphi_1(t)dt = \frac{T}{m+1}. \tag{3.21}$$

Taking into account (3.10), we deduce

$$\int_{\mathcal{H}_T} \varphi^{-\frac{1}{p-1}} |\Delta_{\mathbb{H}}\varphi|^{\frac{p}{p-1}} d\eta dt \leq \frac{C}{m+1} T^{1-\frac{\alpha p}{p-1}+(N+1)\alpha}. \tag{3.22}$$

From the condition (1.2) and using the same argument, we obtain

$$\int_{\mathcal{H}_T} \varphi^{-\frac{q}{p-q}} |\varphi|^{\frac{p}{p-q}} |Div_{\mathbb{H}}(a(\eta))|^{\frac{p}{p-q}} d\eta dt \leq C T^{1-\frac{\tau p}{p-q}+(N+1)\alpha}, \tag{3.23}$$

and

$$\int_{\mathcal{H}_T} \varphi^{-\frac{q}{p-q}} |a(\eta)|^{\frac{p}{p-q}} |\nabla_{\mathbb{H}}\varphi|^{\frac{p}{p-q}} d\eta dt \leq C T^{1-\frac{\alpha p}{2(p-q)}+\frac{\nu p}{p-q}+(N+1)\alpha}. \tag{3.24}$$

Combining (3.15),(3.5),(3.22),(3.23) and (3.24), we get

$$\begin{aligned} &\tfrac{1}{2}\int_{\mathcal{H}_T} |u|^p \varphi d\eta dt + \frac{1}{\lambda_1}\int_{\mathcal{H}_T} G_1(\alpha; g(\eta)) \, {}^C_t D^\alpha_T \varphi d\eta dt \leq \\ &\tfrac{C}{|\lambda_1|}\left(T^{1-\frac{\alpha p}{p-1}+(N+1)\alpha} + |\mu_1| T^{1-\frac{\tau p}{p-q}+(N+1)\alpha} + |\mu_1| T^{1-\frac{\alpha p}{2(p-q)}+\frac{\nu p}{p-q}+(N+1)\alpha}\right). \end{aligned} \tag{3.25}$$

Furthermore, it is not difficult to see that

$$\frac{1}{\lambda_1}\int_{\mathcal{H}_T} G_1(\alpha; g(\eta)) \, {}^C_t D^\alpha_T \varphi d\eta dt = \frac{1}{\lambda_1}\frac{\Gamma(m+1)}{\Gamma(m+2-\alpha)} T^{1-\alpha} \int_{\mathcal{H}_T} G_1(\alpha; g(\eta)) \Phi^\omega(\rho) d\eta dt. \tag{3.26}$$

Hence, we conclude

$$\begin{aligned} &\tfrac{1}{2}\int_{\mathcal{H}_T} |u|^p \varphi d\eta dt + \frac{1}{\lambda_1}\frac{\Gamma(m+1)}{\Gamma(m+2-\alpha)}\int_{\mathcal{H}_T} G_1(\alpha; g(\eta)) \Phi^\omega(\rho) d\eta dt \leq \\ &\tfrac{C}{|\lambda_1|}\left(T^{-\frac{\alpha p}{p-1}+(N+2)\alpha} + |\mu_1| T^{-\frac{\tau p}{p-q}+(N+2)\alpha} + |\mu_1| T^{-\frac{\alpha p}{2(p-q)}+\frac{\nu p}{p-q}+(N+2)\alpha}\right). \end{aligned} \tag{3.27}$$

First, we suppose that

$$\lambda_1 \int_{\mathcal{H}_T} G_1(\alpha; g(\eta)) d\eta > 0. \tag{3.28}$$

This implies that

$$\begin{aligned} &\tfrac{1}{\lambda_1}\tfrac{\Gamma(m+1)}{\Gamma(m+2-\alpha)}\int_{\mathcal{H}_\mathcal{T}} G_1(\alpha; g(\eta))\Phi^{\omega}(\rho)d\eta dt \leq \\ &\tfrac{C}{|\lambda_1|}\left(T^{-\frac{\alpha p}{p-1}+(N+2)\alpha} + |\mu_1|T^{-\frac{\tau p}{p-q}+(N+2)\alpha} + |\mu_1|T^{-\frac{\alpha p}{2(p-q)}+\frac{\nu p}{p-q}+(N+2)\alpha}\right). \end{aligned} \tag{3.29}$$

