International Journal of Maps in Mathematics

*Volume **8**, Issue **1**, 2025, Pages:**192-207***

E-ISSN: 2636-7467

www.simadp.com/journalmim

ON ULTRAMETRIC PSEUDOSPECTRA OF THE DIRECT SUM OF LINEAR OPERATOR PENCILS

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ABSTRACT. In this paper, we introduce the concepts of pseudospectra, condition pseudospectra, determinant spectra and trace pseudospectra of the direct sum of bounded linear operator pencils on ultrametric Banach spaces. We prove numerous results about them and we give some examples to illustrate our work.

Keywords: Ultrametric Banach spaces, pseudospectra, condition pseudospectra, direct sum of operators, linear operator pencils.

2010 Mathematics Subject Classification: Primary 47S10, Secondary 47A05, 47A10.

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1. Introduction and preliminaries

In ultrametric operator theory, Ammar et al. [1] introduced and studied the concept of pseudospectra of closed linear operators on ultrametric Banach spaces. The notion of ultrametric condition pseudospectra of bounded linear operators was introduced by the authors [2].

Throughout this paper, F is an ultrametric Banach space over an ultrametric complete valued field $\mathbb{K}$ with a non-trivial valuation $|\cdot|$, $\mathcal{L}(F)$ denotes the collection of each continuous linear operators on F and $\mathcal{M}_n(\mathbb{K})$ is the collection of any $n\times n$ matrices with coefficients in $\mathbb{K}$. If $S\in\mathcal{L}(F)$, $R(S)$ and $N(S)$ denote the range and the kernel of S respectively. Remember that, an unbounded linear operator $S\ :D(S)\subseteq F\to F$ will be called closed if for each $(x_n)\subset D(S)$ with $\lim\limits_{n\to\infty}\|x_n-x\|=0$ and $\lim\limits_{n\to\infty}\|Ax_n-y\|=0$ for some $x\in F$ and $y\in F$,

Received:2023.11.30 *Revised:2024.10.28* *Accepted:2014.11.13*

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hence $x \in D(A)$ with $y = Sx$. $\mathcal{C}(F)$ is the set of all closed linear operators on F. If $A \in \mathcal{L}(F)$ and B is an unbounded operator, hence $S + B$ is closed if and only if B is closed [4]. For more details on ultrametric pseudospectra and condition pseudospectra of linear operators, we refer to [1], [2], [6], [8], [9], [10], [11], [12], [13] and [14]. We continue by recalling some preliminaries.

Definition 1.1. [4] *Let F be a vector space over $\mathbb{K}$. A function $\|\cdot\| : F \to \mathbb{R}_+$ is an ultrametric norm if:*

(i) *For each $v \in F$, $\|v\| = 0$ if and only if $v = 0$,*

(ii) *For each $v \in F$ and $\lambda \in \mathbb{K}$, $\|\lambda v\| = |\lambda|\|v\|$,*

(iii) *For all $v, y \in F$, $\|v + y\| \leq \max(\|v\|, \|y\|)$.*

Definition 1.2. [4] *An ultrametric normed space is a pair $(F, \|\cdot\|)$ where F is a vector space over $\mathbb{K}$ and $\|\cdot\|$ is an ultrametric norm on F.*

Definition 1.3. [4] *An ultrametric Banach space is a complete ultrametric normed space.*

Proposition 1.1. [4] *The direct sum of two ultrametric Banach spaces is an ultrametric Banach space.*

Definition 1.4. [4] *An ultrametric Banach space F is said to be a free Banach space if there is a set $(v_i)_{i\in I}$ of F indexed by a set I such that all element $v \in F$ can be written uniquely as follows $v = \sum_{i\in I} \lambda_i v_i$ and $\|v\| = \sup_{i\in I} |\lambda_i|\|v_i\|$.*
The family $(v_i)_{i\in I}$ is called an orthogonal basis for F. If, for each $i \in I$, $\|v_i\| = 1$, hence $(v_i)_{i\in I}$ is called an orthonormal basis of F.

Definition 1.5. [4] *Let $S \in \mathcal{L}(F)$. The resolvent set $\rho(S)$ of S is*

$$\rho(S) = \{\lambda \in \mathbb{K} : (S - \lambda I)^{-1} \in \mathcal{L}(F)\}. \tag{1.1}$$

The spectrum $\sigma(S)$ of S is $\mathbb{K}\backslash\rho(S)$.

Lemma 1.1. [5] *Let F be an ultrametric Banach space over a spherically complete field $\mathbb{K}$. If $S \in \Phi(F)$ and $C \in \mathcal{C}_c(F)$, then $S + C \in \Phi(F)$.*

Lemma 1.2. [2] *Let F be an ultrametric Banach space over a spherically complete field $\mathbb{K}$. If $S \in \Phi(F)$, hence for each $C \in \mathcal{C}_c(F)$, we get $S + C \in \Phi(F)$ and $ind(S + K) = ind(S)$.*

We have the following definition.

Definition 1.6. [1] *Let F be an ultrametric Banach space over $\mathbb{K}$, let $S \in \mathcal{L}(F)$ and $\varepsilon > 0$. The pseudospectrum $\sigma_\varepsilon(S)$ of S is*

$$\sigma_\varepsilon(S) = \sigma(S) \cup \{\lambda \in \mathbb{K} : \ \|(S - \lambda I)^{-1}\| > \varepsilon^{-1}\},$$

with the convention $\|(S - \lambda I)^{-1}\| = \infty$ if $\lambda \in \sigma(S)$.

Theorem 1.1. [1] *Let F be an ultrametric Banach space over a spherically complete field $\mathbb{K}$ such that $\|F\| \subseteq |\mathbb{K}|$, let $S \in \mathcal{L}(F)$ and $\varepsilon > 0$. Then*

$$\sigma_\varepsilon(S) = \bigcup_{C \in \mathcal{L}(F): \|C\| < \varepsilon} \sigma(S + C).$$

Theorem 1.2. [2] *Let F be an ultrametric Banach space over a spherically complete field $\mathbb{K}$ and let $S \in \mathcal{L}(F)$. Then*

$$\sigma_e(S) = \bigcap_{C \in \mathcal{C}_c(F)} \sigma(S + C).$$

We generalise the Definition 3.7 of [1]: this definition remains valid for any ultrametric Banach spaces over a non-trivally complete ultrametric valued field $\mathbb{K}$ not only $\mathbb{E}_\omega$.

