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ON $QTAG$-MODULES CONTAINING PROPER h-PURITY

RAFIQUDDIN, AYAZUL HASAN *, AND MOHD HANZLA

Abstract. There are numerous problems of determining the $QTAG$-modules in which every h-pure submodule is isotype or the $QTAG$-modules in which every submodule is isotype. Our global aim here is to find in this direction a new problem by generalizing the h-purity in $QTAG$-modules, and thereby to establish some characterizations of the $QTAG$-modules in which every σ-pure submodule is λ-pure submodule for arbitrary ordinals σ and λ.

Keywords: QTAG-modules, σ-pure submodules, λ-pure submodules

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1. Introduction

The theory of abelian groups studied from time to time by many mathematicians, play a very crucial role in the theory of modules. Many authors interested in module theory have worked on generalizing the theory of abelian groups. The notion of the generalized torsion abelian groups is an important concept in the area of TAG-modules. It was first introduced by Singh [17] in 1976. A module M over a ring R is called a TAG-module if it satisfies the following two conditions while the rings are associated with unity.

"(i) Every finitely generated submodule of any homomorphic image of M is a direct sum of uniserial modules.

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* Corresponding author

Rafiquddin; rafiqamt786@rediffmail.com; https://orcid.org/0000-0001-6843-9798

Ayazul Hasan; ayazulh@jazanu.edu.sa; https://orcid.org/0000-0002-2895-8267

Mohd Hanzla; sheikhhanzla96@gmail.com; https://orcid.org/0000-0002-4144-1043

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(ii) Given any two uniserial submodules U_1 and U_2 of a homomorphic image of M, for any submodule N of U_1, any non-zero homomorphism $\phi : N \to U_2$ can be extended to a homomorphism $\psi : U_1 \to U_2$, provided the composition length $d(U_1/N) \leq d(U_2/\phi(N))$."

It was shown that the theory of these modules very closely paralleled the theory of torsion abelian groups; for this reason they were referred to as TAG-modules. Later on, it was shown that, for almost all applications, one of these conditions was not needed; ignoring this nearly superfluous condition, the slightly more general concept of a $QTAG$-module was initiated by the same author in [18]. Since then, many forms of this notion such as α-modules [4, 10], n-layered modules [15], essentially finitely indecomposable modules [3] and semi-complete modules [6] etc. have been defined and studied by many authors. Moreover, the authors have introduced many new concepts via these types of modules. They have also investigated some of their interesting properties and characterizations of these modules. Not surprisingly, many of the developments parallel the earlier development of the structure of torsion abelian groups. The present work is a natural extension of the torsion abelian groups over to the area of $QTAG$-modules and certainly contributes to the overall knowledge of the structure of $QTAG$-modules.

2. Preliminaries

Throughout the text, we assume that all rings into consideration are associative with unity $(1 \neq 0)$ and modules are unital $QTAG$-modules. By the term "uniserial module" we will mean a module M over a ring R, whose submodules are totally ordered by inclusion, i.e., for any two submodules N and L of M, either $N \subseteq L$ or $L \subseteq N$. Likewise, we shall say M is uniform if intersection of any two of its non-zero submodules is non-zero. In particular, if M is a module and $u \in M$, then let u denote the uniform element and let uR denote the uniform (hence uniserial) module, respectively. Concerning decomposition series, we suppose that all decomposition series are unique. For any module M, the symbol $d(M)$ will denote its decomposition length. In addition, if u is an uniform element of M (i.e., $u \in M$), then $e(u)$ is called the exponent of u, and $e(u) = d(uR)$. As usual, for such a module M, we set the height of u in M as $H_M(u) = \sup\{d(vR/uR) : v \in M,\ u \in vR \text{ and } v \text{ uniform}\}$. For every non-negative integer t, $H_t(M) = \{u \in M |\ H_M(u) \geq t\}$ denotes the t-th copies of M which can be viewed as a submodule of M consisting of all elements of height at least t. In this way, for a module M, the letter M^1 will always denote in the sequel the submodule of M, containing elements of infinite height. Moreover, we denote by $Soc(M)$, the socle of M, i.e., the sum of all simple submodules of M. For any $t \geq 0$, $Soc^t(M)$ is defined inductively as follows: $Soc^0(M) = 0$ and $Soc^{t+1}(M)/Soc^t(M) = Soc(M/Soc^t(M))$.