We have to discuss two cases:

Case 1: $\mu_1 = 0$ and $1 < p < 1 + \frac{1}{N+1}$. In this case passing to the limit as $T \to +\infty$ in (3.29), we obtain

$$\frac{1}{\lambda_1}\int_{\mathbb{H}} G_1(\alpha; g(\eta))d\eta \leq 0,$$

which contradicts (3.28).

Case 2: $\mu_1 \neq 0$ and $N < -2 + p \quad min\left\{\frac{1}{p-1}; \frac{\tau}{\alpha(p-q)}; \frac{\alpha-2\nu}{2\alpha(p-q)}\right\}$. Passing to the limit as $T \to +\infty$ in (3.29), we get

$$\frac{1}{\lambda_1}\int_{\mathbb{H}} G_1(\alpha; g(\eta))d\eta \leq 0,$$

which contradicts (3.28). Next we suppose that

$$\lambda_2 \int_{\mathcal{H}_\mathcal{T}} G_2(\alpha; g(\eta))d\eta > 0. \tag{3.30}$$

Observe that

$$v(\eta; t) = \frac{u(\eta, t)}{i},$$

is a global weak solution to the problem

$$i^{\alpha}\, {}_{0}^{C}D_t^{\alpha} v + \Delta_{\mathbb{H}} v = \lambda'|v|^p + \mu' a\,(\eta)\,.\nabla_{\mathbb{H}}|v|^q,$$

$$v\,(\eta, 0) = \tilde{g}\,(\eta)\,,$$

where

$$\lambda' = \lambda_2 + i(-\lambda_1) = \lambda_1' + i\lambda_2'; \qquad \mu' = \mu_2 + i(-\lambda_1) = \mu_1' + i\mu_2',$$

and

$$\tilde{g}(\eta) = g_2(\eta) + i(-g_1(\eta)) = \tilde{g}_1(\eta) + i\tilde{g}_2(\eta).$$

It can be easily seen that (3.30) is equivalent to

$$\lambda_1' \int_{\mathbb{H}} G_1(\alpha; \tilde{g}(\eta))d\eta > 0.$$

Therefore, from the previous case, if $\mu_2 = 0$ and $1 < p < 1 + \frac{1}{N+1}$ we obtain the contradiction with (3.30). Similarly with the case $\mu_2 \neq 0$ and $N < -2 + p \quad min\left\{\frac{1}{p-1}; \frac{\tau}{\alpha(p-q)}; \frac{\alpha-2\nu}{2\alpha(p-q)}\right\}$, we get the contradiction with (3.30).

3.2. **Case of the system.**

Definition 3.2. *We say that the pair (u,v) is a local weak solution to (1.3) if the equalities*

$$\begin{aligned} &\lambda\int_{\mathcal{H}_T}|v|^p\varphi d\eta dt+i^\alpha\int_{\mathcal{H}_T}g(\eta)\ {}_t^C D_T^\alpha\varphi d\eta dt=\int_{\mathcal{H}_T}u\left(i^\alpha\ {}_t^C D_T^\alpha\varphi+\Delta_{\mathbb{H}}\varphi\right)d\eta dt\\ &+\mu\int_{\mathcal{H}_T}|v|^q\varphi \mathrm{div}_{\mathbb{H}}(a(\eta))d\eta dt+\mu\int_{\mathcal{H}_T}|v|^q a(\eta).\nabla_{\mathbb{H}}\varphi d\eta dt, \end{aligned} \tag{3.31}$$