Definition 1.7. *Let $S \in \mathcal{L}(F)$ and $\varepsilon > 0$. The essential pseudospectrum of S is*

$$\sigma_{e,\varepsilon}(S) = \mathbb{K} \backslash \{\lambda \in \mathbb{K} : S + M - \lambda I \in \Phi_0(F) \text{ for all } M \in \mathcal{L}(F), \ \|M\| < \varepsilon\},$$

where $\Phi_0(F)$ is the set of each bounded Fredholm operators on F of index 0.

We generalise the Theorem 3.8 of [1] as follows.

Theorem 1.3. *Let $S \in \mathcal{L}(F)$ and $\varepsilon > 0$. Hence*

$$\sigma_{e,\varepsilon}(S) = \bigcup_{M \in \mathcal{L}(F): \|M\| < \varepsilon} \sigma_e(S + M).$$

We have the following:

Theorem 1.4. *Let F be an ultrametric Banach space over a spherically complete field $\mathbb{K}$. Let $S \in \mathcal{L}(F)$ and $\varepsilon > 0$. Hence,*

$$\sigma_{e,\varepsilon}(S) = \sigma_{e,\varepsilon}(S + K) \ \text{ for each } \ K \in \mathcal{C}_c(F).$$

Proof. Let $S \in \mathcal{L}(F)$ and $\varepsilon > 0$, let $\lambda \notin \sigma_{e,\varepsilon}(S)$, hence for any $C \in \mathcal{L}(F)$ with $\|C\| < \varepsilon$,

$$S + C - \lambda I \in \Phi(F) \text{ and } ind(S + C - \lambda I) = 0.$$

From Lemma 1.2, for each $K \in \mathcal{C}_c(F)$ and $C \in \mathcal{L}(F)$ such that $\|C\| < \varepsilon$, we have

$$S + C + K - \lambda I \in \Phi(F) \text{ and } ind(S + C + K - \lambda I) = 0. \tag{1.2}$$

By (1.2), we get

$$\lambda \notin \sigma_{e,\varepsilon}(S+K).$$

Then

$$\sigma_{e,\varepsilon}(S+K) \subseteq \sigma_{e,\varepsilon}(S).$$

The opposite inclusion follows from symmetry. □

Remark 1.1. *The Theorem 1.4 showed that the essential pseudospectra of bounded linear operators is invariant under perturbation of completely continuous linear operators on ultrametric Banach space over a spherically complete field* $\mathbb{K}$.

Theorem 1.5. *Let F be an ultrametric Banach space over a spherically complete field $\mathbb{K}$ such that $\|F\| \subseteq |\mathbb{K}|$. Let $S \in \mathcal{L}(F)$ and $\varepsilon > 0$. Then*

$$\sigma_{e,\varepsilon}(S) = \bigcap_{K \in \mathcal{C}_c(F)} \sigma_\varepsilon(S+K).$$

Proof. Let $S \in \mathcal{L}(F)$ and $\varepsilon > 0$, let $\lambda \notin \bigcap_{K \in \mathcal{C}_c(F)} \sigma_\varepsilon(S+K)$, hence there exists $K \in \mathcal{C}_c(F)$ such that $\lambda \notin \sigma_\varepsilon(S+K)$. By Theorem 1.1, we have $\lambda \in \rho(S+K+C)$, for all $C \in \mathcal{L}(F)$ such that $\|C\| < \varepsilon$. We have

$$S+C+K-\lambda I \in \Phi(F) \text{ and } ind(S+C+K-\lambda I) = 0. \tag{1.3}$$

By Lemma 1.1 and Lemma 1.2, we have

$$S+C-\lambda I \in \Phi(F) \text{ and } ind(S+C-\lambda I) = 0. \tag{1.4}$$

We get

$$\lambda \notin \sigma_{e,\varepsilon}(S).$$

Then,

$$\sigma_{e,\varepsilon}(S) \subseteq \bigcap_{K \in \mathcal{C}_c(F)} \sigma_\varepsilon(S+K). \tag{1.5}$$

Conversely, if $\lambda \notin \sigma_{e,\varepsilon}(S)$. Using Theorem 1.3, we have for all $C \in \mathcal{L}(F)$ such that $\|C\| < \varepsilon$, $\lambda \notin \sigma_e(S+C)$. By Theorem 1.2, there is $K \in \mathcal{C}_c(F)$ with $\lambda \notin \sigma(S+K+C)$, hence for all $C \in \mathcal{L}(F)$ such that $\|C\| < \varepsilon$, $\lambda \in \rho(S+K+C)$. Hence

$$\lambda \in \bigcap_{C \in \mathcal{L}(F) : \|C\| < \varepsilon} \rho(S+K+C). \tag{1.6}$$

From Theorem 1.1, $\lambda \notin \sigma_\varepsilon(S+K)$. Consequently,

$$\lambda \notin \bigcap_{K \in \mathcal{C}_c(F)} \sigma_\varepsilon(S+K).$$

Thus

$$\sigma_{e,\varepsilon}(S) = \bigcap_{K \in \mathcal{C}_c(F)} \sigma_\varepsilon(S+K).$$

□

We generalise the Proposition 3.13 of [1] as follows.

Proposition 1.2. *Let F be an ultrametric Banach space over a spherically complete field $\mathbb{K}$ such that $\|F\| \subseteq |\mathbb{K}|$. If $S \in \mathcal{L}(F)$ and $\varepsilon > 0$, then*

(i) $\sigma_{e,\varepsilon}(S) \subset \sigma_\varepsilon(S)$.

(ii) *For each ε_1 and ε_2 with $0 < \varepsilon_1 < \varepsilon_2$, $\sigma_e(S) \subset \sigma_{e,\varepsilon_1}(S) \subset \sigma_{e,\varepsilon_2}(S)$.*

Similarly to the proof of Proposition 3.14 of [1], we have the following:

Proposition 1.3. *Let F be an ultrametric Banach space over a spherically complete field $\mathbb{K}$ such that $\|F\| \subseteq |\mathbb{K}|$. Let $S \in \mathcal{L}(F)$ and $\varepsilon > 0$. Hence*

$$\sigma_e(S) = \bigcap_{\varepsilon > 0} \sigma_{e,\varepsilon}(S).$$

Definition 1.8. [6] *Let $S \in \mathcal{L}(F)$ and $\varepsilon > 0$, the condition pseudospectrum $\Lambda_\varepsilon(S)$ of S is*

$$\Lambda_\varepsilon(S) = \sigma(S) \cup \{\lambda \in \mathbb{K} : \|(S-\lambda I)\|\|(S-\lambda I)^{-1}\| > \frac{1}{\varepsilon}\},$$

with the convention $\|(S-\lambda I)\|\|(S-\lambda I)^{-1}\| = \infty$ if $\lambda \in \sigma(S)$.

