International Journal of Maps in Mathematics

*Volume **8**, Issue **1**, 2025, Pages:**106-124***

E-ISSN: 2636-7467

www.simadp.com/journalmim

INVESTIGATION OF $*$-YAMABE CONFORMAL SOLITON ON LP-KENMOTSU MANIFOLDS

SHASHIKANT PANDEY *, PRIYANKA ALMIA , AND JAYA UPRETI

Abstract. The primary objective of this paper is to examine the $*$-Yamabe conformal soliton, which exhibits a potential vector field that is torse-forming on a Lorentzian Para(LP)-Kenmotsu manifold. Next, we examine the characteristics of the scalar curvature for $*$-Yamabe conformal soliton on the LP-Kenmotsu manifold. The development of the description of the vector field in the context of the $*$-Yamabe conformal soliton has been undertaken. Furthermore, we have improved multiple applications of vector fields, specifically the formation of torse on a LP-Kenmotsu manifold, by utilizing a $*$-Yamabe conformal soliton. At last, we also provide an example for $*$-Yamabe conformal soliton on three-dimensional LP-Kenmotsu manifold.

Keywords: $*$-Yamabe conformal soliton, LP-Kenmotsu manifold, conformal killing vector field, torse forming vector field.

2010 Mathematics Subject Classification: 53B30, 53C15, 53C25.

1. Introduction

In subsequent years, para-kenmotsu manifolds gaining significant values for scholarly interest to, prompting numerous authors to elucidate the manifold's obtaining intresting properties. The introduction of LP-Kenmotsu manifolds, also referred to as Lorentzian almost paracontact metric manifolds in [12].

The idea of Ricci flow, which is an evolution equation for metrics on a Riemannian manifold

Received:2023.11.11 *Revised:2024.02.02* *Accepted:2024.10.21*

* Corresponding author

Shashikant Pandey ⋄ shashi.royal.lko@gmail.com ⋄ https://orcid.org/0000-0002-8128-2884

Priyanka Almia ⋄ almiapriyanka14@gmail.com ⋄ https://orcid.org/0000-0002-1958-8022

Jaya Upreti ⋄ prof.upreti@gmail.com ⋄ https://orcid.org/0000-0001-8615-1819.

was presented in 1982 [10]. The equation governing the Ricci flow is as follows:

$$\frac{\partial g}{\partial t} = -2S, \tag{1.1}$$

on a compact Riemannian manifold M equipped with a Riemannian metric g, where S denote the Ricci tensor on M.

A generalization of an Einstein metric is referred to as a Ricci soliton. A Ricci soliton on the manifold M is defined as a triple (g, Ω, α), where the symbol g represents a Riemannian or semi-Riemannian metric, Ω denotes a vector field known as the potential vector, and real scalar ϑ is such that

$$\pounds_{\Omega} g + 2S + 2\alpha g = 0, \tag{1.2}$$

where $\pounds_{\Omega}$ represents the Lie derivative operator acting on the vector field Ω, S denotes the Ricci tensor, g represents the Riemannian metric, Ω is a vector field, and α is a scalar. The Ricci soliton exhibits three distinct behaviors, namely shrinking, steady, and expanding, which correspond to the values of α being negative, zero, and positive, respectively. Ricci solitons have been the subject of investigation by multiple researchers, as evidenced by the works of various authors [1, 8, 16, 17, 18, 19, 24].

The Yamabe flow was first proposed by Hamilton [11] as a method for producing Yamabe metrics on compact Riemannian manifolds. The Yamabe flow is a process by which a time-dependent metric $g(\cdot, t)$ on a Riemannian or pseudo-Riemannian manifold M evolves according to a specific equation,

$$\frac{\partial}{\partial t} g(t) = -r g(t), \quad g(0) = g_0, \tag{1.3}$$

where r is the scalar curvature of the manifold M.

In the context of two-dimensional spaces, it has been established that the Yamabe flow is mathematically equivalent to the Ricci flow, as stated in [10]. The Ricci flow, characterized by the equation $\frac{\partial}{\partial t} g(t) = -2S(g(t))$, involves the Ricci tensor denoted by S. Nevertheless, it is important to acknowledge that in dimension beyond two, the Ricci and Yamabe flow have distinct characteristics. This is due to the fact that, whereas the Yamabe flow preserves the conformal class of the metric, the Ricci flow does not necessarily possess this property.

On a Riemannian or pseudo-Riemannian manifold (M, g), the definition of a Yamabe soliton may be found. This soliton corresponds to a self-similar solution of the Yamabe flow [2]

$$\frac{1}{2} \pounds_{\Omega} g = (r - \alpha) g, \tag{1.4}$$

where the symbol $\pounds_\Omega g$ represents the Lie derivative of the metric g with respect to the vector field Ω. In this context, r refers to the scalar curvature, and α is a constant. Furthermore, it is stated that a Yamabe soliton exhibits expanding, steady behavior, or shrinking depending on the sign of α, which can be positive, zero, or negative, respectively. If α is a smooth function, the equation (1.4) is referred to as an almost Yamabe soliton according to the [2]. Numerous researchers have examined the solitons on contact manifolds following the emergence of Ricci soliton as well as the Yamabe soliton [6, 7, 9, 21].

The concept of conformal Ricci soliton [20, 21] was introduced by Basu and Bhattacharyya [3] in 2015 as

$$\pounds_\Omega g + 2S = \left[2\alpha - \left(q + \frac{2}{n}\right)\right] g. \tag{1.5}$$

In the above context, the symbol S is employed to designate the Ricci tensor, q represents a scalar non-dynamical field that exhibits temporal variation, α symbolizes a constant, and n signifies the dimension of the manifold.

The earliest proposals for the $*$-Ricci tensor on almost Hermitian manifolds and the $*$-Ricci tensor on real hypersurfaces in non-flat complex space were put forward by Tachibana [23] and Hamada [13], respectively. In their works, the $*$-Ricci tensor is precisely defined as follows:

$$S^*(X,Y) = \frac{1}{2}(Tr\{\varphi \, o \, R(X, \varphi Y)\}), \tag{1.6}$$

for any vector fields X and Y defined on a manifold M^n, let φ be a $(1,1)$-tensor field and the trace operator is denoted by Tr.