and

$$\begin{aligned} &\lambda\int_{\mathcal{H}_T}|u|^k\varphi d\eta dt+i^\beta\int_{\mathcal{H}_T}h(\eta)\ {}_t^C D_T^\beta\varphi d\eta dt=\int_{\mathcal{H}_T}v\left(i^\beta\ {}_t^C D_T^\beta\varphi+\Delta_{\mathbb{H}}\varphi\right)d\eta dt\\ &+\mu\int_{\mathcal{H}_T}|u|^\sigma\varphi \mathrm{div}_{\mathbb{H}}(b(\eta))d\eta dt+\mu\int_{\mathcal{H}_T}|u|^q b(\eta).\nabla_{\mathbb{H}}\varphi d\eta dt, \end{aligned} \tag{3.32}$$

are satisfied for every test function $\varphi\in C^{1,2}_{t,\eta}(\mathcal{H}_T)$ with $\varphi(.,T)=0$.
Moreover, if $T>0$ can be arbitrarily chosen, then (u,v) is said to be a global weak solution to (1.3).

Our second main result is given by the following theorem:

Theorem 3.2. *Let $0<\beta\le\alpha<1$; $p>q>1$; $k>\sigma>1$ and $g,h\in L^1(\mathbb{H})$. Suppose that one of the following cases holds:*

(I)

$$\alpha=\beta,\qquad \lambda_1\int_{\mathbb{H}}\left[G_1(\alpha;g(\eta))+G_2(\beta;h(\eta))\right]d\eta>0, \tag{3.33}$$

and

$$\mu_1=0,\qquad Q<min\left\{\frac{1}{p-1};\frac{1}{k-1}\right\},$$

or

$$\mu_1\neq 0\quad and\quad Q<min\left\{\frac{\tau_1 p-\alpha(p-q)}{\alpha(p-q)},\frac{\alpha q-\nu_1 p}{\alpha(p-q)},\frac{\tau_2 k-\alpha(k-\sigma)}{\alpha(k-\sigma)},\frac{\alpha\sigma-\nu_2 k}{\alpha(k-\sigma)}\right\}.$$

(II)

$$\beta<\alpha,\qquad \lambda_1\int_{\mathbb{H}}G_1(\beta;h(\eta))d\eta>0, \tag{3.34}$$

and

$$\mu_1=0,\qquad Q<\frac{2}{\alpha+\beta}\left\{\frac{\alpha}{k-1},\frac{\beta p-\alpha(p-1)}{p-1}\right\},$$

or $\mu_1\neq 0$ and

$$Q<\frac{2}{\alpha+\beta}$$
$$min\left\{\frac{\tau_1 p-\alpha(p-q)}{p-q},\frac{2\alpha q-(\alpha+\beta+2\nu_1)p}{2(p-q)},\frac{\tau_2 k-\alpha(k-\sigma)}{k-\sigma},\frac{(\alpha+\beta)k-2\alpha(k-\sigma)-2\nu_2 k}{2(k-\sigma)}\right\}.$$

(III)

$$\alpha = \beta, \qquad \lambda_2 \int_{\mathbb{H}} [G_1(\alpha; g(\eta)) + G_2(\beta; h(\eta))] \, d\eta > 0, \tag{3.35}$$

and

$$\mu_2 = 0, \qquad Q < min\left\{\frac{1}{p-1}, \quad \frac{1}{k-1}\right\},$$

or

$$\mu_2 \neq 0, \qquad Q < min\left\{\frac{\tau_1 p - \alpha(p-q)}{\alpha(p-q)}, \frac{\alpha q - \nu_1 p}{\alpha(p-q)}, \frac{\tau_2 k - \alpha(k-\sigma)}{\alpha(k-\sigma)}, \frac{\alpha\sigma - \nu_2 k}{\alpha(k-\sigma)}\right\}.$$

(IV)

$$\alpha < \beta, \qquad \lambda_2 \int_{\mathbb{H}} G_1(\beta; h(\eta)) d\eta > 0, \tag{3.36}$$

and

$$\mu_2 = 0, \qquad Q < \frac{2}{\alpha+\beta} \quad min\left\{\frac{\alpha}{k-1}, \frac{\beta p - \alpha(p-1)}{p-1}\right\},$$

or

$\mu_2 \neq 0$ *and*

$$Q < \frac{2}{\alpha+\beta}$$

$$min\left\{\frac{\tau_1 p - \alpha(p-q)}{p-q}, \frac{2\alpha q - (\alpha+\beta+2\nu_1)p}{2(p-q)}, \frac{\tau_2 k - \alpha(k-\sigma)}{k-\sigma}, \frac{(\alpha+\beta)k - 2\alpha(k-\sigma) - 2\nu_2 k}{2(k-\sigma)}\right\}.$$

Then the system (1.3) admits no global weak solution.