We generalise the results of [14] as follows.

Definition 1.9. *Let $S \in \mathcal{C}(F)$, $B \in \mathcal{L}(F)$ and $\varepsilon > 0$. The pseudospectrum $\sigma_\varepsilon(S,B)$ of (S,B) on F is defined by*

$$\sigma_\varepsilon(S,B) = \sigma(S,B) \cup \{\lambda \in \mathbb{K} : \|(S-\lambda B)^{-1}\| > \varepsilon^{-1}\}.$$

The pseudoresolvent $\rho_\varepsilon(S,B)$ of (S,B) is defined by

$$\rho_\varepsilon(S,B) = \rho(S,B) \cap \{\lambda \in \mathbb{K} : \|(S-\lambda B)^{-1}\| \leq \varepsilon^{-1}\},$$

by convention $\|(S-\lambda B)^{-1}\| = \infty$ if $\lambda \in \sigma(S,B)$.

Proposition 1.4. *Let $S \in \mathcal{C}(F)$, $B \in \mathcal{L}(F)$ and $\varepsilon > 0$, we get*

(i) $\sigma(S,B) = \bigcap_{\varepsilon>0} \sigma_\varepsilon(S,B)$.

(ii) *For any* ε_1 *and* ε_2 *such that* $0<\varepsilon_1<\varepsilon_2$, $\sigma(S,B) \subset \sigma_{\varepsilon_1}(S,B) \subset \sigma_{\varepsilon_2}(S,B)$.

Theorem 1.6. *Let F be an ultrametric Banach space over a spherically complete field $\mathbb{K}$ such that $\|F\| \subseteq |\mathbb{K}|$, let $S \in \mathcal{C}(F), B \in \mathcal{L}(F)$ and $\varepsilon > 0$. Hence*

$$\sigma_\varepsilon(S,B) = \bigcup_{C\in\mathcal{L}(F):\|C\|<\varepsilon} \sigma(S+C,B).$$

The condition pseudospectra of operator pencils is defined as follows:

Definition 1.10. [8] *Let F be an ultrametric Banach space over $\mathbb{K}$, let $B,S \in \mathcal{L}(F)$ and $\varepsilon > 0$. The pseudospectrum $\sigma_\varepsilon(S,B)$ of the linear operator pencil (S,B) on F is*

$$\Lambda_\varepsilon(S,B) = \sigma(S,B) \cup \{\lambda \in \mathbb{K} : \|(S-\lambda B)\|\|(S-\lambda B)^{-1}\| > \varepsilon^{-1}\},$$

with the convention $\|(S-\lambda B)\|\|(S-\lambda B)^{-1}\| = \infty$ if $\lambda \in \sigma(S,B)$.

Proposition 1.5. [8] *Let $B,S \in \mathcal{L}(F)$ and $\varepsilon>0$, we get*

(i) $\sigma(S,B) = \bigcap_{\varepsilon>0} \Lambda_\varepsilon(S,B)$.

(ii) *For any* ε_1 *and* ε_2 *such that* $0<\varepsilon_1<\varepsilon_2$, $\sigma(S,B) \subset \Lambda_{\varepsilon_1}(S,B) \subset \Lambda_{\varepsilon_2}(S,B)$.

Theorem 1.7. [8] *Let F be an ultrametric Banach space over a spherically complete field $\mathbb{K}$ such that $\|F\| \subseteq |\mathbb{K}|$, let $B,S \in \mathcal{L}(F)$ and $\varepsilon > 0$. Hence,*

$$\Lambda_\varepsilon(S,B) = \bigcup_{C\in\mathcal{L}(F):\|C\|<\varepsilon\|S-\lambda B\|} \sigma(S+C,B).$$

2. Main results

Let $(X_i)_{1\le i\le n}$ be a sequence of ultrametric Banach spaces over $\mathbb{K}$. The space $X = \oplus_{i=1}^n X_i$ endowed by for all $i \in \{1,\cdots,n\}$, $x_i \in X_i$, $\|x_1 \oplus x_2 \oplus \cdots \oplus x_n\| = \max_{i\in\{1,\cdots,n\}} \|x_i\|$ is an ultrametric Banach space over $\mathbb{K}$ [4]. One can see that if for each $i \in \{1,\cdots,n\}$, $A_i \in \mathcal{L}(X_i)$, hence $A = A_1 \oplus A_2 \oplus \cdots \oplus A_n \in \mathcal{L}(X)$. We get the following definition.

Definition 2.1. *Let $A_i, T_i \in \mathcal{L}(X_i)$. The spectrum $\sigma(A,T)$ of (A,T) is given by*

$$\sigma(A,T) = \{\lambda \in \mathbb{K} : A-\lambda T \textit{ is not invertible in } \mathcal{L}(\oplus_{i=1}^n X_i)\},$$

where $A = \oplus_{i=1}^n A_i$ and $T = \oplus_{i=1}^n T_i$. The resolvent set of (A,T) is

$$\rho(A,T) = \{\lambda \in \mathbb{K} : (A-\lambda T)^{-1} \in \mathcal{L}(\oplus_{i=1}^n X_i)\}.$$

For $i = 2$, we obtain the following proposition.

Proposition 2.1. *Let X, Y be two ultrametric Banach spaces over $\mathbb{K}$. Let $A, S \in \mathcal{L}(X)$, $B, T \in \mathcal{L}(Y)$. The spectrum of $(A \oplus B) - \lambda(S \oplus T) \in \mathcal{L}(X \oplus Y)$ is given by*

$$\sigma(A \oplus B, S \oplus T) = \sigma(A, S) \cup \sigma(B, T).$$

Proof. If $\lambda \in \sigma(A \oplus B, S \oplus T)$, hence $(A \oplus B) - \lambda(S \oplus T)$ is not invertible, then $A - \lambda S$ is not invertible in $\mathcal{L}(X)$ or $B - \lambda T$ is not invertible in $\mathcal{L}(Y)$, consequently $\lambda \in \sigma(A, S) \cup \sigma(B, T)$. Hence $\sigma(A \oplus B, S \oplus T) \subseteq \sigma(A, S) \cup \sigma(B, T)$. Similarly, we obtain that $\sigma(A, S) \cup \sigma(B, T) \subseteq \sigma(A \oplus B, S \oplus T)$. Consequently,

$$\sigma(A \oplus B, S \oplus T) = \sigma(A, S) \cup \sigma(B, T).$$

□

More generally, one can see that.