We add some basic definitions as well from [6], which is necessary for our successful presentation. The module M is named h-divisible if $M = M^1 = \cap_{t=0}^{\infty} H_t(M)$, or equivalently, if $H_1(M) = M$. The module M is termed separable if $M^1 = 0$. A submodule N of a module M is said to be an h-pure in M if for every non-negative integer t the equality $N \cap H_t(M) = H_t(N)$ hold. The cardinality of the minimal generating set of M is denoted by the symbol $g(M)$ that plays a significant role in our further investigation. By analogy, for all ordinals σ, one can define $f_M(\sigma)$, the σ^{th}-Ulm invariant of M as follows: $f_M(\sigma) = g\big(Soc(H_\sigma(M))/Soc(H_{\sigma+1}(M))\big)$.

In [5, 11], respectively, a submodule N of M is L-high, if $N \cap L = 0$ and N is maximal with respect to this intersection, that is, it is not properly contained in any different submodule of M having the same property.

It is well to note that various results for TAG-modules are also valid for $QTAG$-modules [13]. Our present work is motivated by the many significant results from the reference [14]. It is worthwhile noticing that some of the results are already investigated [7, 8] with h-purity. For the better understanding of the mentioned topic here one must go through the papers [9, 16]. In what follows, all notations and notions are standard and will be in agreement with those used in [1, 2]; for the specific ones, we refer the readers to [19].

3. Chief results

We begin by reviewing some terminology. If σ is an ordinal, and M is a $QTAG$-module, then the infinite height $H_\sigma(M)$ will be defined as $H_\sigma(M) = \cap_{\lambda<\sigma} H_\lambda(M)$ in the sense of [12], by using transfinite induction. Likewise, for any first infinite ordinal ω, the submodule M^1 of M, containing elements of infinite height that hold the equality $M^1 = \cap_{t=1}^{\infty} H_t(M) = H_\omega(M)$. Clearly, $H_t(M)$ is a submodule of M and the intersection $\cap_{t=1}^{\infty} H_t(M)$ form a submodule which is known as first Ulm submodule.

Next, we review the following concepts from [13]. A submodule N of M is said to be σ-pure if, for all ordinal λ, there exists an ordinal σ (depending on N) such that $H_\lambda(M) \cap N = H_\lambda(N)$. Besides, a submodule N of M is named isotype, if it is σ-pure for every ordinal σ. It readily follows that an isotype submodule will be h-pure in M, and hence a summand of M.

The theory of isotypity clearly depends on the theory of h-purity in $QTAG$-modules, and hence upon criteria under which a given h-pure submodule must necessarily be isotype (see, [7]). One important example is the determination of the $QTAG$-modules in which every h-pure submodule is a direct summand. Though it has been stated in a variety of forms by a number of characterizations. In this section we follow a somewhat different path and explore a new problem of determining the $QTAG$-modules in which every σ-pure submodule is λ-pure submodule for arbitrary ordinals σ and λ.

The following elementary, but useful lemma, shed some light about the relationships between Ulm-invariant and h-purity.

Lemma 3.1. *Suppose σ and λ are ordinals such that $1 \leq \sigma < \lambda \leq \infty$ and M is a $QTAG$-module with $f_M(\delta) = 0$ for $\sigma \leq \delta + 1 < \lambda$. If N is a σ-pure submodule of M, then N is also λ-pure.*

Proof. First observe that if $\sigma \leq \alpha < \lambda$ and N is an α-pure submodule, then N is an $(\alpha + 1)$-pure submodule of M. Next, choose N is α-pure and let $a \in N \cap H_{\alpha+1}(M)$. Then $a \in H_\alpha(N) \subset H_\sigma(N)$. Thus $a = b'$, where $d(bR/b'R) = 1$ and $b \in H_{\sigma-1}(N)$. But $a = c'$, where $d(cR/c'R) = 1$, and $c \in H_\alpha(M)$. Therefore, $b = c + x$, where $x \in Soc(H_{\sigma-1}(M))$. By hypothesis on $f_M(\delta)$, we have $Soc(H_{\sigma-1}(M)) \subset Soc(H_\alpha(M))$. Thus

$$b = c + x \in H_\alpha(M) \cap N = H_\alpha(N).$$

Therefore, $a = b' \in H_{\alpha+1}(M)$ such that $d(bR/b'R) = 1$, we are done.

Let us recall the smallest ordinal β such that $H_\beta(M) = 0$, is said to be the length of the $QTAG$-module M.

Inspired and motivated by the above concept, we give a new concept of two parameters involving the Ulm-invariant as follows.