Proof. Let φ be the test function defined by (3.8) where φ_2 is given by

$$\varphi_2(\eta) = \Phi^{\omega}\left(\frac{\tau^2 + |x|^4 + |y|^4}{T^{2(\alpha+\beta)}}\right),$$

where Φ is given by formula(3.9).

Suppose that (u, v) is a global weak solution to (1.3). By using the definition of weak solution, we get

$$\begin{aligned} &Re\left(\lambda \int_{\mathcal{H}_T} |v|^p \varphi d\eta dt + i^{\alpha} \int_{\mathcal{H}_T} g(\eta) \, {}^C_t D^{\alpha}_T \varphi d\eta dt\right) = Re\left(\mu \int_{\mathcal{H}_T} |v|^q a(\eta).\nabla_{\mathbb{H}} \varphi d\eta dt\right) \\ &Re\left(\int_{\mathcal{H}_T} u\left(i^{\alpha} \, {}^C_t D^{\alpha}_T \varphi + \Delta_{\mathbb{H}} \varphi\right) d\eta dt\right) + Re\left(\mu \int_{\mathcal{H}_T} |v|^q \varphi div_{\mathbb{H}}(a(\eta)) d\eta dt\right). \end{aligned} \tag{3.37}$$

First we repeat the same calculation as above, we obtain

$$\begin{aligned} &\int_{\mathcal{H}_T} |v|^p \varphi d\eta dt + \frac{1}{\lambda_1} \int_{\mathcal{H}_T} G_1(\alpha; g(\eta)) \, {}^C_t D^{\alpha}_T \varphi d\eta dt \leq \frac{2}{|\lambda_1|} \int_{\mathcal{H}_T} |u|| \, {}^C_t D^{\alpha}_T \varphi| d\eta dt \\ &+ \frac{1}{|\lambda_1|} \int_{\mathcal{H}_T} |u||\Delta_{\mathbb{H}} \varphi| d\eta dt + \frac{|\mu_1|}{|\lambda_1|} \int_{\mathcal{H}_T} |v|^q \left(|\varphi||Div_{\mathbb{H}}(a(\eta))| + |a(\eta)||\nabla_{\mathbb{H}} \varphi|\right) d\eta dt. \end{aligned} \tag{3.38}$$

Also, using the arguments of the previous theorem, we arrive at

$$\begin{aligned}&\left(1-2\tfrac{\varepsilon|\mu_1|}{|\lambda_1|}\right)\int_{\mathcal{H}_\mathcal{T}}|v|^p\varphi d\eta dt+\frac{1}{\lambda_1}\int_{\mathcal{H}_\mathcal{T}}G_1(\alpha;g(\eta))\ {}_t^C D_T^\alpha\varphi d\eta dt\leq\\&\tfrac{C_\varepsilon}{|\lambda_1|}\left(\int_{\mathcal{H}_\mathcal{T}}\varphi^{-\frac{1}{k-1}}\left(|{}_t^C D_T^\alpha\varphi|^{\frac{k}{k-1}}+|\Delta_\mathbb{H}\varphi|^{\frac{k}{k-1}}\right)d\eta dt+3\frac{\varepsilon}{|\lambda_1|}\int_{\mathcal{H}_\mathcal{T}}|u|^k\varphi d\eta dt\right.\\&\left.+|\mu_1|\int_{\mathcal{H}_\mathcal{T}}\varphi^{-\frac{q}{p-q}}\left(|\varphi|^{\frac{p}{p-q}}|div_\mathbb{H}(a(\eta))|^{\frac{p}{p-q}}+|a(\eta)|^{\frac{p}{p-q}}|\nabla_\mathbb{H}\varphi|^{\frac{p}{p-q}}\right)d\eta dt\right).\end{aligned}\tag{3.39}$$

