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CSI-$\xi^{\perp}$- RIEMANNIAN SUBMERSIONS FROM LORENTZIAN PARA-KENMOTSU MANIFOLDS

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Abstract. The purpose of this article is to examine the characteristics of Clairaut semi-invariant-$\xi^{\perp}$(CSI-$\xi^{\perp}$,in brief) Riemannian submersions from Lorentzian para-Kenmotsu manifolds onto Riemannian manifolds and also enrich this geometrical analysis with specific condition for a semi-invariant $\xi^{\perp}$-Riemannian submersion to be CSI-$\xi^{\perp}$-Riemannian submersion. Furthermore, we discuss some results about these submersions and present a consequent non-trivial example based on this study.

Keywords: Riemannian submersions, Clairaut semi-invariant submersion, Lorentzian para Kenmotsu manifolds

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1. Introduction

Let N_1 be a semi-Riemannian manifold endowed with a semi-Riemannian metric g_{N_1}. A Lorentzian manifold is a subclass of semi-Riemannian manifold. Since the Lorentzian manifold has many applications in science and technology, especially in the theory of relativity and cosmology, therefore it attracts many researchers to do the research in this area. The different classes of Lorentzian manifolds have been studied in ([15], [16], [17], [25], [26]) and by many others.

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The concept of Riemannian submersions is studied extensively together with starting the study of Riemannian geometry. In fact, the theory of Riemannian submersion was initiated by O' Neill [27] in 1966, and it has been further studied by Gray [13], in 1967. Watson [40] popularized the knowledge of Riemannian submersions considering almost Hermitian manifolds in terms of almost Hermitian submersions. The Riemannian submersions play a vital role not only in the differential geometry but also in science and technology. It is noticed that the theory of Riemannian submersions are capable to handle many issues of the singularity theory, Yang-Mills theory, quantum theory, Kaluza-Klein theory, relativity, superstring theories, mechanics, modelling, robotics etc. (see, [4], [7], [6], [10], [11], [18], [19], [20]). For more details, we cite the books ([12], [35]) and the references therein. The Riemannian submersions motivate the researchers to define the semi-Riemannian submersions and Lorentzian submersion [12], almost Hermitian submersions [40], almost contact submersions [26], anti-invariant Riemannian submersions [34], semi-slant submersion [28], conformal anti-invariant submersions ([21], [29]), conformal semi-invariant submersion [22], conformal semi-slant submersions ([22], [30]), para-contact submersions [14], quasi bi-slant submersion ([31], [32], [33]).

In 1972, Bishop [8] presented the hypothesis and conditions of a Clairaut submersion in terms of a natural generalization of a surface of a revolution. Let c is any geodesic defined as $c : I_1 \subset R \to M$, $\phi(s)$ is the angle between $c(s)$ and the meridian curve through $c(s)$, $s \in I_1$. Under these conditions, the product $rsin\phi$ is constant on the revolution surface M along geodesic c. Hence, it is apart from s. Afterwards, this idea has been considered in Lorentzian spaces, timelike and spacelike spaces ([24] [37], [38]).

In 1981, Allison [3] proposed Clairaut submersions in case the total manifold is Lorentzian. In addition, it is discovered that Clairaut submersions are used for static spacetime applications. Furthermore, Clairaut submersions have been generalized in [5]. The concept of anti-invariant Riemannian submersions was initiated by Lee [23] in 2013. On the other hand, Sahin [36] introduced Clairaut Riemannian map and studied it's geometric properties in 2017.

In 2017, Akyol, Sari and Aksoy [1] introduced the notion of semi-invariant $\xi^{\perp}-$Riemannian Submersions as well as semi-slant $\xi^{\perp}-$Riemannian Submersions [2], as a generalization of anti invariant $\xi^{\perp}-$Riemannian Submersions and discussed the geometry of the total space and the base space for the existence of such submersions.

The above studies inspire us to introduce the notion of CSI-$\xi^{\perp}$-Riemannian submersions from the Lorentzian para-Kenmotsu (a subclass of semi-Riemannian) manifolds to the Riemannian manifolds and characterize its geometrical properties. Throughout the paper, we denote the Lorentzian para-Kenmotsu manifold of dimension n by (N_1, g_{N_1}). It is noticed that Akyol et al. [1] has been studied the properties of semi-invariant $\xi^{\perp}-$Riemannian Submersions from a class of Riemannian manifold (almost contact manifold) to a Riemannian manifold but in this paper, we are going to characterize the properties of CSI-$\xi^{\perp}$- Riemannian submersions from a class of semi-Riemannian manifold to a Riemannian manifold, which is an extension of [1].

We exhibit our work as follows: Section 2 contains some basic results of Lorentzian para-Kenmotsu manifold, a non-trivial example of Lorentzian para-Kenmotsu manifold. In Section 3, we give the basic definitions related to semi-invariant $\xi^{\perp}-$ Riemannian Submersions and well-known Lemma. In section 4, we define CSI-$\xi^{\perp}$- Riemannian submersions from the Lorentzian para-Kenmotsu manifolds and discuss some geometrical properties of such submersions. The last section is concerned with a non-trivial example of Lorentzian para-Kenmotsu manifold with CSI-$\xi^{\perp}$- Riemannian submersion.

2. Lorentzian para-Kenmotsu manifolds

Let N_1 be an $n-$dimensional Lorentzian metric manifold if it is endowed with a structure $(\phi, \xi, \eta, g_{N_1})$, where ϕ is a $(1,1)$ tensor field, ξ is a vector field, η is a $1-$form on N_1 and g_{N_1} is a Lorentzian metric satisfying:

$$\phi^2 = I + \eta \otimes \xi, \phi \circ \xi = 0, \eta \circ \phi = 0, \tag{2.1}$$

$$g_{N_1}(\phi W_1, \phi W_2) = g_{N_1}(W_1, W_2) + \eta(W_1)\eta(W_2), g_{N_1}(\phi W_1, W_2) = g_{N_1}(W_1, \phi W_2), \tag{2.2}$$

$$\eta(\xi) = -1, g_{N_1}(W_2, \xi) = \eta(W_2), \tag{2.3}$$

for any vector field W_1, W_2 on N_1, then it is called Lorentzian almost para-contact manifold. In the Lorentzian almost para-contact manifold following relations hold:

$$\Phi(W_1, W_2) = \Phi(W_2, W_1) = g_{N_1}(W_1, \phi W_2), \tag{2.4}$$

where Φ is symmetric $(0,2)$ tensor field and vector fields W_1 and W_2 on N_1.

If ξ is a killing vector field, the para-contact structure is called $K-$para contact.

A Lorentzian almost para-contact manifold N_1 is called Lorentzian para-Kenmotsu manifold [9] if

$$(\nabla_{W_1}\phi)W_2 = -g_{N_1}(\phi W_1, W_2)\xi - \eta(W_2)\phi W_1, \tag{2.5}$$

for any vector field W_1, W_2 on N_1.