If the equation $S^*(X,Y) = \vartheta g(X,Y) + \varrho\eta(X)\eta(Y)$ holds for all vector fields X and Y, where ϑ and ϱ are smooth functions, then the manifold is referred to as a $*$-η-Einstein manifold. Moreover, if $\varrho = 0$, that is, if $S^*(X,Y) = \vartheta g(X,Y)$ for every vector fields X and Y, and thus the manifold can be characterized as $*$-Einstein.

In 2014, Kaimakamis and Panagiotidou [14] proposed the notion of a $*$-Ricci soliton, which can be precisely characterized as

$$\pounds_\Omega g + 2R^* + 2\alpha g = 0, \tag{1.7}$$

where X and Y are vector fields defined on the manifold M^n, and for any constant α.

By utilizing equations (1.4), (1.5), and (1.7) as references, we proceed to establish the concept of a $*$-Yamabe conformal soliton as follows:

Definition 1.1. *A manifold (M, g) of dimension n that is Riemannian or pseudo-Riemannian is considered to admit a ∗-Yamabe conformal soliton if*

$$(\pounds_\Omega g)(X, Y) + \left[2\alpha - 2r^* - \left(q + \frac{2}{n}\right)\right] g(X, Y) = 0, \tag{1.8}$$

for any vector fields X and Y, the Lie derivative of the metric g along the vector field Ω is denoted by $\pounds_\Omega g$. Here, $r^* = Tr(S^*)$ represents the ∗-scalar curvature, and α is a constant. The classification of ∗-Yamabe conformal solitons to be as either shrinking, steady, or shrinking depending on the sign of α, which can be positive, zero, or negative, respectively. If the vector field Ω can be expressed as the gradient of a smooth function h (i.e., $\Omega = grad(h)$) on the manifold M, then the equation (1.8) is referred to as a ∗-Yamabe conformal gradient soliton.

In contrast, in the context of a Riemannian or pseudo-Riemannian manifold (M, g), a vector field υ is said to be torse-forming if it does not have any points where it vanishes [27]. Then,

$$\nabla_X \upsilon = \chi X + \varsigma(X)\upsilon, \tag{1.9}$$

where ∇ denotes the Levi-Civita connection associated with the metric g, χ is a smooth function while ς is a 1-form. Furthermore, the vector field υ is referred to as concircular (see [4, 26]) if the 1-form ς disappears in the same way, in the equation (1.9). The vector field denoted by the symbol υ is known as the concurrent [22, 28] if, in equation (1.9), the 1-form ς vanishes identically and the function χ is equal to 1. The vector field υ is referred to as recurrent if the function χ in equation (1.9) is equal to zero. Finally, when $\chi = \varsigma = 0$ in (1.9), the vector field υ is often known as a parallel vector field.

In 2017, Chen [5] introduced an innovative vector field known as the torque-vector field. A vector field υ is referred to as a torqued vector field if it satisfies equation (1.9) with $\varsigma(\upsilon) = 0$. In the present scenario, the function χ is commonly referred to as the torqued function, while the 1-form ς is regarded as the torqued form of υ.

The framework of the article is as follows:

In the second section, following a concise introduction, we have presented several essential findings that will be utilized in subsequent sections. In Section 3, we have constructed a ∗-Yamabe conformal soliton that admits a LP-Kenmotsu manifold. The properties of the soliton, specifically the Laplacian of the smooth function, have also been established. The manifold has been characterized in cases where the vector field exhibits conformal killing properties. In the subsequent section, we have presented a demonstration of various properties pertaining to vector fields on $*$-Yamabe conformal soliton. Section 4 presents a series of results that establish the existence of torse producing vector fields and $*$-Yamabe conformal solitons on manifolds with a given metric. Section 5 of this study focuses on the practical implementation of the Laplace equation within the fields of gravity and physics. Section 6 presents the construction of an exemplary scenario to provide evidence for the presence of a $*$-Yamabe conformal soliton on a three-dimensional LP-Kenmotsu manifold. Let M be an n-dimensional Lorentzian metric manifold.

2. Preliminaries

In this section, we recall some fundamental notations and formulas of almost para contact metric manifolds.

Let M be an $(2n+1)$-dimensional Lorentzian metric manifold. This means that it is endowed with a structure (M, ϕ, ξ, η, g), where ϕ is a (1,1)-type tensor field, ξ is a Reeb vector field, η is a 1-form on M and g is a Lorentzian metric tensor satisfying [25]

$$\phi^2(X) = X + \eta(X)\xi, \quad \eta(\xi) = -1, \quad \eta\, o\, \phi = 0, \quad \phi\xi = 0, \tag{2.10}$$

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \tag{2.11}$$

$$g(X, \phi Y) = g(\phi X, Y), \tag{2.12}$$

$$g(X, \xi) = \eta(X), \tag{2.13}$$

for all vector fields X, Y . Then (M, ϕ, ξ, η, g) is said to be Lorentzian almost paracontact metric manifold.

Theorem 2.1. [12, 15] *A Lorentzian almost paracontact metric manifold (M, ϕ, ξ, η, g) is called Lorentzian para-Kenmotsu manifold if and only if*

$$(\nabla_X \phi)Y = -g(\phi X, Y)\xi - \eta(Y)\phi X, \tag{2.14}$$

for all vector fields $X, Y \in \Gamma(TM)$, where ∇ and $\Gamma(TM)$ denote the Levi-Civita connection and differentiable vector fields set on M respectively.