Definition 3.1. *Let δ be an ordinal and M a $QTAG$-module such that $1 \leq \delta \leq H_\beta(M)$ and let γ be any ordinal. We define t_δ and r_γ by*

$$t_\delta = \begin{cases} \inf\{t \geq 0 : f_M(\delta - 1 + t) \ \neq 0\}, \text{ if } \delta - 1 \text{ exists} \\ 0, \text{ if } \delta \text{ is a limit ordinal,} \end{cases}$$

and

$$r_\gamma = \inf\{\alpha + 1 : \alpha + 1 < \gamma \text{ and } f_M(\alpha) \neq 0\}.$$

It is fairly to see that t_δ is a finite ordinal. This follows easily that $\delta \leq r_\gamma$ implies $\delta + t_\delta \leq \gamma$, with strict inequality holding when δ is not a limit ordinal.

Before presenting our main attainments, two preliminary technical lemmas are necessary.

Lemma 3.2. *Let N be a submodule of a $QTAG$-module M. Then there exists a submodule L of M containing N such that it satisfies the following conditions:*

(i) L is isotype in M.

(ii) N is isotype in L.

In particular, L is σ-pure in M if and only if N is σ-pure in M, where σ is an arbitrary ordinal.

Proof. In order to show that L is isotype in M, it suffices to show that L is σ-pure in M for an arbitrary ordinal σ, that is to show that L is $(\sigma+1)$-pure. In order to do this, among all uniform element in $L \cap H_{\sigma+1}(M)$, choose a such that $a = b'$, where $d(bR/b'R) = 1$ and $b \in H_\sigma(M)$. Now $a \in N_k$ for some k, so that $b' \in N_k$ where $d(bR/b'R) = 1$. Thus $b \in N_{k+1}$. Therefore, $b \in L \cap H_\sigma(M) = H_\sigma(L)$ and $a = b' \in H_{\sigma+1}(L)$ such that $d(bR/b'R) = 1$, as expected.

As for the second part, we can apply the same idea. Assume that N is σ-pure in L and let $a \in N \cap H_{\sigma+1}(L)$. Then $a = b'$ where $d(bR/b'R) = 1$ and $b \in H_\sigma(L)$. Since $b \in L$, it follows that $nb \in N$ for some non-negative integer n. Therefore, $b \in N \cap H_\sigma(L) = H_\sigma(N)$. Hence, $a = b' \in H_{\sigma+1}(N)$ such that $d(bR/b'R) = 1$, as required.

Conversely, suppose that $N \cap H_\lambda(M) = H_\lambda(N)$ for all $\lambda \leq \sigma$. Let $a \in L \cap H_\lambda(M)$, it is readily checked that $na \in N$ for some non-negative integer n. Thus

$$na \in N \cap H_\lambda(M) = H_\lambda(N) \subset H_\lambda(L).$$

It is only a routine exercise to check that $na \in H_\lambda(L)$ and implies that $a \in H_\lambda(L)$. Thus, we conclude that L is σ-pure in M, as asserted.

Lemma 3.3. *Let γ be an ordinal and M a $QTAG$-module such that $H_\gamma(M)$ contains a non-zero uniform element u with $e(u) = \infty$. For each ordinal δ and for some n let $n_\delta = -1$ if $\delta - 1$ exists and $n_\delta = 0$ otherwise. Then there exists a submodule N_δ of M such that $1 \leq \delta \leq r_\gamma$ and it satisfies the following conditions:*

(i) N_δ is $(\delta + t_\delta)$-pure in M.

(ii) $N_\alpha \subset N_\delta$ if $\alpha < \delta$.

(iii) $N_\delta \cap Soc(H_{\delta+t_\delta+n_\delta}(M)) = Soc(H_\gamma(M))$.

(iv) $u \in N_\delta$

(v) $u \notin H_{\delta+t_\delta+1}(N_\delta)$

In particular, N_δ *is not a* $(\gamma+1)$*-pure submodule of* M*, and* N_δ *is not* γ*-pure in* M *if* γ *is not a limit ordinal.*