Proposition 2.2. *Let $A_i, B_i \in \mathcal{L}(X_i)$. Set $A = \oplus_{i=1}^n A_i$ and $B = \oplus_{i=1}^n B_i$, then*

$$\sigma(A, B) = \bigcup_{i=1}^n \sigma(A_i, B_i)$$

and

$$\rho(A, B) = \bigcap_{i=1}^n \rho(A_i, B_i).$$

Now, we define the pseudospectra of the operator pencil (A, B) on $\oplus_{i=1}^n X_i$ where $A = \oplus_{i=1}^n A_i$, $B = \oplus_{i=1}^n B_i$ and for all $i \in \{1, \cdots, n\}$, $A_i, B_i \in \mathcal{L}(X_i)$, we have the following:

Definition 2.2. *Let $A_i, B_i \in \mathcal{L}(X_i)$ and $\varepsilon > 0$. The pseudospectrum $\sigma_\varepsilon(\oplus_{i=1}^n A_i, \oplus_{i=1}^n B_i)$ of the bounded linear operator pencil $(\oplus_{i=1}^n A_i, \oplus_{i=1}^n B_i)$ on $\oplus_{i=1}^n X_i$ is given by*

$$\sigma_\varepsilon(A, B) = \sigma(\oplus_{i=1}^n A_i, \oplus_{i=1}^n B_i) \cup \{\lambda \in \bigcap_{i=1}^n \rho(A_i, B_i) : \sup_{i \in \{1, \cdots, n\}} \|(A_i - \lambda B_i)^{-1}\| > \varepsilon^{-1}\},$$

where $A = \oplus_{i=1}^n A_i$ and $B = \oplus_{i=1}^n B_i$.

Remark 2.1. *It is easy to check that $\sigma_\varepsilon(\oplus_{i=1}^n A_i, \oplus_{i=1}^n B_i) = \bigcup_{i=1}^n \sigma_\varepsilon(A_i, B_i)$.*

Proposition 2.3. *Let $(X_i)_{1 \leq i \leq n}$ be a sequence of ultrametric Banach spaces over $\mathbb{K}$, let $A_i, B_i \in \mathcal{L}(X_i)$ and $\varepsilon > 0$. Set $S = \oplus_{i=1}^n A_i$ and $B = \oplus_{i=1}^n B_i$, then*

(i) $\sigma(S, B) = \bigcap_{\varepsilon > 0} \sigma_\varepsilon(\oplus_{i=1}^n A_i, \oplus_{i=1}^n B_i)$.

(ii) *If $0 < \varepsilon_1 < \varepsilon_2$, hence $\sigma(S, B) \subset \sigma_{\varepsilon_1}(S, B) \subset \sigma_{\varepsilon_2}(S, B)$.*

Proof. (i) From Definition 2.2, for each $\varepsilon > 0$, $\sigma(S,B) \subset \sigma_\varepsilon(S,B)$, hence $\sigma(S,B) \subset \bigcap_{\varepsilon>0} \sigma_\varepsilon(S,B)$. Conversely, if $\lambda \in \bigcap_{\varepsilon>0} \sigma_\varepsilon(S,B)$, since

$$\bigcap_{\varepsilon>0} \sigma_\varepsilon(S,B) = \sigma(S,B) \cup \bigcap_{\varepsilon>0} \{\lambda \in \bigcap_{i=1}^{n} \rho(A_i,B_i) : \sup_{i\in\{1,\cdots,n\}} \|(A_i - \lambda B_i)^{-1}\| > \varepsilon^{-1}\}$$

and $\bigcap_{\varepsilon>0} \{\lambda \in \bigcap_{i=1}^{n} \rho(A_i,B_i) : \sup_{i\in\{1,\cdots,n\}} \|(A_i - \lambda B_i)^{-1}\| > \varepsilon^{-1}\} = \emptyset$ because of for all $i \in \{1,\cdots,n\}$, $(A_i - \lambda B_i)^{-1}$ are bounded linear operators, hence we get a contradiction, thus $\lambda \in \sigma(S,B)$.

(ii) If $0 < \varepsilon_1 < \varepsilon_2$ and $\lambda \in \sigma_{\varepsilon_1}(S,B)$, consequently, $\sup_{i\in\{1,\cdots,n\}} \|(A_i - \lambda B_i)^{-1}\| > \varepsilon_1^{-1} > \varepsilon_2^{-1}$, hence $\lambda \in \sigma_{\varepsilon_2}(S,B)$.

□

Lemma 2.1. *Let $A_i, B_i \in \mathcal{L}(X_i)$ and $\varepsilon > 0$, hence*

$$\sup_{\lambda\in\sigma(\oplus_{i=1}^n A_i, \oplus_{i=1}^n B_i)} |\lambda| \le \sup_{\lambda\in\sigma_\varepsilon(\oplus_{i=1}^n A_i, \oplus_{i=1}^n B_i)} |\lambda|.$$

Proof. Set $S = \oplus_{i=1}^n A_i$ and $B = \oplus_{i=1}^n B_i$, since $\sigma(S,B) \subseteq \sigma_\varepsilon(S,B)$, we get

$$\sup_{\lambda\in\sigma(S,B)} |\lambda| \le \sup_{\lambda\in\sigma_\varepsilon(S,B)} |\lambda|.$$

□

Put $r_\varepsilon(\oplus_{i=1}^n A_i, \oplus_{i=1}^n B_i) = \sup_{\lambda\in\sigma_\varepsilon(\oplus_{i=1}^n A_i, \oplus_{i=1}^n B_i)} |\lambda|$, we have the following:

Lemma 2.2. *Let $A_i, B_i \in \mathcal{L}(X_i)$ and $\varepsilon > 0$, hence $r_\varepsilon(\oplus_{i=1}^n A_i, \oplus_{i=1}^n B_i) = \sup_{i\in\{1,\cdots,n\}} r_\varepsilon(A_i, B_i)$.*

Proof. By Remark 2.1, $\sigma_\varepsilon(\oplus_{i=1}^n A_i, \oplus_{i=1}^n B_i) = \bigcup_{i=1}^{n} \sigma_\varepsilon(A_i,B_i)$. It is easy to see that

$$r_\varepsilon(\oplus_{i=1}^n A_i, B_i) = \sup_{i\in\{1,\cdots,n\}} r_\varepsilon(A_i, B_i).$$

□

We get the following examples.