Similarly, we have

$$\begin{aligned}&\left(1-2\tfrac{\varepsilon|\mu_1|}{|\lambda_1|}\right)\int_{\mathcal{H}_\mathcal{T}}|u|^k\varphi d\eta dt+\frac{1}{\lambda_1}\int_{\mathcal{H}_\mathcal{T}}G_1(\beta,h(\eta))\ {}_t^C D_T^\beta\varphi d\eta dt\leq\\&\tfrac{C_\varepsilon}{|\lambda_1|}\left(\int_{\mathcal{H}_\mathcal{T}}\varphi^{-\frac{1}{p-1}}\left(|{}_t^C D_T^\beta\varphi|^{\frac{p}{p-1}}+|\Delta_\mathbb{H}\varphi|^{\frac{p}{p-1}}\right)d\eta dt+3\frac{\varepsilon}{|\lambda_1|}\int_{\mathcal{H}_\mathcal{T}}|v|^p\varphi d\eta dt.\right.\\&\left.+|\mu_1|\int_{\mathcal{H}_\mathcal{T}}\varphi^{-\frac{\sigma}{k-\sigma}}\left(|\varphi|^{\frac{k}{k-\sigma}}|Div_\mathbb{H}(b(\eta))|^{\frac{k}{k-\sigma}}+|b(\eta)|^{\frac{k}{k-\sigma}}|\nabla_\mathbb{H}\varphi|^{\frac{k}{k-\sigma}}d\eta dt\right)\right).\end{aligned}\tag{3.40}$$

Next, adding (3.39) to (3.40) and taking $\varepsilon=\frac{|\lambda_1|}{2(2|\mu_1|-3|)}$, we get

$$\begin{aligned}&\frac{1}{\lambda_1}\int_{\mathcal{H}_\mathcal{T}}G_1(\alpha;g(\eta))\ {}_t^C D_T^\alpha\varphi d\eta dt+\frac{1}{\lambda_1}\int_{\mathcal{H}_\mathcal{T}}G_1(\beta,h(\eta))\ {}_t^C D_T^\beta\varphi d\eta dt\\&+\frac{1}{2}\int_{\mathcal{H}_\mathcal{T}}\left(|v|^p+|u|^k\right)\varphi d\eta dt\leq\frac{C_\varepsilon}{|\lambda_1|}\left(\int_{\mathcal{H}_\mathcal{T}}\varphi^{-\frac{1}{k-1}}\left(|{}_t^C D_T^\alpha\varphi|^{\frac{k}{k-1}}+|\Delta_\mathbb{H}\varphi|^{\frac{k}{k-1}}\right)\right)d\eta dt\\&+|\mu_1|\int_{\mathcal{H}_\mathcal{T}}\varphi^{-\frac{q}{p-q}}\left(|\varphi|^{\frac{p}{p-q}}|Div_\mathbb{H}(a(\eta))|^{\frac{p}{p-q}}+|a(\eta)|^{\frac{p}{p-q}}|\nabla_\mathbb{H}\varphi|^{\frac{p}{p-q}}\right)d\eta dt\\&+\int_{\mathcal{H}_\mathcal{T}}\varphi^{-\frac{1}{p-1}}\left(|{}_t^C D_T^\beta\varphi|^{\frac{p}{p-1}}+|\Delta_\mathbb{H}\varphi|^{\frac{p}{p-1}}\right)d\eta dt\\&+|\mu_1|\int_{\mathcal{H}_\mathcal{T}}\varphi^{-\frac{\sigma}{k-\sigma}}\left(|\varphi|^{\frac{k}{k-\sigma}}|Div_\mathbb{H}(b(\eta))|^{\frac{k}{k-\sigma}}|b(\eta)|^{\frac{k}{k-\sigma}}|\nabla_\mathbb{H}\varphi|^{\frac{k}{k-\sigma}}d\eta dt\right).\end{aligned}\tag{3.41}$$