Corollary 2.1. *Let (M, ϕ, ξ, η, g) be a Lorentzian para-Kenmotsu manifold. Then, we have*

$$\nabla_X \xi = -X - \eta(X)\xi. \tag{2.15}$$

In a LP-Kenmotsu manifold, the following relations are hold,

$$\eta(R(X,Y)Z) = g(R(X,Y)Z,\xi) = g(Y,Z)\eta(X) - g(X,Z)\eta(Y), \tag{2.16}$$

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y, \tag{2.17}$$

$$R(X,\xi)Y = \eta(Y)X - g(X,Y)\xi, \tag{2.18}$$

where R denotes the Riemannian Curvature tensor.

$$S(X,\xi) = 2n\eta(X), \tag{2.19}$$

$$S(\phi X,\phi Y) = S(X,Y) + 2n\eta(X)\eta(Y), \tag{2.20}$$

$$(\nabla_X\eta)Y = -g(X,Y) - \eta(X)\eta(Y), \tag{2.21}$$

for any vector fields X, Y, $Z \in \Gamma(M)$.

It is now understood,

$$\pounds_\xi g(X,Y) = g(\nabla_X\xi, Y) + g(X, \nabla_Y\xi), \tag{2.22}$$

for any vector fields X, Y, $Z \in \Gamma(M)$.

Subsequently, by employing equations (2.15) and (2.22), we obtain

$$\pounds_\xi g(X,Y) = -2[g(X,Y) + \eta(X)\eta(Y)]. \tag{2.23}$$

Proposition 2.1. *The ∗-Ricci tensor on a (2n+1)-dimensional LP-Kenmotsu manifold is expressed as*

$$S^*(X,Y) = S(X,Y) + 2ng(X,Y). \tag{2.24}$$

Furthermore, by selecting $X = \sigma^i$ and $Y = \sigma^i$ in the aforementioned equation, where σ^i governs the elements of a local orthonormal frame and doing a summation across the range of i from 1 to $2n+1$, we may get the following result

$$r^* = r + 4n^2 + 2n, \tag{2.25}$$

where r^* represents the ∗-scalar curvature of the manifold M.

3. Main Results

Proposition 3.1. *If the metric g of a LP-Kenmotsu manifold with $(2n+1)$ dimension satisfies the $*$-Yamabe conformal soliton (g, ξ, α), where the reeb vector field ξ, subsequently the soliton can be put into distinct categories as either expanding, steady, or shrinking, the outcome is contingent upon the inequality* $r + 4n^2 + \frac{1}{2}\left(q + \frac{2}{2n+1}\right) \lesseqgtr 0$.

Proof. Let us consider a manifold M of the dimension $(2n+1)$ that is equipped with a LP- Kenmotsu structure. When we substitute $\Omega = \xi$ into the equation of the $*$ - Yamabe conformal soliton (2.15) on the manifold M, we obtain

$$(\pounds_{\xi} g)(X,Y) + \left[2\alpha - 2r^* - \left(q + \frac{2}{2n+1}\right)\right] g(X,Y) = 0, \tag{3.26}$$

for every vector fields X and Y belonging to the set of vector fields on M.

From (2.23) and (2.25), the above equation can be rewritten

$$\left[\alpha - r - 4n^2 - 2n - \frac{1}{2}\left(q + \frac{2}{2n+1}\right) - 1\right] g(X,Y) - \eta(X)\eta(Y) = 0. \tag{3.27}$$

Substituting $Y = \xi$ into the above equation and referring to equation (2.10), we obtain

$$\left[\alpha - r - 4n^2 - 2n - \frac{1}{2}\left(q + \frac{2}{2n+1}\right)\right] \eta(X) = 0. \tag{3.28}$$

Since $\eta(X) \neq 0$, the equation presented above can be expressed as

$$\alpha = r + 4n^2 + 2n + \frac{1}{2}\left(q + \frac{2}{2n+1}\right). \tag{3.29}$$

The proof is concluded. □

Corollary 3.1. *If the metric g of a flat para-Kenmotsu manifold with $(2n+1)$ dimension satisfies the $*$-Yamabe conformal soliton, then, the soliton exhibits expanding, steady, or shrinking behavior depending on the expression value* $4n^2 + 2n + \frac{1}{2}\left(q + \frac{2}{2n+1}\right) \lesseqgtr 0$, *for the reeb vector field ξ.*

Proof. Considering the fact that the manifold has a flat, denoted by $r = 0$, we can deduce from equation (3.29) that α is equal to $4n^2 + 2n + \frac{1}{2}\left(q + \frac{2}{2n+1}\right)$.

Therefore, the evidence presented substantiates the claim. □

Proposition 3.2. *If the metric g of a LP-Kenmotsu manifold with $(2n+1)$ dimension satisfies the $*$-Yamabe conformal soliton (g, Ω, α), where Ω is the gradient of a smooth function h, then the Laplacian equation satisfied by h is*

$$\triangle(h) = -(2n+1)\left[\alpha - r - 4n^2 - 2n - \frac{1}{2}\left(q + \frac{2}{2n+1}\right)\right].$$

Proof. In this investigation, we will take into account a ∗-Yamabe conformal soliton (g, Ω, α) that is defined on a $(2n+1)$-dimensional LP-Kenmotsu manifold M as

$$(\pounds_{\Omega} g)(X,Y) + \left[2\alpha - 2r^{*} - \left(q + \frac{2}{2n+1}\right)\right] g(X,Y) = 0 \tag{3.30}$$

for each vector fields X, $Y \in \Gamma(M)$.

By substituting $X = Y = \sigma^{i}$ into the aforementioned equation, where σ^{i} denotes a local orthonormal frame, and a summation from $i = 1$ to $2n+1$, with the aid of equation (2.25), we obtain

$$div\Omega + (2n+1)\left[\alpha - r - 4n^{2} - 2n - \frac{1}{2}\left(q + \frac{2}{2n+1}\right)\right] = 0. \tag{3.31}$$

Given that the vector field Ω can be expressed as the gradient of a smooth function h on the manifold M, we can rewrite equation (3.31) as follows:

$$\triangle(h) = -(2n+1)\left[\alpha - r - 4n^{2} - 2n - \frac{1}{2}\left(q + \frac{2}{2n+1}\right)\right]. \tag{3.32}$$

The symbol $\triangle(h)$ denotes the Laplacian equation, which is satisfied by the function h.
The proof is concluded. □