Proof. The proof is by induction on δ. Assume that for each ordinal $\alpha < \delta$, there exists a submodule N_α and satisfies $(i)-(v)$. If δ is a limit ordinal, then

$$N_\delta = \cup_{\alpha<\delta} N_\alpha.$$

Certainly, if the submodule N_δ exists, then N_δ satisfies $(i)-(v)$. If $\delta-1$ exists and $t_{\delta-1} > 0$, we have $N_\delta = N_{\delta-1}$. It follows that N_α satisfies $(i)-(v)$, since $t_\delta = t_{\delta-1}-1$. If $\delta-1$ exists and $t_{\delta-1} = 0$, then we can construct N_δ from $N_{\delta-1}$. Since $f_M(\delta+t_\delta-1) \neq 0$, there exists an uniform element $v \in Soc(M)$ such that $H_M(v) = \delta+t_\delta-1$. Note also that $\delta+t_\delta < \gamma$, since $\delta \leq r_\gamma$. Then for any submodule P of M containing v, we have

$$Soc(H_{\delta+t_\delta-1}(M)) = Soc(H_\gamma(M)) \oplus P.$$

Thus, for $0 \neq w \in H_{\delta+t_\delta}(M)$ such that $u = w'$ and $d(wR/w'R) = 1$. This, in tern, implies that, there exists a submodule Q of M containing $v+w$ such that $Q = \langle N_{\delta-1}, a\rangle$, where $a = v+w$.

We first claim that $P \cap Q = 0$, if this failed, then there exist elements $b \in P$ and $c \in N_{\delta-1}$ and an integer k such that $b = c + a' \neq 0$, where $d(aR/a'R) = k$. If $k = 0$, then $u = a' = -c'$ such that $d(aR/a'R) = 1$, $d(cR/c'R) = 1$ and

$$c \in N_{\delta-1} \cap H_{\delta+t_\delta-1}(M) \subset H_{\delta-1}(N_{\delta-1}).$$

Thus $u \in H_\delta(N_{\delta-1})$, which is a contradiction that satisfies (v). On the other hand if $k > 0$, then $b = u' + c \in N_{\delta-1} \in P \cap N_{\delta-1}$ where $d(uR/u'R) = k-1$. But $P \cap N_{\delta-1} = 0$ because $N_{\delta-1}$ satisfies (iii). This gives the desired claim that $P \cap Q = 0$.

Suppose now that N_δ is a P-high submodule of M containing Q. Then $N_{\delta-1} \subset N_\delta$, which satisfies (ii). In fact, the checking of (i) is elementary for N_δ. As for (iii), using the fact that $Soc(H_\gamma(M)) \subset N_{\delta-1}$ for N_δ. Observe that N_δ also satisfies (iv) because $a \in N_\delta$ and $a' = u$ where $d(aR/a'R) = 1$. In order to see that (v) is valid, let us suppose that $u \in H_{\delta+t_\delta+1}(N_\delta)$. Then $u = x'$ where $d(xR/x'R) = 1$ and $x \in H_{\delta+t_\delta}(N)$. Thus $a = x+y$, where $y \in N_\delta \cap Soc(H_{\delta+t_\delta-1}(M))$ and $H_M(a) = \delta+t_\delta-1$. Therefore, $y \in H_\gamma(M)$ because of (iii). But $a = x + y \in H_{\delta+t_\delta}(M)$. This is a contradiction. Hence N_δ must satisfy (v), as promised.

We construct now a submodule N_1 of M, imitating the method of N_δ as demonstrated in the above paragraph. Therefore to finish the induction, we choose $v \in Soc(M)$ such that $H_M(v) = t_1$ and $Soc(H_{t_1}(M)) = Soc(H_\gamma(M)) \oplus P$ with $v \in P$. Let $Q = \langle Soc(H_\gamma(M)), a\rangle$, where $a = v+w$. If $0 \neq b \in P \cap Q$, then $b = c+ta$, where $c \in Soc(H_\gamma(M))$ and t is a positive integer. Bearing in mind this construction, it is apparent that $P \cap Q = 0$. Finally, we let N_1 be a P-high submodule of M, and a routine computations reveals that N_1 satisfies $(i) - (v)$. The proof is completed.

We next give an explicit definition of our main term.

Definition 3.2. *Let σ and λ be ordinals, we say a QTAG-module M is $(\sigma,\ \lambda)$-module if every σ-pure submodule of M is λ-pure.*

Now we have all the ingredients needed to establish the following.

Theorem 3.1. *Suppose σ and λ are ordinals with $\lambda > 0$ and M is a QTAG-module. Then M is a $(\sigma,\ \lambda)$-module if and only if M is h-divisible.*

Proof. Foremost, assume that M is h-divisible, that is $H_1(M) = M$. Knowing this, we yield that there is a submodule N of M such that $N \cap H_1(M) \subset H_\delta(N)$ for any ordinal $\delta > 0$. Hence, in particular, every σ-pure submodule of M is λ-pure and we are done.