At this stage, we use the scaled variable

$$\tilde{\tau}=T^{-(\alpha+\beta)}\tau,\quad\tilde{x}=T^{-\frac{(\alpha+\beta)}{2}}x,\quad\tilde{y}=T^{-\frac{(\alpha+\beta)}{2}}y,\tag{3.42}$$

to obtain

$$\begin{aligned}&\frac{1}{2}\int_{\mathcal{H}_\mathcal{T}}\left(|v|^p+|u|^k\right)\varphi d\eta dt+\frac{1}{\lambda_1}\int_{\mathcal{H}_\mathcal{T}}G_1(\alpha;g(\eta))\ {}_t^C D_T^\alpha\varphi d\eta dt\\&+\frac{1}{\lambda_1}\int_{\mathcal{H}_\mathcal{T}}G_1(\beta,h(\eta))\ {}_t^C D_T^\beta\varphi d\eta dt\leq C\left(T^{1-\frac{2k}{k-1}+(N+1)(\alpha+\beta)}+T^{1-\frac{\alpha k}{k-1}+(N+1)(\alpha+\beta)}\right)\\&+C\left(T^{1-\frac{2p}{p-1}+(N+1)(\alpha+\beta)}+T^{1-\frac{\beta p}{p-1}+(N+1)(\alpha+\beta)}\right)\\&+|\mu_1|\left(T^{1-\frac{\tau_1 p}{p-q}+(N+1)(\alpha+\beta)}+T^{1-\frac{\theta p}{p-q}+\frac{\nu_1 p}{p-q}+(N+1)(\alpha+\beta)}\right)\\&+|\mu_1|\left(T^{1-\frac{\tau_2 k}{k-\sigma}+(N+1)(\alpha+\beta)}+T^{1-\frac{\theta k}{k-\sigma}+\frac{\nu_2 k}{k-\sigma}+(N+1)(\alpha+\beta)}\right).\end{aligned}\tag{3.43}$$

Furthermore, we have

$$\frac{1}{\lambda_1}\int_{\mathcal{H}_\mathcal{T}}G_1(\alpha;g(\eta))\ {}_t^C D_T^\alpha\varphi d\eta dt=\frac{1}{\lambda_1}\frac{\Gamma(m+1)}{\Gamma(m+2-\alpha)}T^{1-\alpha}\int_{\mathcal{H}_\mathcal{T}}G_1(\alpha;g(\eta))\Phi^\omega(\rho)d\eta dt,\tag{3.44}$$

and

$$\frac{1}{\lambda_1}\int_{\mathcal{H}_\mathcal{T}} G_1(\beta,h(\eta))\ {}_t^C D_T^\beta \varphi d\eta dt = \frac{1}{\lambda_1}\frac{\Gamma(m+1)}{\Gamma(m+2-\beta)}T^{1-\beta}\int_{\mathcal{H}_\mathcal{T}} G_1(\beta;h(\eta))\Phi^\omega(\rho)d\eta dt. \quad (3.45)$$

Therefore, we get

$$\begin{aligned}&\frac{1}{\lambda_1}\frac{\Gamma(m+1)}{\Gamma(m+2-\alpha)}\int_{\mathcal{H}_\mathcal{T}} G_1(\alpha;g(\eta))\Phi^\omega(\rho)d\eta dt\\ &+\frac{1}{\lambda_1}\frac{\Gamma(m+1)}{\Gamma(m+2-\beta)}T^{\alpha-\beta}\int_{\mathcal{H}_\mathcal{T}} G_1(\beta;h(\eta))\Phi^\omega(\rho)d\eta dt\\ &\le C\left(T^{\theta_1}+T^{\theta_2}+T^{\theta_3}+T^{\theta_4}\right)+C|\mu_1|\left(T^{\theta_5}+T^{\theta_6}+T^{\theta_7}+T^{\theta_8}\right),\end{aligned} \quad (3.46)$$

where

$$\theta_1 = \alpha - \frac{2k}{k-1} + (N+1)(\alpha+\beta), \qquad \theta_2 = \alpha - \frac{\alpha k}{k-1} + (N+1)(\alpha+\beta),$$