Based on the aforementioned theorem, it is possible to assert

Remark 3.1. *Consider a $(2n+1)$-dimensional LP-Kenmotsu manifold with metric g. Let this metric satisfy the ∗-Yamabe conformal soliton (g, Ω, α). The vector field Ω exhibits solenoidal behavior if and only if the scalar curvature becomes $\alpha - 4n^{2} - 2n - \frac{1}{2}(q + \frac{2}{2n+1})$.*

Proof. Given the vector field Ω is solenoidal, meaning that its divergence is zero, i.e., $div\Omega = 0$, equation (3.31) yields

$$r = \alpha - 4n^{2} - 2n - \frac{1}{2}(q + \frac{2}{2n+1}). \tag{3.33}$$

On the other hand, let us consider the scalar curvature of the manifold, denoted as r. It can be expressed as $r = \alpha - 4n^{2} - 2n - \frac{1}{2}\left(q + \frac{2}{2n+1}\right)$.
Subsequently, by referencing equation (3.31), it can be deduced that the divergence of vector field Ω is zero, thereby indicating that Ω possesses solenoidal characteristics. Therefore, the evidence provided confirms the claim. □

Definition 3.1. *If the following relation holds,*

$$(\pounds_{\Omega} g)(X,Y) = 2\Phi g(X,Y), \tag{3.34}$$

then the vector field ξ is called as a conformal Killing vector field. Where Φ represents a function of the coordinates, specifically a conformal scalar.
Furthermore, in the case where Φ is not always the same, the conformal killing vector field Ω is often denoted as proper. Furthermore, in the case where Φ remains constant, the vector field Ω is referred to as a homothetic vector field. Conversely, when the constant Φ takes on a non-zero value, Ω is characterized as a proper homothetic vector field. If Φ is equal to zero in the aforementioned equation, the symbol Ω is commonly used to denote a vector field known as a killing vector field.

Based on the aforementioned definition, it is possible to assert

Proposition 3.3. *If the metric g of a LP-Kenmotsu manifold with $(2n+1)$ dimension conforms to the $*$-Yamabe conformal soliton (g, Ω, α). Subsequently, Ω is*
(i) proper vector field if the expression $r + 4n^2 - \alpha + 2n + \frac{1}{2}\big(q + \frac{2}{2n+1}\big)$ is not constant,
(ii) homothetic vector field if the expression $r + 4n^2 - \alpha + 2n + \frac{1}{2}\big(q + \frac{2}{2n+1}\big)$ is constant,
(iii) proper homothetic vector field if the expression $r + 4n^2 - \alpha + 2n + \frac{1}{2}\big(q + \frac{2}{2n+1}\big)$ has a non-zero value,
(iv) killing vector field if the expression $\alpha = r + 4n^2 + 2n + \frac{1}{2}\big(q + \frac{2}{2n+1}\big)$
where Ω represents a conformal Killing vector field.

Proof. Consider a $(2n+1)$-dimensional LP-Kenmotsu manifold M equipped with a $*$-Yamabe conformal soliton (g, Ω, α), where Ω represents a conformal killing vector field. When utilizing equations (1.8), (2.25), and (3.34) and substituting $Y = \xi$, the following result is obtained

$$\left[\Phi + \alpha - r - 4n^2 - 2n - \frac{1}{2}\left(q + \frac{2}{2n+1}\right)\right]\eta(X) = 0. \tag{3.35}$$

Given that $\eta(X) \neq 0$, we can deduce that

$$\Phi = r + 4n^2 - \alpha + 2n + \frac{1}{2}\left(q + \frac{2}{2n+1}\right). \tag{3.36}$$

The proof is concluded. □

Proposition 3.4. *If the metric g of a LP-Kenmotsu manifold with $(2n+1)$ dimension conforms to the $*$-Yamabe conformal soliton (g, Ω, α), then we have the vector Ω and its metric dual 1-form $\emptyset$ fulfills the given relation,*

$$(divH)\Omega + S(Y, \Omega) = 0,$$

and

$$\nabla_X|\Omega|^2+2g(FX,\Omega)-(\pounds_\Omega g)(X,\Omega)=0,$$

where S stands for the Ricci tensor.

Proof. By utilizing the Lie derivative property, we are able to express

$$(\pounds_\Omega g)(X,Y)=g(\nabla_X\Omega,Y)+g(\nabla_Y\Omega,X), \tag{3.37}$$

for arbitrary vector fields X and Y.

Subsequently, by utilizing (2.25) and (3.37), (1.8) can be expressed as

$$g(\nabla_X\Omega,Y)+g(\nabla_Y\Omega,X)+\left[2\alpha-2(r+4n^2+2n)-\left(q+\frac{2}{2n+1}\right)\right]g(X,Y)=0. \tag{3.38}$$

The symbol $\o$ represents a 1-form, which possesses a metric equivalence to the vector field Ω. Consequently, it can be expressed as $\o(X)=g(X,\Omega)$, where X denotes any vector field. The mathematical representation of the exterior derivative $d\o$ may be stated in the following manner

$$2(d\o)(X,Y)=g(\nabla_X\Omega,Y)-g(\nabla_Y\Omega,X). \tag{3.39}$$

Given that $d\o$ is skew-symmetric, we can proceed to define a tensor field H of type $(1,1)$ by

$$(d\o)(X,Y)=g(X,HY), \tag{3.40}$$

then H is skew self-adjoint, it satisfies the property of $g(X,HY)=-g(HX,Y)$.