Example 2.1. *Let $(A_k)_{1\le k\le n}$ and $(B_k)_{1\le k\le n}$ be two linear operators defined on $\mathbb{K}^2$ by*

$$A_k = \begin{pmatrix} \lambda_k & 0 \\ 0 & \mu_k \end{pmatrix}$$

and

$$B_k = \begin{pmatrix} \alpha_k & 0 \\ 0 & \beta_k \end{pmatrix},$$

where $\lambda_k, \mu_k \in \mathbb{K}$, $\alpha_k, \beta_k \in \mathbb{K}\backslash\{0\}$ *for each* $k \in \{1, \cdots, n\}$. *Then*

$$\sigma(\oplus_{k=1}^n A_k, \oplus_{k=1}^n B_k) = \bigcup_{k=1}^n \left\{\frac{\lambda_k}{\alpha_k}, \frac{\mu_k}{\beta_k}\right\}$$

and

$$\sigma_\varepsilon(\oplus_{k=1}^n A_k, \oplus_{k=1}^n A_k) = \bigcup_{k=1}^n \left\{\frac{\lambda_k}{\alpha_k}, \frac{\mu_k}{\beta_k}\right\} \cup \{\lambda \in \mathbb{K} : \sup_{1\leq k\leq n} \|(A_k - \lambda B_k)^{-1}\| > \frac{1}{\varepsilon}\}.$$

Example 2.2. *Let* F *be an ultrametric free Banach space over* $\mathbb{K}$ *with an orthogonal basis* $(e_m)_{m\in\mathbb{N}}$. *Consider the linear operators* $(A_k)_{1\leq k\leq n}$ *and* $(B_k)_{1\leq k\leq n}$ *given by for all* $x \in F$ *and for each* $k \in \{1, \cdots, n\}$, $A_k x = \lambda_k x$ *and* $B_k x = \mu_k x$ *where* $\lambda_k \in \mathbb{K}$ *and* $\mu_k \in \mathbb{K}\backslash\{0\}$. *Put* $A = \oplus_{k=1}^n A_k$ *and* $B = \oplus_{k=1}^n B_k$. *One can see that*

$$\sigma(A, B) = \bigcup_{k=1}^n \{\frac{\lambda_k}{\mu_k}\}$$

and for all $k \in \{1, \cdots, n\}$ *and for each* $\lambda \in \rho(A_k, B_k)$, $\|(A_k - \lambda B_k)^{-1}\| = \frac{1}{|\lambda_k - \lambda\mu_k|}$, *then*

$$\sigma_\varepsilon(A_k, B_k) = \{\frac{\lambda_k}{\mu_k}\} \cup B(\frac{\lambda_k}{\mu_k}, \frac{\varepsilon}{|\mu_k|}).$$

Consequently,

$$\sigma_\varepsilon(A, B) = \bigcup_{k=1}^n \{\frac{\lambda_k}{\mu_k}\} \cup \bigcup_{k=1}^n B(\frac{\lambda_k}{\mu_k}, \frac{\varepsilon}{|\mu_k|}).$$

Definition 2.3. *Let* $(X_i)_{1\leq i\leq n}$ *be a sequence of ultrametric Banach spaces over* $\mathbb{K}$ *and let* $A_i, B_i \in \mathcal{L}(X_i)$ *and* $\varepsilon > 0$. *The condition pseudospectrum of the bounded linear operator pencil* $(\oplus_{i=1}^n A_i, \oplus_{i=1}^n B_i)$ *on* $\oplus_{i=1}^n X_i$ *is*

$$\Lambda_\varepsilon(S, B) = \sigma(S, B) \cup \{\lambda \in \mathbb{K} : \sup_{i\in\{1,\cdots,n\}} \|(A_i - \lambda B_i)\| \sup_{i\in\{1,\cdots,n\}} \|(A_i - \lambda B_i)^{-1}\| > \varepsilon^{-1}\},$$

where $S = \oplus_{i=1}^n A_i$ *and* $B = \oplus_{i=1}^n B_i$. *With the convention* $\sup_{i\in\{1,\cdots,n\}} \|(A_i - \lambda B_i)\| \sup_{i\in\{1,\cdots,n\}} \|(A_i - \lambda B_i)^{-1}\| = \infty$ *if* $\lambda \in \sigma(\oplus_{i=1}^n A_i, \oplus_{i=1}^n B_i)$.

Remark 2.2. *It is easy to see that* $\bigcup_{i=1}^n \Lambda_\varepsilon(A_i, B_i) \subset \Lambda_\varepsilon(\oplus_{i=1}^n A_i, \oplus_{i=1}^n B_i)$.

Proposition 2.4. *Let* $A_i, B_i \in \mathcal{L}(X_i)$ *and* $\varepsilon > 0$, *then*

(i) $\sigma(\oplus_{i=1}^n A_i, \oplus_{i=1}^n B_i) = \bigcap_{\varepsilon>0} \Lambda_\varepsilon(\oplus_{i=1}^n A_i, \oplus_{i=1}^n B_i)$.

(ii) *If* $0 < \varepsilon_1 < \varepsilon_2$, *then* $\sigma(\oplus_{i=1}^n A_i, \oplus_{i=1}^n B_i) \subset \Lambda_{\varepsilon_1}(\oplus_{i=1}^n A_i, \oplus_{i=1}^n B_i) \subset \Lambda_{\varepsilon_2}(\oplus_{i=1}^n A_i, \oplus_{i=1}^n B_i)$.

Proof. Put $S = \oplus_{i=1}^n A_i$ and $B = \oplus_{i=1}^n B_i$.

(i) From Definition 2.4, for each $\varepsilon > 0$, $\sigma(S, B) \subset \Lambda_\varepsilon(S, B)$. Conversely, if $\lambda \in \bigcap_{\varepsilon>0} \Lambda_\varepsilon(S, B)$ and $\lambda \notin \sigma(S, B)$. Using $\lim_{\varepsilon\to\to 0} \sup_{i\in\{1,\cdots,n\}} \|(A_i - \lambda B_i)\| \sup_{i\in\{1,\cdots,n\}} \|(A_i - \lambda B_i)^{-1}\| = \infty$, we get a contradiction.