In the Lorentzian para-Kenmotsu manifold, we have

$$\nabla_{W_2}\xi = -W_2 - \eta(W_2)\xi, \tag{2.6}$$

$$(\nabla_{W_1}\eta)W_2 = -g_{N_1}(W_1, W_2) - \eta(W_1)\eta(W_2), \tag{2.7}$$

where ∇ denotes the operation of covariant differentiation (Levi-Civita connection) with respect to the Lorentzian metric g_{N_1}.

In a Lorentzian para Kenmotsu manifold, it is clear that

$$rank\phi = n - 1.$$

Example 2.1. [39] *We consider $(2n+1)$ dimensional manifold $R^{2n+1} = \{(x^1, x^2, ..., x^n, y^1, y^2, ...y^n, z) = (x^i, y^i, z) \in R^{2n+1}, (x^i, y^i, z \in R, i = 1, 2, ..., n)\}$. Consider R^{2n+1} with the following structure:*

$$\phi(X_i) = Y_i, \phi(Y_i) = X_i, \phi(\xi) = 0,$$

which are linearly independent at each point of N_1. Let g_{N_1} is Lorentzian metric defined by

$$g_{N_1} = -(\eta \otimes \eta) + e^{2Z}\sum_{i=1}^{n}(dx^i \otimes dx^i + dy^i \otimes dy^i),$$

$$\phi^2 X = X + \eta(X)\xi, g_{N_1}(X, \xi) = \eta(X),$$

for all vector fields X on R^{2n+1}.

Then, $(R^{2m+1}, \phi, \xi, \eta, g_{N_1})$ is a Lorentzian para-Kenmotsu manifold. The vector fields $X_i = e^{-Z}\frac{\partial}{\partial x^i}, Y_i = e^{-Z}\frac{\partial}{\partial y^i}$ and $\xi = \frac{\partial}{\partial z}$ form a $\phi-$basis for Lorentzian para-Kenmotsu manifold R^{2n+1}, where $i = 1, 2, ..., n$.

3. Semi-invariant $\xi^{\perp}-$ Riemannian submersions

An essential background of Riemannian submersions ($F : N_1 \to N_2$) and definition of semi-invariant $\xi^{\perp}-$ Riemannian submersions are given at this section. It is well-known that the fundamental tensors $\mathcal{T}$ and $\mathcal{A}$, define by O'Neill's [27] by

$$\mathcal{A}_{Z_1}U_1 = \mathcal{H}\nabla_{\mathcal{H}Z_1}\mathcal{V}U_1 + \mathcal{V}\nabla_{\mathcal{H}Z_1}\mathcal{H}U_1, \tag{3.8}$$

$$\mathcal{T}_{Z_1}U_1 = \mathcal{H}\nabla_{\mathcal{V}Z_1}\mathcal{V}U_1 + \mathcal{V}\nabla_{\mathcal{V}Z_1}\mathcal{H}U_1 \tag{3.9}$$

for any vector fields Z_1, U_1 on N_1, where ∇ is the Levi-Civita connection of g_{N_1}.
From equations (3.8) and (3.9), we have

$$\nabla_{Y_1} Z_2 = \mathcal{T}_{Y_1} Z_2 + \mathcal{V}\nabla_{Y_1} Z_2, \tag{3.10}$$

$$\nabla_{Y_1} U_1 = \mathcal{T}_{Y_1} U_1 + \mathcal{H}\nabla_{Y_1} U_1, \tag{3.11}$$

$$\nabla_{U_1} Y_1 = \mathcal{A}_{U_1} Y_1 + \mathcal{V}\nabla_{U_1} Y_1, \tag{3.12}$$

$$\nabla_{U_1} W_2 = \mathcal{H}\nabla_{U_1} W_2 + \mathcal{A}_{U_1} W_2 \tag{3.13}$$

for all $Y_1, Z_2 \in \Gamma(\ker F_*)$ and $U_1, W_2 \in \Gamma(\ker F_*)^\perp$, where $\mathcal{H}\nabla_{Y_1} U_1 = \mathcal{A}_{U_1} Y_1$, if U_1 is basic. It is easy to notice that $\mathcal{A}$ performs on the horizontal distribution and estimates the interference to the integrability of this distribution and $\mathcal{T}$ performs on the fibers as the second fundamental form. .

Here F between two Riemannian manifolds called totally geodesic if

$$(\nabla F_*)(U_1, W_2) = 0, \text{ for all } U_1, W_2 \in \Gamma(TN_1) \tag{3.14}$$

and F is called totally umbilical if [6]

$$\mathcal{T}_{Y_1} Y_2 = g_{N_1}(Y_1, Y_2) H \tag{3.15}$$

for all $Y_1, Y_2 \in \Gamma(\ker F_*)$, where H represents the mean curvature vector field of fibers.

The the second fundamental form of F is given by

$$(\nabla F_*)(W_1, W_2) = \nabla^F_{W_1} F_*(W_2) - F_*(\nabla^{N_1}_{W_1} W_2) \tag{3.16}$$

for vector field $W_1, W_2 \in \Gamma(TN_1)$, where ∇^F denotes the pullback connection [6] and it is easy to see that the second fundamental form is symmetric.

Lemma 3.1. [6] *Let (N_1, g_{N_1}) and (N_2, g_{N_2}) are two Riemannian manifolds. If $F : N_1 \to N_2$ Riemannian submersion between Riemannian manifolds, then for any horizontal vector fields Y_1, Y_2 and vertical vector fields Z_1, Z_2, we have*

(a) $(\nabla F_*)(Y_1, Y_2) = 0$,
(b) $(\nabla F_*)(Z_1, Z_2) = -F_*(\mathcal{T}_{Z_1} Z_2) = -F_*(\nabla^{N_1}_{Z_1} Z_2)$,
(c) $(\nabla F_*)(Y_1, Z_1) = -F_*(\nabla^{N_1}_{Y_1} Z_1) = -F_*(\mathcal{A}_{Y_1} Z_1)$.

Definition 3.1. [35] *Let (N_1, g_{N_1}) be an almost Hermitian manifold and (N_2, g_{N_2}) be a Riemannian manifold. Then we say that F is a semi-invariant Riemannian submersion if there is a distribution $D_1 \subseteq \ker F_*$ such that*

$$\ker F_* = D_1 \oplus D_2, J(D_1) = D_1, J(D_2) \subseteq (\ker F_*)^{\perp}.$$

We can write

$$(\ker F_*)^{\perp} = J(D_2) \oplus \mu,$$

where, μ is an invariant subbundle of $(\ker F_*)^{\perp}$.