Equation (3.40) can be expressed as

$$(d\o)(X,Y)=-g(HX,Y). \tag{3.41}$$

By utilizing (3.41), (3.39) can be transformed

$$g(\nabla_X\Omega,Y)-g(\nabla_Y\Omega,X)=-2g(HX,Y). \tag{3.42}$$

By combining equations (3.42) and (3.38) in simultaneously performing a common factorization of the variable Y, we obtain

$$\nabla_X\Omega=-HX-\left[\alpha-(r+4n^2+2n)-\frac{1}{2}\left(q+\frac{2}{2n+1}\right)\right]X. \tag{3.43}$$

By substituting the aforementioned equation into the expression $R(X,Y)\Omega=\nabla_X\nabla_Y\Omega-\nabla_Y\nabla_X\Omega-\nabla_{[X,Y]}\Omega$, we obtain

$$R(X,Y)\Omega=(\nabla_Y H)X-(\nabla_X H)Y. \tag{3.44}$$

Upon observing that $d\o$ is closed, we can derive

$$g(X,(\nabla_Z H)Y)+g(Y,(\nabla_X H)Z)+g(Z,(\nabla_Y H)X)=0. \tag{3.45}$$

By taking the inner product of equation (3.44) with respect to Z, we obtain

$$g(R(X,Y)\Omega,Z)=g((\nabla_Y H)X,Z)-g((\nabla_X H)Y,Z). \tag{3.46}$$

Given that H is skew-adjoint, it follows that $\nabla_X H$ is also skew-adjoint. Subsequently, by employing (3.45), (3.46) can be expressed as

$$g(R(X,Y)\Omega,Z)=g(X,(\nabla_Z H)Y). \tag{3.47}$$

By substituting $X=Z=\sigma^i$ into the given equation, where σ^i governs the elements of a local orthonormal frame and doing a summation across the range of i from 1 to $2n+1$, we may get the following result

$$div(H)Y-S(Y,\Omega)=0, \tag{3.48}$$

where $divH$ refers to the a divergence of the tensor field denoted by H.

In this step, we will calculate the covariant derivative of the squared g-norm of Ω utilizing (3.43) in the following manner:

$$\nabla_X|\Omega|^2=2g(\nabla_X\Omega,\Omega)$$

$$=-2g(HX,\Omega)-\left[2\alpha-2(r+4n^2+2n)-\left(q+\frac{2}{2n+1}\right)\right]g(X,\Omega). \tag{3.49}$$

Once again, employing equations (1.8) and (2.25), we obtain

$$(\pounds_\Omega g)(X,Y)=-\left[2\alpha-2r-8n^2-4n-\left(q+\frac{2}{2n+1}\right)\right]g(X,Y). \tag{3.50}$$

Subsequently, by employing (3.50), (3.49) can be transformed

$$\nabla_X|\Omega|^2+2g(HX,\Omega)-(\pounds_\Omega g)(X,\Omega)=0. \tag{3.51}$$

And thus, the evidence is presented. □

4. Some Results on the LP-Kenmotsu Manifold for the ∗-Yamabe conformal soliton with Torse-forming vector field

By utilizing the equation (1.9) that describes the torse forming vector field, it is possible to assert

Proposition 4.1. *If the metric g of a LP-Kenmotsu manifold with $(2n+1)$ dimension satisfies the $*$-Yamabe conformal soliton (g, Ω, α), where a torse forming vector field υ, we can express $\alpha = r + 4n^2 - \chi + 2n + \frac{1}{2}\left(q + \frac{2}{2n+1}\right) - \frac{\varsigma(\upsilon)}{(2n+1)}$ and the soliton exhibits expanding, steady, shrinking according as $r + 4n^2 - \chi + 2n + \frac{1}{2}\left(q + \frac{2}{2n+1}\right) - \frac{\varsigma(\upsilon)}{(2n+1)} \lesseqqgtr 0$.*

Proof. Given that (g, υ, α) be a $*$- Yamabe conformal soliton defined on a $(2n+1)$-dimensional LP-Kenmotsu manifold M, where a torse-forming vector field υ, we can derive the following equations from (1.8) and (2.25):

$$(\pounds_{\upsilon} g)(X,Y) + \left[2\alpha - 2r - 8n^2 - 4n - \left(p + \frac{2}{2n+1}\right)\right] g(X,Y) = 0. \tag{4.52}$$

The notation $(\pounds_{\upsilon} g)(X,Y)$ represents the Lie derivative of the metric g with respect to the vector field υ.

Now, utilizing (2.15), we obtain

$$\begin{aligned}(\pounds_{\upsilon} g)(X,Y) &= g(\nabla_X \upsilon, Y) + g(X, \nabla_Y \upsilon) \\ &= 2\chi g(X,Y) + \varsigma(X) g(\upsilon, Y) + \varsigma(Y) g(\upsilon, X),\end{aligned} \tag{4.53}$$

for every X, $Y \in M$.

Subsequently, by referring to equations (4.52) and (4.53), we obtain.

$$\left[r + 4n^2 - \alpha - \chi + 2n + \frac{1}{2}\left(q + \frac{2}{2n+1}\right)\right] g(X,Y) = \frac{1}{2}\Big[\varsigma(X) g(\upsilon, Y) + \varsigma(Y) g(X, \upsilon)\Big]. \tag{4.54}$$

By contracting the above equation throughout X and Y, we obtain

$$\alpha = r + 4n^2 - \chi + 2n + \frac{1}{2}\left(q + \frac{2}{2n+1}\right) - \frac{\varsigma(\upsilon)}{(2n+1)}. \tag{4.55}$$

The proof is now concluded. □

Based on the aforementioned theorem, it is possible to assert that,

Corollary 4.1. *If the metric g of a LP-Kenmotsu manifold with $(2n+1)$ dimension satisfies the $*$-Yamabe conformal soliton (g, Ω, α), where a torse forming vector field υ, Consequently, if υ is*

(i) concircular, the equation becomes $\alpha = r + 4n^2 - \chi + 2n + \frac{1}{2}\left(q + \frac{2}{2n+1}\right)$. The soliton is

either expanding, steady, shrinking depending on the value of the expression $r + 4n^2 - \chi + 2n + \frac{1}{2}\big(q + \frac{2}{2n+1}\big) \lesseqqgtr 0$.

(ii) concurrent, the equation becomes $\alpha = r + 4n^2 - 1 + 2n + \frac{1}{2}\big(q + \frac{2}{2n+1}\big)$. *The soliton is either expanding, steady, shrinking depending on the value of the expression* $r + 4n^2 - 1 + 2n + \frac{1}{2}\big(q + \frac{2}{2n+1}\big) \lesseqqgtr 0$.