$$\theta_3 = \alpha - \frac{2p}{p-1} + (N+1)(\alpha+\beta), \qquad \theta_4 = \alpha - \frac{\beta p}{p-1} + (N+1)(\alpha+\beta),$$

$$\theta_5 = \alpha - \frac{\tau_1 p}{p-q} + (N+1)(\alpha+\beta), \qquad \theta_6 = \alpha - \frac{(\alpha+\beta)p}{2(p-q)} + \frac{\tau_1 p}{p-q} + (N+1)(\alpha+\beta),$$

$$\theta_7 = \alpha - \frac{\tau_2 k}{k-\sigma} + (N+1)(\alpha+\beta), \qquad \theta_8 = \alpha - \frac{(\alpha+\beta)k}{2(k-\sigma)} + \frac{\tau_2 k}{k-\sigma} + (N+1)(\alpha+\beta).$$

Suppose now that

$$\alpha=\beta \quad \text{and} \qquad \lambda_1\int_{\mathbb{H}}\left[G_1(\alpha;g(\eta))+G_2(\beta;h(\eta))\right]d\eta>0. \quad (3.47)$$

We distinguish two cases:

Case 1: $\mu_1 = 0$, $Q < min\{\frac{1}{p-1};\frac{1}{k-1}\}$. In this case, passing to the limit as $T \longrightarrow +\infty$ in (3.46), we obtain a contradiction with (3.47).

Case 2: $\mu_1 \neq 0$; $Q < min\left\{\frac{\tau_1 p-\alpha(p-q)}{\alpha(p-q)};\frac{\alpha q-\nu_1 p}{\alpha(p-q)};\frac{\tau_2 k-\alpha(k-\sigma)}{\alpha(k-\sigma)};\frac{\alpha\sigma-\nu_2 k}{\alpha(k-\sigma)}\right\}$. Similarly, passing to the limit as $T \longrightarrow +\infty$ in (3.46), we obtain a contradiction with (3.47).

Suppose now that

$$\beta<\alpha; \qquad \lambda_1\int_{\mathbb{H}} G_1(\beta;h(\eta))d\eta>0. \quad (3.48)$$

We have to distinguish to cases:

Case 1: $\mu_1 = 0$, $Q < \frac{2}{\alpha+\beta} \quad min\left\{\frac{\alpha}{k-1};\frac{\beta p-\alpha(p-1)}{p-1}\right\}$. In this case, passing to the limit as $T \longrightarrow +\infty$ in (3.46), we get a contradiction with (3.48).

Case 2: $\mu_1 \neq 0, Q < \frac{2}{\alpha+\beta} \quad min\left\{\frac{\tau_1 p-\alpha(p-q)}{p-q};\frac{2\alpha q-(\alpha+\beta+2\nu_1)p}{2(p-q)};\frac{\tau_2 k-\alpha(k-\sigma)}{k-\sigma};\frac{(\alpha+\beta)k-2\alpha(k-\sigma)-2\nu_2 k}{2(k-\sigma)}\right\}$.

Similarly, passing to the limit as $T \longrightarrow +\infty$ in (3.46), we obtain a contradiction with (3.48). Next, we consider the case $\lambda_2 \neq 0$. Observe that

$$(U(\eta;t),\ V(\eta;t)) = \left(\frac{u(\eta;t)}{i},\ \frac{v(\eta;t)}{i}\right),$$

is a global weak solution to the system

$$\begin{aligned} i^{\alpha}\ {}_{0}^{C}D_t^{\alpha}U + \Delta_{\mathbb{H}}U &= \lambda'|V|^p + \mu' a\left(\eta\right).\nabla_{\mathbb{H}}|V|^q, \\ i^{\beta}\ {}_{0}^{C}D_t^{\beta}V + \Delta_{\mathbb{H}}V &= \lambda'|U|^k + \mu' b\left(\eta\right).\nabla_{\mathbb{H}}|U|^{\sigma}, \\ U\left(\eta,0\right) = \tilde{g}\left(\eta\right); &\qquad V\left(\eta,0\right) = \tilde{h}\left(\eta\right), \end{aligned}$$

where

$$\lambda' = \lambda_2 + i(-\lambda_1) = \lambda_1' + i\lambda_2', \mu' = \mu_2 + i(-\lambda_1) = \mu_1' + i\mu_2',$$

$$\tilde{g}(\eta) = g_2(\eta) + i(-g_1(\eta)) = \tilde{g}_1(\eta) + i\tilde{g}_2(\eta), \tilde{h}(\eta) = h_2(\eta) + i(-h_1(\eta)) = \tilde{h}_1(\eta) + i\tilde{h}_2(\eta).$$