Definition 3.2. [23] *Let $F : (N_1, \phi, \xi, \eta, g_{N_1}) \to (N_2, g_{N_2})$ be a Riemannian submersion in such a manner that ξ is normal to $(\ker F_*)$ and $(\ker F_*)$ is anti-invariant with respect to ϕ. Then F is called an anti-invariant $\xi^{\perp}$-Riemannian submersion.*

Definition 3.3. [1] *Let $F : (N_1, \phi, \xi, \eta, g_{N_1}) \to (N_2, g_{N_2})$ be a Riemannian submersion from an almost para-contact metric manifold onto a Riemannian manifold. F is called a semi-invariant $\xi^{\perp}$-Riemannian submersion if $D_1 \subset (\ker F_*)$ is such that*

$$(\ker F_*) = D_1 \oplus D_2, \phi(D_1) = D_1, \phi(D_2) \subset (\ker F_*)^{\perp}.$$

4. CSI-$\xi^{\perp}$-Riemannian submersions from a Lorentzian para-Kenmotsu manifolds

In this section, we define and study CSI-$\xi^{\perp}$-Riemannian submersion from Lorentzian para-Kenmotsu manifolds onto a Riemannian manifolds.

In the theory of Riemannian submersions, Bishop [8] defines the notion of Clairaut submersion:

Definition 4.1. *Let α is any geodesic on N_1, r is a positive function on N_1 and $\theta(t)$ is the angle between $\dot{\alpha}$ and the horizontal space at $\alpha(t)$ for any t. If the function $(r \circ \alpha)\sin\theta$ is constant on N_1, then a Riemannian submersion $F : (N_1, g_{N_1}) \to (N_2, g_{N_2})$ is called a Clairaut submersion.*

Theorem 4.1. [8] *Let $F : (N_1, g_{N_1}) \to (N_2, g_{N_2})$ be a Riemannian submersion with connected fibers. Then, F is a Clairaut Riemannian submersion with $r = e^f$ if each fiber is totally umbilical and has the mean curvature vector field $H = -\nabla f$ is the gradient of the function f with respect to g_{N_1}.*

Definition 4.2. *A semi-invariant $\xi^{\perp}$-Riemannian submersion F from a Lorentzian para-Kenmotsu manifold $(N_1, \phi, \xi, \eta, g_{N_1})$ onto a Riemannian manifold (N_2, g_{N_2}) is called Csi-$\xi^{\perp}$-Riemannian submersion if it satisfies the condition of Clairaut Riemannian submersion i.e., if each fiber is totally umbilical with mean curvature vector field $H = -\nabla f$ with respect to g_{N_1}, then F is a Clairaut Riemannian submersion with $r = e^f$.*

Now, using definition (4.1), we have

$$(\ker F_*) = D_1 \oplus D_2, \phi(D_1) = D_1, \phi(D_2) \subseteq (\ker F_*)^{\perp}.$$

Thus for any $V_1 \in (\ker F_*)$, we put

$$V_1 = PV_1 + QV_1, \tag{4.17}$$

where $PV_1 \in \Gamma(D_1)$ and $QV_1 \in \Gamma(D_2)$.

In addition, for $Y_1 \in (\ker F_*)$, we get

$$\phi Y_1 = \psi Y_1 + \omega Y_1, \tag{4.18}$$

where $\phi Y_1 \in \Gamma(D_1)$ and $\omega Y_1 \in \Gamma(\phi D_2)$.

$\Gamma(\ker F_*)^{\perp}$ is decomposed as

$$\Gamma(\ker F_*)^{\perp} = \phi(D_2) \oplus \mu.$$

Here μ is invariant and contains ξ.

Also for $X_2 \in \Gamma(\ker F_*)^{\perp}$, we have

$$\phi X_2 = BX_2 + CX_2, \tag{4.19}$$

where $BX_2 \in \Gamma(D_2)$ and $CX_2 \in \Gamma(\mu)$.

Lemma 4.1. *Let F be a semi-invariant $\xi^{\perp}$-Riemannian submersion from a Lorentzian para-Kenmotsu manifold $(N_1, \phi, \xi, \eta, g_{N_1})$ onto a Riemannian manifold (N_2, g_{N_2}). Then, we get*

$$\mathcal{V}\nabla_{Y_1}\psi Z_1 + \mathcal{T}_{Y_1}\omega Z_1 = B\mathcal{T}_{Y_1}Z_1 + \psi\mathcal{V}\nabla_{Y_1}Z_1, \tag{4.20}$$

$$\mathcal{T}_{Y_1}\psi Z_1 + \mathcal{H}\nabla_{Y_1}\omega Z_1 + g_{N_1}(\psi Y_1, Z_1)\xi = C\mathcal{T}_{Y_1}Z_1 + \omega\mathcal{V}\nabla_{Y_1}Z_1, \tag{4.21}$$

$$\mathcal{V}\nabla_{V_1}BW_1 + \mathcal{A}_{V_1}CW_1 + \eta(W_1)BV_1 = B\mathcal{H}\nabla_{V_1}W_1 + \psi\mathcal{A}_{V_1}W_1, \tag{4.22}$$

$$\mathcal{A}_{V_1}BW_1 + \mathcal{H}\nabla_{V_1}CW_1 + \eta(W_1)CV_1 + g_{N_1}(CV_1, W_1)\xi = C\mathcal{H}\nabla_{V_1}W_1 + \omega\mathcal{A}_{V_1}W_1, \tag{4.23}$$

$$\mathcal{V}\nabla_{Y_1}BV_1 + \mathcal{T}_{Y_1}CV_1 + \eta(V_1)\psi Y_1 = \psi\mathcal{T}_{Y_1}V_1 + B\mathcal{H}\nabla_{Y_1}V_1, \tag{4.24}$$

$$\mathcal{T}_{Y_1}BV_1 + \mathcal{H}\nabla_{Y_1}CV_1 + \eta(V_1)\omega Y_1 + g_{N_1}(\omega Y_1, V_1)\xi = \omega\mathcal{T}_{Y_1}V_1 + C\mathcal{H}\nabla_{Y_1}V_1, \tag{4.25}$$

$$\mathcal{V}\nabla_{V_1}\psi Y_1 + \mathcal{A}_{V_1}\omega Y_1 = B\mathcal{A}_{V_1}Y_1 + \psi\mathcal{V}\nabla_{V_1}Y_1, \tag{4.26}$$

$$\mathcal{A}_{V_1}\psi Y_1 + \mathcal{H}\nabla_{V_1}\omega Y_1 + g_{N_1}(BV_1, Y_1)\xi = C\mathcal{A}_{V_1}Y_1 + \omega\mathcal{V}\nabla_{V_1}Y_1, \tag{4.27}$$

where $Y_1, Z_1 \in \Gamma(\ker F_*)$ *and* $V_1, W_1 \in \Gamma(\ker F_*)^\perp$.

Proof. Using equations (2.5), (3.10)-(3.13), (4.18) and (4.19), we get Lemma 4.1.