(iii) recurrent, the equation may be written as $\alpha = r + 4n^2 + 2n + \frac{1}{2}\big(q + \frac{2}{2n+1}\big) - \frac{\varsigma(\upsilon)}{(2n+1)}$. *The soliton is either expanding, steady, shrinking depending on the value of the expression* $r + 4n^2 + 2n + \frac{1}{2}\big(q + \frac{2}{2n+1}\big) - \frac{\varsigma(\upsilon)}{(2n+1)} \lesseqqgtr 0$.

(iv) parallel, the equation becomes $\alpha = r + 4n^2 + 2n + \frac{1}{2}\big(q + \frac{2}{2n+1}\big)$. *The soliton is either expanding, steady, shrinking depending on the value of the expression* $r + 4n^2 + 2n + \frac{1}{2}\big(q + \frac{2}{2n+1}\big) \lesseqqgtr 0$.

(v) torqued, the equation becomes $\alpha = r + 4n^2 - \chi + 2n + \frac{1}{2}\big(q + \frac{2}{2n+1}\big)$. *The soliton is either expanding, steady, shrinking depending on the value of the expression* $r + 4n^2 - \chi + 2n + \frac{1}{2}\big(q + \frac{2}{2n+1}\big) \lesseqqgtr 0$.

Proof. In (4.55), if the 1-form ς disappears the same way, meaning that υ becomes concircular, then α can be expressed as $\alpha = r + 4n^2 - \chi + 2n + \frac{1}{2}\big(q + \frac{2}{2n+1}\big)$.

In the given equation (4.55), if the 1-form ς disappears the same way and the function χ is equal to 1, then υ becomes concurrent. Consequently, the expression for α can be simplified to $r + 4n^2 - 1 + 2n + \frac{1}{2}\big(q + \frac{2}{2n+1}\big)$.

In equation (4.55), when the function χ is equal to zero, meaning that υ becomes recurrent, the expression for α is given by $r + 4n^2 + 2n + \frac{1}{2}\big(q + \frac{2}{2n+1}\big) - \frac{\varsigma(\upsilon)}{(2n+1)}$.

When $\chi = \varsigma = 0$ in equation (4.55), which means that υ becomes parallel, the expression for α simplifies to $r + 4n^2 + 2n + \frac{1}{2}\big(q + \frac{2}{2n+1}\big)$.

Lastly, in equation (4.55), when $\varsigma(\upsilon) = 0$, In this context, the symbol υ can be interpreted as a torqued vector field. Consequently, the expression $\alpha = r + 4n^2 - \chi + 2n + \frac{1}{2}\big(q + \frac{2}{2n+1}\big)$ can be derived.

Thus, the evidence is provided. □

5. Applications of Laplace Equation in Physics and Gravity

The Laplace equation is a second-order partial differential equation that finds extensive application in physics due to its solution, known as harmonic functions. These functions arise in various contexts, including the determination of electrical, magnetic, and gravitational potentials, steady-state temperatures, and hydrodynamics problems.

- Both the real and imaginary components of a complex analytic function satisfy the Laplace equation. That is if $z = x + iy$ and $f(x, y) = u(x, y) + iv(x, y)$, the essential requirement for the function $f(z)$ to possess analyticity is that the real part u and the imaginary part v satisfy the C-R equation, $u_x = v_y$, $u_y = -v_x$, the symbols u_x and u_y represent the first partial derivatives of the function u with respect to the variables x and y, respectively and v_x and v_y represent the first partial derivatives of the function v with respect to the variables x and y, respectively. Consequently, $u_{yy} = (-v_x)_y = -(v_y)_x = -(u_{xx})$. Hence, the function u fulfills the Laplace equation.
- In the given situation, whereby a specific area demonstrates a charge density of zero, while allowing for non-zero charge densities at its limits, the electric potential V inside that zone conforms to the Laplace equation. By solving the Laplace equation, we can determine the electric potential, a crucial quantity that allows us to easily calculate the electric field using the equation $E = \nabla V$. Consequently, we can determine the force experienced by a charge using the equation $F = qE$. In the field of physics, numerous intriguing scenarios arise wherein our focus lies on the potential within regions characterized by a zero charge density. Conventional examples include both the inside and outside regions of a charged hollow sphere, as well as the exterior portion of charged metal plates. The fact that each of the situations has a unique combination of boundary conditions is one of the things that makes the Laplace equation so intriguing.

In a broad context, the gravitational and electric potentials, represented by the symbol V, adhere to Poisson's equation, which is expressed as $\nabla^2 V = L(x, y, z)$, where $L(x, y, z)$ represents the given charged density. The equation of Laplace and the equation of Poisson are two of the simplest examples of a type of partial differential equations known as elliptical PDEs.Laplace was the pioneer in introducing a multitude of intriguing mathematical techniques that have been employed to solve electrical partial differential equations (PDEs).

- In the field of electrostatics, as per Maxwell's equation, a two-dimensional electric fluid (u_1, u_2), which is not dependent on time, fulfills the following conditions:

$$\nabla \times (u_1, u_2, 0) = ((u_2)_x - (u_1)_y)\hat{k_1}$$

where $\hat{k_1}$ is the standard unit vector and

$$\nabla \cdot (u_1, u_2) = Q.$$

The symbol Q is used to denote the charge density.

The equation of Laplace is applicable to three-dimensional scenarios in the fields of electrostatics and fluid dynamics, similar to its utilization in two-dimensional contexts.

- Furthermore, it is worth noting that this phenomenon also finds applications in the field of gravity. Let $\tilde{g}$, $\tilde{\rho}$, and G denote the gravitational field, mass density, and gravitational constant, respectively. The differential expression representing Gauss's law for gravity may be stated as follows:

$$\nabla \cdot \tilde{g} = -4\pi G\tilde{\rho}.$$

Furthermore, the equation $\nabla^2 V = 4\pi G\tilde{\rho}$ represents Poisson's gravitational field equations. The physical significance of this scenario can be understood by considering Proposition 3.2 and the equation (3.32). This equation represents a Laplace equation with a potential vector field of gradient type.