(ii) If $0 < \varepsilon_1 < \varepsilon_2$ and $\lambda \in \Lambda_{\varepsilon_1}(S, B)$, thus $\sup_{i\in\{1,\cdots,n\}} \|(A_i-\lambda B_i)\| \sup_{i\in\{1,\cdots,n\}} \|(A_i-\lambda B_i)^{-1}\| > \varepsilon_1^{-1} > \varepsilon_2^{-1}$, then $\lambda \in \Lambda_{\varepsilon_2}(\oplus_{i=1}^n A_i, \oplus_{i=1}^n B_i)$.

□

We conclude the following:

Lemma 2.3. *Let $A_i, B_i \in \mathcal{L}(X_i)$ and $\varepsilon > 0$, hence*

$$\sup_{\lambda\in\sigma(\oplus_{i=1}^n A_i,\oplus_{i=1}^n B_i)} |\lambda| \leq \sup_{\lambda\in\Lambda_\varepsilon(\oplus_{i=1}^n A_i,\oplus_{i=1}^n B_i)} |\lambda|.$$

Proof. Since $\sigma(\oplus_{i=1}^n A_i, \oplus_{i=1}^n B_i) \subseteq \Lambda_\varepsilon(\oplus_{i=1}^n A_i, \oplus_{i=1}^n B_i)$, then

$$\sup_{\lambda\in\sigma(\oplus_{i=1}^n A_i,\oplus_{i=1}^n B_i)} |\lambda| \leq \sup_{\lambda\in\Lambda_\varepsilon(\oplus_{i=1}^n A_i,\oplus_{i=1}^n B_i)} |\lambda|.$$

□

Now, we introduce a new definition of condition pseudospectra of the direct sum of the operator pencil $(\oplus_{i=1}^n A_i, \oplus_{i=1}^n B_i)$ as follows.

Definition 2.4. *Let $(X_i)_{1\leq i\leq n}$ be a sequence of ultrametric Banach spaces over $\mathbb{K}$ and let $A_i, B_i \in \mathcal{L}(X_i)$ and $\varepsilon > 0$. The condition pseudospectrum of the bounded linear operator pencil $(\oplus_{i=1}^n A_i, \oplus_{i=1}^n B_i)$ on $\oplus_{i=1}^n X_i$ is*

$$\Lambda'_\varepsilon(S, B) = \sigma(S, B) \cup \{\lambda \in \mathbb{K} : \sup_{i\in\{1,\cdots,n\}} \|(A_i - \lambda B_i)\|\|(A_i - \lambda B_i)^{-1}\| > \varepsilon^{-1}\},$$

where $S = \oplus_{i=1}^n A_i$ and $B = \oplus_{i=1}^n B_i$.

Remark 2.3. (i) *It is easy to see that $\Lambda'_\varepsilon(\oplus_{i=1}^n A_i, \oplus_{i=1}^n B_i) = \bigcup_{i=1}^n \Lambda'_\varepsilon(A_i, B_i)$.*

(ii) $\sigma(\oplus_{i=1}^n A_i, \oplus_{i=1}^n B_i) = \bigcap_{\varepsilon>0} \Lambda'_\varepsilon(\oplus_{i=1}^n A_i, \oplus_{i=1}^n B_i)$.

(iii) *If $0 < \varepsilon_1 < \varepsilon_2$, then $\sigma(\oplus_{i=1}^n A_i, \oplus_{i=1}^n B_i) \subset \Lambda'_{\varepsilon_1}(\oplus_{i=1}^n A_i, \oplus_{i=1}^n B_i) \subset \Lambda'_{\varepsilon_2}(\oplus_{i=1}^n A_i, \oplus_{i=1}^n B_i)$.*

(iv) *For any $\varepsilon > 0$, hence $\sup_{\lambda\in\sigma(\oplus_{i=1}^n A_i,\oplus_{i=1}^n B_i)} |\lambda| \leq \sup_{\lambda\in\Lambda'_\varepsilon(\oplus_{i=1}^n A_i,\oplus_{i=1}^n B_i)} |\lambda|$.*

(v) *The condition pseudospectrum $\Lambda'_\varepsilon(\oplus_{i=1}^n A_i, \oplus_{i=1}^n B_i)$ of the pencil $(\oplus_{i=1}^n A_i, \oplus_{i=1}^n B_i)$ gives nice properties than $\Lambda_\varepsilon(\oplus_{i=1}^n A_i, \oplus_{i=1}^n B_i)$.*

We obtain the following examples.

Example 2.3. *Let* $(A_k)_{1\leq k\leq n}$ *and* $(B_k)_{1\leq k\leq n}$ *be two linear operators defined on* $\mathbb{K}^2$ *by*

$$A_k = \begin{pmatrix} \alpha_k & 0 \\ 0 & -\alpha_k \end{pmatrix}$$

and

$$B_k = \begin{pmatrix} 0 & \lambda_k \\ -\lambda_k & 0 \end{pmatrix},$$

where $\alpha_k \in \mathbb{K}$, $\lambda_k \in \mathbb{K}\backslash\{0\}$ *for any* $k \in \{1, \cdots, n\}$. *Set* $A = \oplus_{k=1}^n A_k$ *and* $B = \oplus_{k=1}^n B_k$, *then*

$$\sigma(\oplus_{k=1}^n A_k, \oplus_{k=1}^n B_k) = \bigcup_{k=1}^n \{\frac{-\alpha_k}{\lambda_k}, \frac{\alpha_k}{\lambda_k}\}$$

and

$$\Lambda_\varepsilon(A, B) = \bigcup_{k=1}^n \{\frac{-\alpha_k}{\lambda_k}, \frac{\alpha_k}{\lambda_k}\} \cup \{\lambda \in \mathbb{K} : \sup_{1\leq k\leq n} \|A_k - \lambda B_k\| \sup_{1\leq k\leq n} \|(A_k - \lambda B_k)^{-1}\| > \frac{1}{\varepsilon}\},$$

where for all $k \in \{1, \cdots, n\}$, $\|A_k - \lambda B_k\| = \max\{|\alpha_k|, |\lambda\lambda_k|\}$ *and for each* $\lambda \in \rho(A_k, B_k)$,

$$\|(A_k - \lambda B_k)^{-1}\| = \max\{\frac{|\alpha_k|}{|(\lambda\lambda_k)^2 - \alpha_k^2|}, \frac{|\lambda\lambda_k|}{|(\lambda\lambda_k)^2 - \alpha_k^2|}\}.$$