Lemma 4.2. *Let* F *be a semi-invariant* $\xi^\perp$*-Riemannian submersion from a Lorentzian para-Kenmotsu manifold* $(N_1, \phi, \xi, \eta, g_{N_1})$ *onto a Riemannian manifold* (N_2, g_{N_2})*. If* $\alpha : I_2 \subset R \to N_1$ *is a regular curve and* $Z_1(t)$ *and* $U_1(t)$ *are the vertical and horizontal components of the tangent vector field* $\dot\alpha = E$ *of* $\alpha(t)$*, respectively, then* α *is a geodesic if and only if along* α *the following equations hold:*

$$\mathcal{V}\nabla_{\dot\alpha}\psi Z_1 + \mathcal{V}\nabla_{\dot\alpha}BU_1 + (\mathcal{T}_{Z_1} + \mathcal{A}_{U_1})\omega Z_1 + (\mathcal{T}_{Z_1} + \mathcal{A}_{U_1})CU_1 + \eta(U_1)(\psi Z_1 + BZ_1) = 0,$$

$$\mathcal{H}\nabla_{\dot\alpha}\omega Z_1 + \mathcal{H}\nabla_{\dot\alpha}CU_1 + (\mathcal{T}_{Z_1} + \mathcal{A}_{U_1})\psi Z_1 + (\mathcal{T}_{Z_1} + \mathcal{A}_{U_1})BU_1 + g_{N_1}(\phi\dot\alpha, \dot\alpha)\xi + \eta(U_1)(\omega Z_1 + CZ_1) = 0.$$

Proof. Let $\alpha : I_2 \to N_1$ is a regular curve on N_1 and $\dot\alpha(t)$ is the tangent vector field. If $Z_1(t)$ and $U_1(t)$ are the vertical and horizontal parts of the tangent vector field, respectively. Then $\dot\alpha(t) = Z_1(t) + U_1(t)$. From equations (2.5), (3.10)-(3.13), (4.18) and (4.19), we get

$$\begin{aligned}
&\phi\nabla_{\dot\alpha}\dot\alpha \\
= \;& \nabla_{\dot\alpha}\phi\dot\alpha - (\nabla_{\dot\alpha}\phi)\dot\alpha, \\
= \;& \nabla_{Z_1}\psi Z_1 + \nabla_{Z_1}\omega Z_1 + \nabla_{Z_1}BU_1 + \nabla_{Z_1}CU_1 + \nabla_{U_1}\psi Z_1 + \nabla_{U_1}\omega Z_1 + \\
& \nabla_{U_1}BU_1 + \nabla_{U_1}CU_1 + g_{N_1}(\phi\dot\alpha, \dot\alpha)\xi + \eta(U_1)(\psi Z_1 + BZ_1) + \eta(U_1)(\omega Z_1 + CZ_1), \\
= \;& \mathcal{T}_{Z_1}\psi Z_1 + \mathcal{V}\nabla_{Z_1}\psi Z_1 + \mathcal{T}_{Z_1}\omega Z_1 + \mathcal{H}\nabla_{Z_1}\omega Z_1 + \mathcal{T}_{Z_1}BU_1 + \mathcal{V}\nabla_{Z_1}BU_1 + \\
& \mathcal{T}_{Z_1}CU_1 + \mathcal{H}\nabla_{Z_1}CU_1 + \mathcal{A}_{U_1}\psi Z_1 + \mathcal{V}\nabla_{U_1}\psi Z_1 + \mathcal{H}\nabla_{U_1}\omega Z_1 + \mathcal{A}_{U_1}\omega Z_1 + \\
& \mathcal{A}_{U_1}BU_1 + \mathcal{V}\nabla_{U_1}BU_1 + \mathcal{H}\nabla_{U_1}CU_1 + \mathcal{A}_{U_1}CU_1 + g_{N_1}(\phi\dot\alpha, \dot\alpha)\xi + \\
& \eta(U_1)(\psi Z_1 + BZ_1) + \eta(Z_2)(\omega Z_1 + CZ_1).
\end{aligned}$$

Taking the vertical and horizontal components in above equation, we have

$$\mathcal{V}\phi\nabla_{\dot\alpha}\dot\alpha = \mathcal{V}\nabla_{\dot\alpha}\psi Z_1 + \mathcal{V}\nabla_{\dot\alpha}BU_1 + (\mathcal{T}_{Z_1} + \mathcal{A}_{U_1})\omega Z_1 + (\mathcal{T}_{Z_1} + \mathcal{A}_{U_1})CU_1 + \eta(U_1)(\psi Z_1 + BZ_1),$$

$$\begin{aligned}
&\mathcal{H}\phi\nabla_{\dot\alpha}\dot\alpha \\
= \;& \mathcal{H}\nabla_{\dot\alpha}\omega Z_1 + \mathcal{H}\nabla_{\dot\alpha}CU_1 + (\mathcal{T}_{Z_1} + \mathcal{A}_{U_1})\psi Z_1 + (\mathcal{T}_{Z_1} + \mathcal{A}_{U_1})BU_1 + \\
& g_{N_1}(\phi\dot\alpha, \dot\alpha)\xi + \eta(U_1)(\omega Z_1 + CZ_1),
\end{aligned}$$

Hence, α is a geodesic on N_1 if and only if $\mathcal{V}\phi\nabla_{\dot{\alpha}}\dot{\alpha}$ and $\mathcal{H}\phi\nabla_{\dot{\alpha}}\dot{\alpha}$ both are vanish, which gives our result.

Theorem 4.2. *Let F be a semi-invariant $\xi^{\perp}$-Riemannian submersion from a Lorentzian para-Kenmotsu manifold $(N_1, \phi, \xi, \eta, g_{N_1})$ onto a Riemannian manifold (N_2, g_{N_2}). Then F is a Csi-$\xi^{\perp}$-Riemannian submersion with $r = e^f$ if and only if*

$$\begin{aligned} & (g_{N_1}(\nabla f, Z_1) - \eta(Z_1)|)||V_1||^2 \\ = \;& -g_{N_1}(\mathcal{V}\nabla_{\dot{\alpha}}BZ_1, \psi V_1) - g_{N_1}(\mathcal{H}\nabla_{\dot{\alpha}}CZ_1, \omega V_1) - g_{N_1}((\mathcal{T}_{V_1} + \mathcal{A}_{Z_1})CZ_1, \psi V_1) - \\ & g_{N_1}((\mathcal{T}_{V_1} + \mathcal{A}_{Z_1})BZ_1, \omega V_1) \end{aligned}$$

where $\alpha : I_2 \to N_1$ is a geodesic on N_1 and V_1, Z_1 are vertical and horizontal components of $\dot{\alpha}(t)$.