The Laplace equation for gravitational fields, $\nabla^2 V = 0$, is true in empty space with $\tilde{\rho} = 0$.

6. Example

We examine the 3-dimensional manifold $M = \{(l_1, l_2, l_3) \in \mathbb{R}^3, (l_1, l_2, l_3) \neq (0, 0, 0)\}$, in the context of $\mathbb{R}^3$, the coordinates (l_1, l_2, l_3) are referred to as standard coordinates. The vector fields referred to in the context are

$$\sigma^1 = l_3 \frac{\partial}{\partial l_1}, \qquad \sigma^2 = l_3 \frac{\partial}{\partial l_2}, \qquad \sigma^3 = l_3 \frac{\partial}{\partial l_3}$$

are linearly independent at every point of M.

Let g denote the Riemannian metric that is defined by

$$g(\sigma^1, \sigma^2) = g(\sigma^2, \sigma^3) = g(\sigma^3, \sigma^1) = 0,$$

$$g(\sigma^1, \sigma^1) = g(\sigma^2, \sigma^2) = 1, \; g(\sigma^3, \sigma^3) = -1.$$

Let 1-form η that is defined by $\eta(Z) = g(Z, \sigma^3)$, for any $Z \in \Gamma(M)$, Let $\Gamma(M)$ denote the collection of all differentiable vector fields on the manifold M and ϕ denote the $(1,1)$-tensor field that is defined by

$$\phi\sigma^1 = -\sigma^1, \;\; \phi\sigma^2 = -\sigma^2, \;\; \phi\sigma^3 = 0.$$

By utilizing the property of linearity for both ϕ and g, we can deduce that

$$\eta(\sigma^3) = -1, \;\; \phi^2 X = X + \eta(X)\sigma^3, \;\; g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y),$$

for every X, $Y \in \chi(M)$. Therefore, when $\sigma^3 = \xi$, the tuple (ϕ, ξ, η, g) establishes a Lorentzian almost paracontact metric structure on the manifold M.

Consider the Levi-Civita connection denoted by ∇, which is associated with the Riemannian metric g. Subsequently, we possess

$$[\sigma^1, \sigma^2] = 0, \quad [\sigma^1, \sigma^3] = -\sigma^1, \quad [\sigma^2, \sigma^3] = \sigma^2.$$

The connection ∇ associated with the metric g is defined in the following manner

$$2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) - g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]),$$

which is referred to as Koszul's formula.

Using Koszul's formula, it is straightforward to compute

$$\nabla_{\sigma^1}\sigma^1 = \sigma^3, \quad \nabla_{\sigma^1}\sigma^2 = 0, \quad \nabla_{\sigma^1}\sigma^3 = -\sigma^1,$$
$$\nabla_{\sigma^2}\sigma^1 = 0, \quad \nabla_{\sigma^2}\sigma^2 = \sigma^3, \quad \nabla_{\sigma^2}\sigma^3 = -\sigma^2,$$
$$\nabla_{\sigma^3}\sigma^1 = 0, \quad \nabla_{\sigma^3}\sigma^2 = 0, \quad \nabla_{\sigma^3}\sigma^3 = 0.$$

Based on the aforementioned, it can be deduced that the manifold fulfills the equation $\nabla_X \xi = -X - \eta(X)\xi$, where $\xi = \sigma^3$. Therefore, the manifold under consideration can be classified as a LP-Kenmotsu manifold.

In addition to this, the Riemannian curvature tensor, denoted by R, can be represented as

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

Hence,

$$R(\sigma^1, \sigma^2)\sigma^2 = -\sigma^1, \quad R(\sigma^1, \sigma^3)\sigma^3 = -\sigma^1, \quad R(\sigma^2, \sigma^1)\sigma^1 = -\sigma^2,$$
$$R(\sigma^2, \sigma^3)\sigma^2 = -\sigma^2, \quad R(\sigma^3, \sigma^1)\sigma^1 = -\sigma^3, \quad R(\sigma^3, \sigma^2)\sigma^2 = -\sigma^3,$$
$$R(\sigma^1, \sigma^2)\sigma^3 = 0, \quad R(\sigma^2, \sigma^3)\sigma^1 = 0, \quad R(\sigma^3, \sigma^1)\sigma^2 = 0.$$

Next, the Ricci tensor S is expressed as

$$S(\sigma^1, \sigma^1) = -2, \quad S(\sigma^2, \sigma^2) = -2, \quad S(\sigma^3, \sigma^3) = -2. \tag{6.56}$$

Furthermore, the scalar curvature transforms

$$r = \sum_{i=1}^{3} S(\sigma_i, \sigma_i) = -2. \tag{6.57}$$

By utilizing equations (2.24) and (6.56),

$$S^*(\sigma^1,\sigma^1)=0,\quad S^*(\sigma^2,\sigma^2)=0,\quad S^*(\sigma^3,\sigma^3)=-4. \tag{6.58}$$

Hence,

$$r^*=Tr(S^*)=4. \tag{6.59}$$

Let us consider the potential vector field as $\Omega=2l_1\frac{\partial}{\partial l_1}+2l_2\frac{\partial}{\partial l_2}+l_3\frac{\partial}{\partial l_3}$.
Then $(\pounds_\Omega g)(\sigma^1,\sigma^1)=-2g(\pounds_\Omega\sigma^1,\sigma^1)=2$.
similarly, $(\pounds_\Omega g)(\sigma^1,\sigma^1)=2,\quad (\pounds_\Omega g)(\sigma^3,\sigma^3)=0$.
Therefore, we have

$$\sum_{i=1}^{3}(\pounds_\Omega g)(\sigma^i,\sigma^i)=4. \tag{6.60}$$

Now putting $X=Y=\sigma^i$ in the (1.8), by performing a summation across the range of i from 1 to 3 and using equations (6.59) and (6.60), the resulting expression is derived.

$$\alpha=\frac{3q+14}{6} \tag{6.61}$$

The aforementioned α, as defined, fulfills equation (3.31), therefore it can be deduced that g sets up an $*$-Yamabe conformal soliton on the 3-dimensional LP-Kenmotsu manifold M.