We begin with the following:

Definition 2.5. [3] *Let* $S, B \in \mathcal{M}_n(\mathbb{K}), \varepsilon > 0$, *the* ε*-determinant spectrum* $d_\varepsilon(S, B)$ *of the matrix pencil* (S, B) *is*

$$d_\varepsilon(S, B) = \{\lambda \in \mathbb{K} : |\det(S - \lambda B)| \leq \varepsilon\}.$$

We get the following:

Proposition 2.5. *Let* $S, B \in \mathcal{M}_n(\mathbb{K}), \varepsilon > 0$, *then*

$$d_{\varepsilon_1}(S) \cap d_{\varepsilon_2}(B) \subset d_\varepsilon(S \oplus B),$$

where $\varepsilon = \varepsilon_1\varepsilon_2$.

Proof. Let $\lambda \in d_{\varepsilon_1}(S) \cap d_{\varepsilon_2}(B)$, thus $\lambda \in d_{\varepsilon_1}(S)$ and $\lambda \in d_{\varepsilon_2}(B)$, hence $|\det(S - \lambda I)| \leq \varepsilon_1$ and $|\det(B - \lambda I)| \leq \varepsilon_2$. Consequently,

$$\begin{aligned} |\det(S \oplus B - \lambda(I \oplus I))| &= |\det(S - \lambda I)\det(B - \lambda I)| \\ &= |\det(S - \lambda I)||\det(B - \lambda I)| \\ &\leq \varepsilon_1\varepsilon_2, \end{aligned}$$

thus $\lambda \in d_\varepsilon(S \oplus B)$ where $\varepsilon = \varepsilon_1\varepsilon_2$, hence $d_{\varepsilon_1}(S) \cap d_{\varepsilon_2}(B) \subset d_\varepsilon(S \oplus B)$ where $\varepsilon = \varepsilon_1\varepsilon_2$. □

Proposition 2.6. *Let $S, B \in \mathcal{M}_n(\mathbb{K})$ and $\varepsilon > 0$, hence*

(i) *For all $0 < \varepsilon_1 \leq \varepsilon_2$, we have $d_{\varepsilon_1}(S \oplus B) \subseteq d_{\varepsilon_2}(S \oplus B)$,*

(ii) *For each $\mu \in \mathbb{K}$, $d_\varepsilon(\mu(I \oplus I), I \oplus I) = \{\lambda \in \mathbb{K} : |\lambda - \mu| \leq \varepsilon^{\frac{1}{2n}}\}$,*

(iii) *If $\det(B) \neq 0$ and for each $\mu \in \mathbb{K}$, then $d_\varepsilon(\mu(I \oplus B), I \oplus B) = \{\lambda \in \mathbb{K} : |\lambda - \mu| \leq (\frac{\varepsilon}{|\det(B)|})^{\frac{1}{2n}}\}$.*

Proof. (i) For all $0 < \varepsilon_1 \leq \varepsilon_2$, let $\lambda \in d_{\varepsilon_1}(S \oplus B)$, hence $|\det(S \oplus B - \lambda(I \oplus I))| \leq \varepsilon_1 \leq \varepsilon_2$, thus $\lambda \in d_{\varepsilon_2}(S \oplus B)$.

(ii) Let $\mu \in \mathbb{K}$ hence $|\det(\mu I - \lambda I)| = |\lambda - \mu|^n$, then

$$\begin{aligned} d_\varepsilon(\mu(I \oplus I), I \oplus I) &= \{\lambda \in \mathbb{K} : |\det(\mu(I \oplus I) - \lambda(I \oplus I))| \leq \varepsilon\} \\ &= \{\lambda \in \mathbb{K} : |\lambda - \mu|^{2n} \leq \varepsilon\} \\ &= \{\lambda \in \mathbb{K} : |\lambda - \mu| \leq \varepsilon^{\frac{1}{2n}}\}. \end{aligned}$$

(iii) We have

$$\begin{aligned} d_\varepsilon(\mu(I \oplus B), I \oplus B) &= \{\lambda \in \mathbb{K} : |\det(\mu(I \oplus B) - \lambda(I \oplus B))| \leq \varepsilon\} \\ &= \{\lambda \in \mathbb{K} : |\det((\mu - \lambda)I)||\det((\mu - \lambda)B)| \leq \varepsilon\} \\ &= \{\lambda \in \mathbb{K} : |\lambda - \mu|^{2n}|\det(B)| \leq \varepsilon\} \\ &= \{\lambda \in \mathbb{K} : |\lambda - \mu| \leq (\frac{\varepsilon}{|\det(B)|})^{\frac{1}{2n}}\}. \end{aligned}$$

□

Example 2.4. *If*

$$S = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \text{ and } B = \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix} \in \mathcal{M}_2(\mathbb{Q}_p),$$

where $a, b, c, d \in \mathbb{Q}_p$. Hence for any $\varepsilon > 0$,

$$\begin{aligned} d_\varepsilon(S \oplus B) &= \{\lambda \in \mathbb{Q}_p : |\det(S \oplus B - \lambda(I \oplus I))| \leq \varepsilon\} \\ &= \{\lambda \in \mathbb{Q}_p : |\det(S - \lambda I)||\det(B - \lambda I))| \leq \varepsilon\} \\ &= \{\lambda \in \mathbb{Q}_p : |(a - \lambda)(b - \lambda)(c - \lambda)(d - \lambda)| \leq \varepsilon\}. \end{aligned}$$

Example 2.5. *If*

$$S = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathcal{M}_2(\mathbb{Q}_p).$$

Hence for each $\varepsilon > 0$,

$$\begin{aligned} d_\varepsilon(S \oplus B) &= \{\lambda \in \mathbb{Q}_p : |\det(S \oplus B - \lambda(I \oplus I))| \le \varepsilon\} \\ &= \{\lambda \in \mathbb{Q}_p : |\det(S - \lambda I)||\det(B - \lambda I)| \le \varepsilon\} \\ &= \{\lambda \in \mathbb{Q}_p : |\lambda(1-\lambda)(2-\lambda)(1+\lambda)| \le \varepsilon\}. \end{aligned}$$

We obtain the following proposition.