Proof. Let $\alpha : I_2 \to N_1$ be a geodesic on N_1 with $V_1(t) = \mathcal{V}\dot{\alpha}(t)$ and $Z_1(t) = \mathcal{H}\dot{\alpha}(t)$. Let $\theta(t)$ denote the angle in $[0, \pi]$ between $\dot{\alpha}(t)$ and $Z_1(t)$. Assuming $\nu = ||\dot{\alpha}(t)||^2$ then we get

$$g_{N_1}(V_1(t), V_1(t)) = \nu \sin^2 \theta(t), \tag{4.28}$$

$$g_{N_1}(Z_1(t), Z_1(t)) = \nu \cos^2 \theta(t). \tag{4.29}$$

Now, differentiating (4.28), we get

$$\frac{d}{dt} g_{N_1}(V_1(t), V_1(t)) = 2\nu \cos\theta(t) \sin\theta(t) \frac{d\theta}{dt}.$$

Using equation (2.2), we get

$$g_{N_1}(\phi\nabla_{\dot{\alpha}}V_1, \phi V_1) = \nu \cos\theta(t) \sin\theta(t) \frac{d\theta}{dt}. \tag{4.30}$$

Now, using equation (2.5), we get

$$\nabla_{\dot{\alpha}}\phi V_1 = \phi\nabla_{\dot{\alpha}}V_1 + g_{N_1}(\phi\dot{\alpha}, V_1)\xi,$$

$$\begin{aligned} & g_{N_1}(\phi\nabla_{\dot{\alpha}}V_1, \phi V_1) \\ = \;& g_{N_1}(\nabla_{\dot{\alpha}}\phi V_1, \phi V_1), \\ = \;& g_{N_1}(\mathcal{V}\nabla_{\dot{\alpha}}\psi V_1, \psi V_1) + g_{N_1}(\mathcal{H}\nabla_{\dot{\alpha}}\omega V_1, \omega V_1) + g_{N_1}((\mathcal{A}_{Z_1} + \mathcal{T}_{V_1})\psi V_1, \omega V_1) + \\ & g_{N_1}((\mathcal{A}_{Z_1} + \mathcal{T}_{V_1})\omega V_1, \psi V_1). \end{aligned}$$

Using Lemma 4.2 in above equation, we get

$$\begin{aligned} & g_{N_1}(\phi\nabla_{\dot\alpha}V_1,\phi V_1) \\ = \; & -g_{N_1}(\mathcal{V}\nabla_{\dot\alpha}BZ_1,\psi V_1)-g_{N_1}(\mathcal{H}\nabla_{\dot\alpha}CZ_1,\omega V_1)- \\ & g_{N_1}((\mathcal{T}_{V_1}+\mathcal{A}_{Z_1})CZ_1,\psi V_1)-g_{N_1}((\mathcal{T}_{V_1}+\mathcal{A}_{Z_1})BZ_1,\omega V_1)- \\ & \eta(Z_1)g_{N_1}(V_1,V_1). \end{aligned} \tag{4.31}$$

From equations (4.30) and (4.31), we have

$$\begin{aligned} & \nu\cos\theta(t)\sin\theta(t)\frac{d\theta}{dt} \\ = \; & -g_{N_1}(\mathcal{V}\nabla_{\dot\alpha}BZ_1,\psi V_1)-g_{N_1}(\mathcal{H}\nabla_{\dot\alpha}CZ_1,\omega V_1)- \\ & g_{N_1}((\mathcal{T}_{V_1}+\mathcal{A}_{Z_1})CZ_1,\psi V_1)-g_{N_1}((\mathcal{T}_{V_1}+\mathcal{A}_{Z_1})BZ_1,\omega V_1)- \\ & \eta(Z_1)g_{N_1}(V_1,V_1). \end{aligned} \tag{4.32}$$

Moreover, F is a Csi-$\xi^{\perp}$-Riemannian submersion with $r=e^{f}$ if and only if

$$\begin{gathered} \frac{d}{dt}(e^{f\circ\alpha}\sin\theta)=0 \\ e^{f\circ\alpha}(\cos\theta\frac{d\theta}{dt}+\sin\theta\frac{df}{dt})=0. \end{gathered} \tag{4.33}$$

Multiplying with non-zero factor $\upsilon\sin\theta$ on both sides, we have

$$\begin{aligned} -\upsilon\cos\theta\sin\theta\frac{d\theta}{dt} &= \upsilon\sin^2\theta\frac{df}{dt}, \\ \upsilon\cos\theta\sin\theta\frac{d\theta}{dt} &= -g_{N_1}(V_1,V_1)\frac{df}{dt}, \\ \upsilon\cos\theta\sin\theta\frac{d\theta}{dt} &= -g_{N_1}(\nabla f,\dot\alpha)||V_1||^2, \\ \upsilon\cos\theta\sin\theta\frac{d\theta}{dt} &= -g_{N_1}(\nabla f,Z_1)||V_1||^2. \end{aligned} \tag{4.34}$$

Thus, from equations (4.32) and (4.34), we have

$$\begin{aligned} & (g_{N_1}(\nabla f,Z_1)-\eta(Z_1))||V_1||^2 \\ = \; & g_{N_1}(\mathcal{V}\nabla_{\dot\alpha}BZ_1,\psi V_1)+g_{N_1}(\mathcal{H}\nabla_{\dot\alpha}CZ_1,\omega V_1)+g_{N_1}((\mathcal{T}_{V_1}+\mathcal{A}_{Z_1})CZ_1,\psi V_1)+ \\ & g_{N_1}((\mathcal{T}_{V_1}+\mathcal{A}_{Z_1})BZ_1,\omega V_1). \end{aligned}$$

Hence the theorem 4.2 is proved.

Corollary 4.1. *Let F be a semi-invariant $\xi^{\perp}$-Riemannian submersion from a Lorentzian para-Kenmotsu manifold $(N_1, \phi, \xi, \eta, g_{N_1})$ onto a Riemannian manifold (N_2, g_{N_2}). Then, we get*

$$g_{N_1}(\nabla f, \xi) = -1.$$

Theorem 4.3. *Let F be a Csi-$\xi^{\perp}$-Riemannian submersion from a Lorentzian para-Kenmotsu manifold $(N_1, \phi, \xi, \eta, g_{N_1})$ onto a Riemannian manifold (N_2, g_{N_2}) with $r = e^f$. Then, we get*

$$\mathcal{A}_{\phi V_1} \phi Y_1 = Y_1(f) V_1 \tag{4.35}$$

for $Y_1 \in \Gamma(\mu)$ and $V_1 \in \Gamma(D_2)$, such that ϕY_1 is basic.