7. Conclusion

In this study, we examine intriguing findings regarding the Lorentzian Para-Kenmotsu metric, specifically its characterization as a $*$-conformal soliton with a torse forming vector field. Additionally, we provide an illustrative example of such a manifold. In this study, we also present derived outcomes concerning $*$-Yamabe conformal solitons featuring a torse-forming vector field on the respective manifold. In addition, we explore various applications of the Laplace equation in the fields of Physics and Gravity.

Acknowledgments

We would like to extend our appreciation to the referees for generously dedicating their time and providing us with their insightful comments. The second author expresses gratitude for the research fellowship (SRF) provided by the Department of Science and Technology (DST), New Delhi (No. DST/INSPIRE Fellowship/2019/IF190040).

References

[1] Ali, M., & Ahsan, Z. (2012). Ricci solitons and the symmetries of the spacetime manifold of general relativity. Global J. Advanced Research on classical and Modern Geometries, 1, 75-84.

[2] Barbosa, E., & Ribeiro, E. Jr. (2013). On conformal solutions of the Yamabe flow Arch. Math. 101, 79-89.

[3] Basu, N., & Bhattacharyya, A. (2015). Conformal Ricci soliton in Kenmotsu manifold. Global Journal of Advanced Research on classical and Modern Geometries, 4, 15-21.

[4] Chen, B. Y. (2014). A simple characterization of generalized Robertson-Walker space-times. Gen. Relativity Gravitation, 46, 1-5.

[5] Chen, B. Y. (2017). Classification of torqued vector fields and its applications to Ricci solitons. Kragujevac J. Math, 41, 239-250.

[6] Cao, H. D., Sun, X., & Zhang Y. (2012). On the structure of gradient Yamabe solitons. Math. Res. Lett., 19, 767-774.

[7] Cho, J. T., & Kimura, M. (2009). Ricci solitons and real hypersurfaces in a complex space form. Tohoku Math. J., 61, 205-212.

[8] De, U. C. (2010). Ricci soliton and gradient Ricci soliton on P-Sasakian manifolds. The Aligarh Bull. of Maths., 29, 29-34.

[9] Ghosh, A. (2020). Yamabe soliton and Quasi Yamabe soliton on Kenmotsu manifold. Mathematica Slovaca, 70, 151-160.

[10] Hamilton, R. S. (1982). Three manifold with positive Ricci curvature. J. Diff. Geom., 17, 255-306.

[11] Hamilton, R. S. (1988). The Ricci flow on surfaces. Contemporary Mathematics, 71, 237-261.

[12] Haseeb, H., & Prasad, R. (2021). Certain results on Lorentzian para Kenmotsu manifolds. Bol. Soc. Paran. Mat., 39(3), 201-220.

[13] Hamada, T. (2002). Real hypersurfaces of complex space form in terms of Ricci $*$-tensor. Tokyo J. Math., 25, 473-483.

[14] Kaimakamis, G., & Panagiotidou, K. (2014). $*$-Ricci solitons of real hypersurface in non-flat complex space forms. Journal of Geometry and Physics, 76, 408-413.

[15] Kenmotsu, K. (1972). A class of almost contact Riemannian manifolds.Tohoku Math. J., 24, 93-103.

[16] Pandey, S., Singh, A., & Bahadir, O. (2022). Some Geometric properties of η-Ricci solitons on three-dimensional quasi-para-Sasakian manifolds. Balkan Journal of Geometry and its Applications, 27, 89-102.

[17] Pandey, S., Singh, A., & Bahadir, O. (2022). Certain results of Ricci soliton on 3-dimensional Lorentzian para Sasakian manifolds. International Journal of Maps in Mathematics, 5(2), 139-153.

[18] Pandey, S., Singh, A., & Prasad, R. (2022). Some Geometric Properties of η_* Ricci Solitons on α-Lorentzian Sasakian Manifolds. Kyungpook Math. J., 62, 737-749.

[19] Pandey, S., Singh, A., & Prasad, R. (2022). Eta Star-Ricci solitons on Sasakian manifolds. Differential geometry-dynamical systems, 24, 164-176.

[20] Roy, S., Dey, S., & Bhattacharyya, A. (2020). Conformal Ricci solitons on 3-dimensional trans-Sasakain manifold. Jordan Journal of Mathematics and statistics, 13, 89-109.

[21] Roy, S., Dey, S., & Bhattacharyya, A. (2022). ∗-conformal η-Ricci soliton on Sasakian manifold. Asian-European Journal of Mathematics, 15, 2250035.

[22] Schouten, J. A. (1954). Ricci Calculus. Berlin.

[23] Tachibana, S. (1959). On almost-analytic vectors in almost Kählerian manifolds. Tohoku Math. J., 11, 247-265.

[24] Tarun, M., De, U. C., & Yildiz, A. (2012). Ricci solitons and gradient Ricci solitons in three-dimensional trans-Sasakian manifolds. Filomat, 26, 363-370.

[25] Tripathi, M. M., & De, U. C. (2001). Lorentzian almost paracontact manifolds and their submanifolds. The Pure and Applied Mathematics, 8.2, 101-125.

[26] Yano, K. (1940). Concircular geometry I. Concircular transformations. Proc. Imp. Acad. Tokyo., 16, 195-200.

[27] Yano, K. (1944). On the torse-forming directions in Riemannian spaces. Proc. Imp. Acad. Tokyo, 20, 340-345.

[28] Yano, K. & Chen, B. Y. (1971). On the concurrent vector fields of immersed manifolds. Kodai Math. Sem. Rep., 23, 343-350.

(S. Pandey) Department of Mathematics and Astronomy, University of Lucknow, Lucknow 226007, Uttar Pradesh, India

(P. Almia) Department of Mathematics,Soban Singh Jeena Campus, Kumaun University, Nainital, Uttarakhand, 263001, India

(P. Almia) Department of Mathematics, Graphic Era Hill University, Dehradun, Uttarakhand, 248002, India

(J. Upreti) Department of Mathematics, Soban Singh Jeena Campus, SSJ University, Almora, Uttarakhand, 263601, India