Proposition 2.7. *Let* $D_1, D_2 \in \mathcal{L}(\mathbb{Q}_p^n)$ *be two diagonal operators with for each* $i \in \{1, \cdots, n\}$, $D_1 e_i = \lambda_i e_i$ *and* $D_2 e_i = \mu_i e_i$ *with* $\lambda_i, \mu_i \in \mathbb{Q}_p$, $\lambda_i \neq \lambda_{i+1}$ *and* $\mu_i \neq \mu_{i+1}$. *Hence*

$$d_\varepsilon(D_1 \oplus D_2) = \{\lambda \in \mathbb{Q}_p : |\lambda_1 - \lambda| \cdots |\lambda_n - \lambda||\lambda - \mu_1| \cdots |\lambda - \mu_n| \le \varepsilon\}.$$

Proof. We have

$$\text{for each } i \in \{1, \cdots, n\},\ (D_1 - \lambda)e_i = (\lambda_i - \lambda)e_i$$

and

$$\text{for each } i \in \{1, \cdots, n\},\ (D_2 - \lambda)e_i = (\mu_i - \lambda)e_i,$$

where $(e_k)_{1 \le k \le n}$ is a basis of $\mathbb{Q}_p^n$.

Thus $|\det(D_1 - \lambda I)| = |\lambda_1 - \lambda| \cdots |\lambda_n - \lambda|$ and $|\det(D_2 - \lambda I)| = |\mu_1 - \lambda| \cdots |\mu_n - \lambda|$. Hence for any $\varepsilon > 0$,

$$\begin{aligned} d_\varepsilon(D_1 \oplus D_2) &= \{\lambda \in \mathbb{Q}_p : |\det(D_1 \oplus D_2 - \lambda(I \oplus I))| \le \varepsilon\} \\ &= \{\lambda \in \mathbb{Q}_p : |\det(D_1 - \lambda I)\det(D_2 - \lambda I)| \le \varepsilon\} \\ &= \{\lambda \in \mathbb{Q}_p : |\lambda - \lambda_1| \cdots |\lambda - \lambda_n||\mu_1 - \lambda| \cdots |\mu_n - \lambda| \le \varepsilon\}. \end{aligned}$$

□

We have:

Definition 2.6. [3] *Let* $S, B \in \mathcal{M}_n(\mathbb{K}), \varepsilon > 0$, *the trace pseudospectrum* $Tr_\varepsilon(S, B)$ *of the matrix pencil* (S, B) *is*

$$Tr_\varepsilon(S, B) = \sigma(S, B) \cup \{\lambda \in \mathbb{K} : |tr(S - \lambda B)| \le \varepsilon\}.$$

As the classical setting, we have.

Proposition 2.8. *Let* $(A_i)_{1\leq i\leq n} \in \mathcal{M}_n(\mathbb{K})$*, then the trace of the diagonal block matrix* $\oplus_{i=1}^{n} A_i$ *is*

$$tr(\oplus_{i=1}^{n} A_i) = \sum_{i=1}^{n} tr(A_i).$$

We conclude the following:

Proposition 2.9. *Let* $(A_i)_{1\leq i\leq n} \in \mathcal{M}_n(\mathbb{K})$ *and* $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n > 0$*. Then*

$$\bigcap_{i=1}^{n} Tr_{\varepsilon_i}(A_i) \subset Tr_{\varepsilon}(\oplus_{i=1}^{n} A_i),$$

where $\varepsilon = \max_{1\leq i\leq n} \varepsilon_i$*.*

Proof. If $\lambda \in \bigcap_{i=1}^{n} Tr_{\varepsilon_i}(A_i)$, then for each $i \in \{1; \cdots, n\}$, $\lambda \in Tr_{\varepsilon_i}(A_i)$, hence for all $i \in \{1; \cdots, n\}$, $\lambda \in \sigma(A_i) \cup \{\lambda \in \mathbb{K} : |tr(A_i - \lambda I)| \leq \varepsilon_i\}$, since $\lambda \in \bigcap_{i=1}^{n} \sigma(A_i) \subset \bigcup_{i=1}^{n} \sigma(A_i)$ and from Proposition 2.8, we have

$$\begin{aligned} |tr(\oplus_{i=1}^{n} A_i - \lambda(\oplus_{i=1}^{n} I))| &= |\sum_{i=1}^{n} tr(A_i - \lambda I)| \\ &\leq \max_{1\leq i\leq n} |tr(A_i - \lambda I)| \\ &\leq \max_{1\leq i\leq n} \varepsilon_i, \end{aligned}$$

hence $\lambda \in Tr_{\varepsilon}(\oplus_{i=1}^{n} A_i)$, where $\varepsilon = \max_{1\leq i\leq n} \varepsilon_i$.

□

Theorem 2.1. *Let* $S, B \in \mathcal{M}_n(\mathbb{K})$ *and* $\varepsilon > 0$*. Hence*

(i) *If* $0 < \varepsilon_1 \leq \varepsilon_2$, $Tr_{\varepsilon_1}(S \oplus B) \subset Tr_{\varepsilon_2}(S \oplus B)$,

(ii) *For all* $\beta \in \mathbb{K}$ *and* A *is invertible and* $Tr(A) \neq 0$*, we get*

$$Tr_{\varepsilon}(\beta(A \oplus A), A \oplus A) = \Big\{\lambda \in \mathbb{K} : |\lambda - \beta| \leq \frac{\varepsilon}{|2tr(A)|}\Big\}.$$

Proof. (i) It follows from Definition 2.6.

(iii) If $\beta, \lambda \in \mathbb{K}$, hence

$$|tr(\beta(A \oplus A) - \lambda(A \oplus A))| = |\lambda - \beta||2tr(A)| \leq \varepsilon.$$

Thus

$$Tr_{\varepsilon}(\beta(A \oplus A), A \oplus A) = \Big\{\lambda \in \mathbb{K} : |\lambda - \beta| \leq \frac{\varepsilon}{|2tr(A)|}\Big\}.$$

□

We finish with the following example.

Example 2.6. *Let* $\varepsilon > 0$. *If* $A = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$ *and* $B = \begin{pmatrix} 2 & 5 \\ 0 & 1 \end{pmatrix} \in \mathcal{M}_2(\mathbb{Q}_2)$. *Then*

$$Tr_{\varepsilon}(A \oplus B) = (\{1,3\} \cup \{1,2\}) \cup \{\lambda \in \mathbb{Q}_p : |7 - 4\lambda| \leq \varepsilon\}.$$

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