Proof. Let F be Csi-$\xi^{\perp}$-Riemannian submersion from a Lorentzian para-Kenmotsu manifold onto a Riemannian manifold. For $X_1, X_2 \in \Gamma(D_2)$, using equation (3.15) and Theorem 4.1, we get

$$\mathcal{T}_{X_1} X_2 = -g_{N_1}(X_1, X_2) grad f. \tag{4.36}$$

Now, we take inner product with ϕV_1 in equation (4.36),

$$g_{N_1}(\mathcal{T}_{X_1} X_2, \phi V_1) = -g_{N_1}(X_1, X_2) g_{N_1}(grad f, \phi V_1), \tag{4.37}$$

for all $V_1 \in \Gamma(D_2)$.

From equations (2.2) and (2.5), we obtain

$$g_{N_1}(\nabla_{X_1} \phi X_2, V_1) = -g_{N_1}(X_1, X_2) g_{N_1}(grad f, \phi V_1).$$

Here ∇ is metric connection, so we can use equations (3.15) and (4.37) in above equation and get

$$g_{N_1}(X_1, V_1) g_{N_1}(grad f, \phi X_2) = -g_{N_1}(X_1, X_2) g_{N_1}(grad f, \phi V_1). \tag{4.38}$$

Now, we take $V_1 = X_2$ and obtain the following equatin by interchanging the role of X_1 and X_2,

$$g_{N_1}(X_2, X_2) g_{N_1}(grad f, \phi X_1) = g_{N_1}(X_1, X_2) g_{N_1}(grad f, \phi X_2). \tag{4.39}$$

Using equation (4.39) with $V_1 = X_1$ in (4.38), we have

$$g_{N_1}(grad f, \phi X_1) = \frac{(g_{N_1}(X_1, X_2))^2}{||X_1||^2 ||X_2||^2} g_{N_1}(grad f, \phi X_1). \tag{4.40}$$

If $grad f \in \Gamma(\phi(D_2))$, then equation (4.40) and the condition of equality in the Schwarz inequality implies that either f is constant on $\phi(D_2)$ or the fibers are one dimensional.

On the other hand, using equation (2.5), we get

$$g_{N_1}(\phi\nabla_{X_1}V_1,\phi Y_1)=g_{N_1}(\nabla_{X_1}\phi V_1,\phi Y_1)$$

for $Y_1\in\Gamma(\mu)$ and $Y_1\neq\xi$. Now, using equation (2.2), we obtain

$$g_{N_1}(\nabla_{X_1}\phi V_1,\phi Y_1)=g_{N_1}(\nabla_{X_1}V_1,Y_1).$$

Using equations (2.2) and (2.5) in above equation, we get

$$g_{N_1}(\nabla_{X_1}\phi V_1,\phi Y_1)=-g_{N_1}(X_1,V_1)g_{N_1}(gradf,Y_1).$$

Since ϕY_1 is basic and using the fact that $\mathcal{H}\nabla_{X_1}\phi V_1=\mathcal{A}_{\phi V_1}X_1$, we get

$$\begin{aligned}
g_{N_1}(\nabla_{X_1}\phi V_1,\phi Y_1) &= -g_{N_1}(X_1,V_1)g_{N_1}(gradf,Y_1),\\
g_{N_1}(\mathcal{A}_{\phi V_1}X_1,\phi Y_1) &= -g_{N_1}(X_1,V_1)g_{N_1}(gradf,Y_1),\\
g_{N_1}(\mathcal{A}_{\phi V_1}\phi Y_1,X_1) &= g_{N_1}(X_1,V_1)g_{N_1}(gradf,Y_1)\\
g_{N_1}(\mathcal{A}_{\phi V_1}\phi Y_1,X_1) &= g_{N_1}(X_1,V_1)g_{N_1}(\nabla f,Y_1). \qquad (4.41)
\end{aligned}$$

Since $\mathcal{A}_{\phi V_1}\phi Y_1$ and V_1 are vertical and ∇f is horizontal, we obtain equation (4.35).

Theorem 4.4. *Let F be a Csi-$\xi^{\perp}$-Riemannian submersion from a Lorentzian para-Kenmotsu manifold $(N_1,\phi,\xi,\eta,g_{N_1})$ onto a Riemannian manifold (N_2,g_{N_2}) with $r=e^f$ and* $\dim(D_2)>1$. *Then, for all $Y_1\in\Gamma(D_2)$ and $V_1\in\Gamma(\ker F_*)^{\perp}$,*

$$\overset{F}{\nabla}_{V_1}F_*(\phi Y_1)=V_1(f)F_*(\phi Y_1).$$

Proof. Let F be a Csi-$\xi^{\perp}$-Riemannian submersion from a Lorentzian para-Kenmotsu manifold onto a Riemannian manifold. Since each fiber is totally umbilical with mean curvature vector field $H=-gradf$, then from theorem (4.1), we have

$$\begin{aligned}
-g_{N_1}(\nabla_{Y_1}V_1,Y_2) &= g_{N_1}(\nabla_{Y_1}Y_2,V_1),\\
-g_{N_1}(\nabla_{Y_1}V_1,Y_2) &= -g_{N_1}(Y_1,Y_2)g_{N_1}(gradf,V_1),
\end{aligned}$$

for all $Y_1,Y_2\in\Gamma(D_2)$ and $V_1\in\Gamma(\ker F_*)^{\perp}$.

Using equations (2.2),(2.5) and (3.15) in above equation, we get

$$g_{N_1}(\nabla_{V_1}\phi Y_1,\phi Y_2)=g_{N_1}(\phi Y_1,\phi Y_2)g_{N_1}(gradf,V_1). \qquad (4.42)$$

Since F is the semi-invariant Riemannian submersion and using equation (4.42), we have

$$g_{N_2}(F_*(\nabla_{V_1}\phi Y_1),F_*(\phi Y_2))=g_{N_2}(F_*(\phi Y_1),F_*(\phi Y_2))g_{N_1}(gradf,V_1). \qquad (4.43)$$

From (3.16) in (4.43), we obtain

$$g_{N_2}(\overset{F}{\nabla}_{V_1} F_*(\phi Y_1), F_*(\phi Y_2)) = g_{N_2}(F_*(\phi Y_1), F_*(\phi Y_2)) g_{N_1}(gradf, V_1), \tag{4.44}$$

which implies $\overset{F}{\nabla}_{V_1} F_*(\phi Y_1) = V_1(f) F_*(\phi Y_1)$, for all $Y_1 \in \Gamma(D_2)$ and $V_1 \in \Gamma(\ker F_*)^{\perp}$, hence the proof.