Therefore, from the previous study, if one of the cases **(III)** or **(IV)** holds, we obtain a contradiction.

References

[1] Alsaedi, A., Ahmad, B., and Kirane, M., Nonexistence of global solutions of nonlinear space- fractional equations on the Heisenberg group, Electronic Journal of Diferential Equations, (2015), 01-10.

[2] Al-Salti, N., Kerbal, S., Non-Local Elliptic Systems on the Heisenberg Group, Electronic Journal of Differential Equations, 09, (2016), 1-7.

[3] Azman, I., Jleli, M., Kirane, M., Samet, B., Nonexistence of global solutions for fractional temporal Schödinger equations and systems, Electronic Journal of Diferential Equations, Vol.276, (2017), 1-17.

[4] Baras, P., Pierre.P., Critére d'existence de solutions positives pour des équations semilinéaires non monotones, Annales de l'Institut H. Poincarée Analyse Non Linéaire,2, (1985), 185-212.

[5] El-Hamidi, A., Kirane, M., Nonexistence results of solutions to systems of semilinear differential inequalities on the Heisenberg group, Abstract and Applied Analysis,2, (2004), 155-164.

[6] Folland, G. B., Stein, E. M., Estimates for the ∂_h complex and analysis on the Heisenberg Group, Comm. Pure Appl. Math, 27, (1974), 492-522.

[7] Fujita, H., On the blowing-up of solutions to the Cauchy problems for $u_t = \Delta u + u^{1+\alpha}$, J. Fac. Sci. Univ. Tokyo, Sect. IA, 13, (1966), 109-124.

[8] Hakem, A.,Berbiche, M.,. Blow up results for fractional differential equations and systems, Publication de l'institut mathématique, Nouvelle série, 107(93), (2013),173-186.

[9] Kellil, R., Kirane, M., Nonexistence of Global Weak Solutions of a System of Wave Equations on the Heisenberg Group, J. Nonlinear Sci. Appl (2015), 1-9.

[10] Kilbas, A. A., Srivastava, H.M., Trujillo, J.J., Theory and Applications of Fractional Differential Equations, vol 204. Elsevier Science B.V., Amsterdam (2006).

[11] Kirane, M., Ragoub,L., Nonexistence results for a pseudo-hyperbolic equation in the Heisenberg group, Electronic Journal of Differential Equations, 110, (2015), 1-9.

[12] Mitidieri, E., Pohozaev, S. I., A priori estimates and blow-up of solutions to nonlinear partial differential equations and inequalities, Proc. Steklov. Inst. Math, 234, (2001), 1-383.

[13] Mitidieri, E., Pohozaev, S. I., Nonexistence of weak solutions for some degenerate elliptic and parabolic problems on $\mathbb{R}^N$, J. Evol. Equations 1, (2001), 189-220.

[14] Podlubny, P., Fractional Differential Equations, Academic press, New York, 1999.

[15] Pohozaev, S. I., Tesei, E., Blow-up of nonnegative solutions to quasilinear parabolic inequalities, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei , 9 Mat. Appl, 11(2), (2000), 99-109.

[16] Zhang, Q. S. The critical exponent of a reaction difusion equation on some lie groups, Math. Z., 228, (1998), 51-72.

[17] Zhang, Q. S. Blow-up results for nonlinear parabolic equations on manifolds, Duke Math. J, 97, (1999) 515-539.

[18] Zhang, Q. S. A blow up result for a nonlinear wave equation with damping: the critical case, C. R. Acad. Sci. Paris, 333(02) (2001), 109-114.

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