Theorem 4.5. *Let F be a Csi-$\xi^{\perp}$-Riemannian submersion with $r = e^f$ from a Lorentzian para-Kenmotsu manifold $(N_1, \phi, \xi, \eta, g_{N_1})$ onto a Riemannian manifold (N_2, g_{N_2}). If $\mathcal{T}$ is not equal to zero identically, then the invariant distribution D_1 cannot defined a totally geodesic foliation on N_1.*

Proof. For $W_1, W_2 \in \Gamma(D_1)$ and $X_1 \in \Gamma(D_2)$, using equations (2.2), (2.5), (3.11) and (3.15), we get

$$\begin{aligned} g_{N_1}(\nabla_{W_1} W_2, X_1) &= g_{N_1}(\nabla_{W_1} \phi W_2, \phi X_1), \\ &= g_{N_1}(\mathcal{T}_{W_1} \phi W_2, \phi X_1), \\ &= -g_{N_1}(W_1, \phi W_2) g_{N_1}(gradf, \phi X_1). \end{aligned}$$

Thus, the assertion can be seen from above equation and the fact that $gradf \in \phi(D_2)$.

Theorem 4.6. *Let F be a Csi-$\xi^{\perp}$-Riemannian submersion with $r = e^f$ from a Lorentzian para-Kenmotsu manifold $(N_1, \phi, \xi, \eta, g_{N_1})$ onto a Riemannian manifold (N_2, g_{N_2}) with $r = e^h$. Then, D_2 is not totally geodesic foliation on N_1.*

Proof. For $Z_1, Z_2 \in \Gamma(D_2)$ and $\xi \in \Gamma(\ker \pi_*)^{\perp}$, using (2.6), we get

$$g_{N_1}(\nabla_{Z_1} Z_2, \xi) = -g_{N_1}(\nabla_{Z_1} \xi, Z_2) = g_{N_1}(Z_1, Z_2) \neq 0.$$

Hence D_2 is not totally geodesic foliation on N_1.

Using Theorems (4.5) and (4.6), one can give the following Theorem.

Theorem 4.7. *Let F be a Csi-$\xi^{\perp}$-Riemannian submersion with $r = e^f$ from a Lorentzian para-Kenmotsu manifold $(N_1, \phi, \xi, \eta, g_{N_1})$ onto a Riemannian manifold (N_2, g_{N_2}) with $r = e^f$. Then, $(\ker \pi_*)$ is not totally geodesic foliation on N_1.*

Theorem 4.8. *Let F be a Csi-$\xi^{\perp}$-Riemannian submersion with $r = e^f$ from a Lorentzian para-Kenmotsu manifold $(N_1, \phi, \xi, \eta, g_{N_1})$ onto a Riemannian manifold (N_2, g_{N_2}). Then, the anti-invariant distribution D_2 one-dimensional.*

Proof. Since F is a Clairaut proper semi-invariant submersion, then either $\dim(D_2) = 1$ or $\dim(D_2) > 1$. If $\dim(D_2) > 1$, then we can choose $Z_1, Z_2 \in \Gamma(D_2)$ such that $\{Z_1, Z_2\}$ is orthonormal. From equations (2.5), (3.10), (4.18) and (4.19), we get

$$\begin{aligned}\mathcal{T}_{Z_1}\phi Z_2 + \mathcal{H}\nabla_{Z_1}\phi Z_2 &= \nabla_{Z_1}\phi Z_2,\\ \mathcal{T}_{Z_1}\phi Z_2 + \mathcal{H}\nabla_{Z_1}\phi Z_2 &= B\mathcal{T}_{Z_1}Z_2 + C\mathcal{T}_{Z_1}Z_2 + \psi\mathcal{V}\nabla_{Z_1}Z_2 + \omega\mathcal{V}\nabla_{Z_1}Z_2.\end{aligned}$$

Now, we take inner product with Z_1 in above equation and obtain

$$g_{N_1}(\mathcal{T}_{Z_1}\phi Z_2, Z_1) = g_{N_1}(B\mathcal{T}_{Z_1}Z_2, Z_1) + g_{N_1}(\psi\mathcal{V}\nabla_{Z_1}Z_2, Z_1). \quad (4.45)$$

From equation (2.5), (3.10) and (3.15), we have

$$g_{N_1}(\mathcal{T}_{Z_1}Z_1, \phi Z_2) = -g_{N_1}(\mathcal{T}_{Z_1}\phi Z_2, Z_1) = -g_{N_1}(gradf, \phi Z_2) = g_{N_1}(\mathcal{T}_{Z_1}Z_2, \phi Z_1). \quad (4.46)$$

From above equation, we obtain

$$\begin{aligned}g_{N_1}(gradf, \phi Z_2) &= g_{N_1}(\mathcal{T}_{Z_1}Z_2, \phi Z_1),\\ g_{N_1}(gradf, \phi Z_2) &= g_{N_1}(Z_1, Z_2)g_{N_1}(gradf, \phi Z_1),\\ g_{N_1}(gradf, \phi Z_2) &= 0.\end{aligned}$$

So, we get

$$gradf \perp \phi(D_2).$$

Therefore, the dimension of D_2 must be one.

5. Example

Let N_1 be a 5-dimensional space given by the following:

$$\mathbb{R}^5 = \{(x_1, x_2, y_1, y_2, z) \in R^5 | (x_1, x_2, y_1, y_2) \neq (0,0,0,0) \text{ and } z \neq 0\}.$$

Let η be a 1-form defined by $\eta = dz$. The vector field ξ is given by $\frac{\partial}{\partial z}$ and its Lorentzian metric g_{N_1} and tensor field ϕ are given by

$$g_{N_1} = e^{2z}(dx_1)^2 + e^{2z}(dx_2)^2 + e^{2z}(dy_1)^2 + e^{2z}(dy_2)^2 - (dz)^2,$$

$$\phi = \begin{bmatrix} 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ 1 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

This gives a Lorentzian para-Kenmotsu structure $(\phi, \xi, \eta, g_{N_1})$ on N_1.

A ϕ-basis for this structure can be given by $\{e_1 = e^{-z}\frac{\partial}{\partial x_1}, e_2 = e^{-z}\frac{\partial}{\partial x_2}, e_3 = e^{-z}\frac{\partial}{\partial y_1}, e_4 = e^{-z}\frac{\partial}{\partial y_2}, e_5 = \xi = \frac{\partial}{\partial z}\}$.

Let N_2 be $\{(u_1, u_2) \in R^2 | u_2 = z \neq 0\}$. We choose the Riemannian metric $g_{N_2} = e^{2z}(du_1)^2 + (du_2)^2$ on N_2.

Now, we define the map $F : (N_1, \phi, \xi, \eta, g_{N_1}) \to (N_2, g_{N_2})$ by the following:

$$F(x_1, x_2, y_1, y_2, z) = (\frac{x_2 + y_2}{\sqrt{2}}, z).$$

By direct calculations, we have

$$\begin{aligned}
\ker F_* &= span\{X_1 = e_1, X_2 = (e_2 - e_4), X_3 = e_3\}, \\
D_1 &= span\{X_1 = e_1, X_3 = e_3\}, D_2 = span\{X_2 = (e_2 - e_4)\}, \\
(\ker F_*)^{\perp} &= span\{V_1 = (e_2 + e_4), V_2 = \xi = e_5\}.
\end{aligned}$$

After some computations, we find that

$$\begin{aligned}
F_*(V_1) = \sqrt{2}e^{-z}\frac{\partial}{\partial u_1}, \\
F_*(V_2) = \frac{\partial}{\partial u_2}, \\
g_{N_1}(V_i, V_j) = g_{N_2}(F_* V_i, F_* V_j)
\end{aligned} \tag{5.47}$$

for all $V_i, V_j \in \Gamma(\ker F_*)^{\perp}, i, j = 1, 2$. Thus F is semi-invariant $\xi^{\perp}$-Riemannian submersion.

Now, we will obtain smooth function f on $\mathbb{R}^5$ which satisfy the condition $\mathcal{T}_X X = g_1(X, X)\nabla f$, for all $X \in \Gamma(\ker \pi_*)$.

Using the given Kenmotsu structure, we find

$$\begin{aligned}
[e_1, e_1] &= [e_2, e_2] = [e_3, e_3] = [e_4, e_4] = [e_5, e_5] = 0, \qquad (5.48) \\
[e_1, e_2] &= 0, [e_1, e_3] = 0, [e_1, e_4] = 0, [e_1, e_5] = e_1, \\
[e_2, e_3] &= 0, [e_2, e_4] = 0, [e_2, e_5] = e_2, [e_3, e_4] = 0, \\
[e_3, e_5] &= e_3, [e_4, e_5] = e_4,
\end{aligned}$$

The Levi-Civita connection ∇ of the metric g_{N_1} is given by the Koszul′s formula which is

$$\begin{aligned} & 2g_{N_1}(\nabla_X Z, W) \\ = \;& Xg_{N_1}(Z,W) + Zg_{N_1}(W,X) - Wg_{N_1}(X,Z) + g_{N_1}([X,Z],W) - \\ & g_{N_1}([Z,W],X) + g_{N_1}([W,X],Z). \end{aligned} \tag{5.49}$$

Using (5.48) and (5.49), we get

$$\begin{aligned} \nabla_{e_1}e_1 &= \nabla_{e_2}e_2 = \nabla_{e_3}e_3 = \nabla_{e_4}e_4 = \frac{\partial}{\partial z}, \\ \nabla_{e_1}e_2 &= \nabla_{e_1}e_3 = \nabla_{e_1}e_4 = \nabla_{e_2}e_1 = \nabla_{e_2}e_3 = \nabla_{e_2}e_4 = 0, \\ \nabla_{e_3}e_1 &= \nabla_{e_3}e_2 = \nabla_{e_3}e_4 = \nabla_{e_4}e_1 = \nabla_{e_4}e_2 = \nabla_{e_4}e_3 = 0. \end{aligned} \tag{5.50}$$

Therefore

$$\begin{aligned} \nabla_{X_1}X_1 &= \nabla_{e_1}e_1 = \frac{\partial}{\partial z}, \nabla_{X_2}X_2 = \nabla_{e_2-e_4}e_2 - e_4 = 2\frac{\partial}{\partial z} \\ \nabla_{X_3}X_3 &= \nabla_{e_3}e_3 = \frac{\partial}{\partial z}, \nabla_{X_1}X_2 = 0, \nabla_{X_1}X_3 = 0, \\ \nabla_{X_2}X_3 &= 0, \nabla_{X_2}X_1 = 0, \nabla_{X_3}X_1 = 0, \nabla_{X_3}X_2 = 0. \end{aligned} \tag{5.51}$$

Now, we have

$$\mathcal{T}_X X = \mathcal{T}_{\lambda_1 X_1 + \lambda_2 X_2 + \lambda_3 X_3}\lambda_1 X_1 + \lambda_2 X_2 + \lambda_3 X_3, \lambda_1, \lambda_2, \lambda_3 \in R.$$

$$\begin{aligned} \mathcal{T}_X X = \;& \lambda_1^2 \mathcal{T}_{X_1}X_1 + \lambda_2^2 \mathcal{T}_{X_2}X_2 + \lambda_3^2 \mathcal{T}_{X_3}X_3 + \\ & \lambda_1\lambda_2 \mathcal{T}_{X_1}X_2 + \lambda_1\lambda_3 \mathcal{T}_{X_1}X_3 + \lambda_2\lambda_3 \mathcal{T}_{X_2}X_3 + \\ & \lambda_1\lambda_2 \mathcal{T}_{X_2}X_1 + \lambda_1\lambda_3 \mathcal{T}_{X_3}X_1 + \lambda_2\lambda_3 \mathcal{T}_{X_3}X_2. \end{aligned} \tag{5.52}$$

Using (5.51), we obtain

$$\begin{aligned} \mathcal{T}_{X_1}X_1 &= \frac{\partial}{\partial z}, \mathcal{T}_{X_2}X_2 = 2\frac{\partial}{\partial z}, \mathcal{T}_{X_3}X_3 = \frac{\partial}{\partial z}, \\ \mathcal{T}_{X_1}X_2 &= 0, \mathcal{T}_{X_1}X_3 = 0, \mathcal{T}_{X_2}X_3 = 0, \mathcal{T}_{X_2}X_1 = 0, \\ \mathcal{T}_{X_3}X_1 &= 0, \mathcal{T}_{X_3}X_2 = 0. \end{aligned} \tag{5.53}$$

Next, using (5.52) and (5.53), we get

$$\mathcal{T}_X X = (\lambda_1^2 + 2\lambda_2^2 + \lambda_3^2 + \lambda_3^2)\frac{\partial}{\partial z}.$$

Since $X = \lambda_1 X_1 + \lambda_2 X_2 + \lambda_3 X_3$, so $g_{N_1}(\lambda_1 X_1 + \lambda_2 X_2 + \lambda_3 X_3, \lambda_1 X_1 + \lambda_2 X_2 + \lambda_3 X_3) = \lambda_1^2 + 2\lambda_2^2 + \lambda_3^2$. For any smooth function f on R^5, ∇f with respect to the metric g_{N_1} is given by

$\nabla f = e^{-2Z}\frac{\partial f}{\partial x_1}\frac{\partial}{\partial x_1} + e^{-2Z}\frac{\partial f}{\partial x_2}\frac{\partial}{\partial x_2} + e^{-2Z}\frac{\partial f}{\partial y_1}\frac{\partial}{\partial y_1} + e^{-2Z}\frac{\partial f}{\partial y_2}\frac{\partial}{\partial y_2} + \frac{\partial f}{\partial z}\frac{\partial}{\partial z}.$

Therefore, $\nabla f = \frac{\partial}{\partial z}$ for the function $f = z$. Now, we can see that $\mathcal{T}_X X = g_{N_1}(X, X)\nabla f$ and by Theorem (4.1), it is clear that F is a CSI-$\xi^{\perp}$-Riemannian submersions.

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