Next, we deal with the converse implication. Assume that M is a $(\sigma,\ \lambda)$-module. If $H_1(M) \neq M$, then there exists an uniform element u containing $H_1(M)$ such that $e(u) = \infty$. Let $N = \langle u\rangle$. Then N is not h-pure in M. Henceforth, according to Lemma 3.2 , there is a submodule L of M such that L is not λ-pure for any $\lambda > 0$. Since L is isotype in M, we have $H_1(M) = M$, as required.

And so, we come to the following.

Theorem 3.2. *Suppose σ and λ are ordinals, $1 \leq \sigma < \lambda \leq \infty$ and M is a QTAG-module. If σ is a limit ordinal, then M is a (σ, λ)-module if and only if the following hold:*
(i) $H_\beta(Soc(H_k(M)) < \sigma$, for some $k > 0$
(ii) $H_\sigma(M) = U \oplus H_1(M)$, where U is a direct sum of uniserial modules of exponent k.

Proof. In virtue of Lemma 3.3 , the necessity is true. Suppose (ii) is not hold, then there exists an element $x \in H_{\sigma+1}(M)$ such that $e(x) = \infty$. After this, let us assume that (i)

is not hold, then $\sigma = r_\sigma \leq r_{\sigma+1}$. If we replace $\delta = \sigma$ and $\gamma = \sigma + 1$ in Lemma 3.3 , we get that a σ-pure submodule N_σ of M which is not $(\sigma + 1)$-pure. Henceforth, all the conditions are satisfied for M to be a $(\sigma,\ \lambda)$-module.

The sufficiency of (i) being self-evident from Lemma 3.1 , where we replace σ by $H_\beta(Soc(H_k(M))+1$. Let us assume that (ii) is hold and let N be a σ-pure submodule of M with $\sigma < \alpha \leq \lambda$. Without loss of generality, we assume that $y \in N \cap H_\alpha(M)$. Then $y \in N \cap H_1(M)$, since $H_\alpha(M) = H_1(M)$. From the δ-purity of N, we have $y \in H_\delta(N)$ for every ordinal δ. Consequently, $y \in H_\alpha(N)$. Thus N is λ-pure, as expected.

We continue in this way by the following.

Theorem 3.3. *Suppose σ and λ are ordinals, $1 \leq \sigma < \lambda \leq \infty$ and M is a $QTAG$-module. If $\sigma - 1$ exists, then M is a $(\sigma,\ \lambda)$-module if and only if the following hold:*
(i) $f_M(\delta) = 0$ if δ satisfies $\sigma \leq \delta + 1 < \lambda$.
(ii) $H_{\sigma-1}(M) = U \oplus V \oplus H_1(M)$, where U and V are direct sum of uniserial modules of exponent k and $k + 1$ respectively, for some $k > 0$.

Proof. First assume that M is a (σ, λ)-module such that (i) is not hold. Suppose now (ii) holds. Then $\sigma \leq H_\beta(Soc(H_k(M))$, for some k and an ordinal β. Thus by Definition 3.1 , there exists a parameter t_σ such that $f_{(H_{\sigma-1}(M))}(\delta) = 0$, for some $\delta < t_\sigma$. Let x be an uniform element of $H_{\sigma+t_\sigma+1}(M)$ such that $e(x) = \infty$. Then by Lemma 3.3 , there exists a $(\sigma + t_\sigma)$-pure submodule N_σ of $H_{\sigma-1}(M)$, which is not $(\sigma + t_\sigma + 1)$-pure. But this is impossible because $\sigma + t_\sigma + 1 \leq \lambda$. Utilizing the preceding point, it is straight forward to compute that

$$H_{\sigma+t_\sigma+1}(M) = H_{t_\sigma+2}(H_{\sigma-1}(M))$$

is a $QTAG$-module and besides it is direct sum of uniserial module. Let $k = t_\sigma + 1$. Then $f_M(\delta) \neq 0$ if $k-1 \leq \delta \leq k$ and the above condition on $H_{k+1}(H_{\sigma-1}(M))$ holds (ii), as needed.

Concerning the sufficiency, the first condition is straight forward from Lemma 3.1. As for the second condition, let N be a σ-pure submodule of M. Then N is $(\sigma + k - 1)$-pure, in conjunction with Lemma 3.1 , since $f_M(\delta) = 0$ for $\sigma \leq \delta + 1 < \sigma + k - 1$. In fact, for every ordinal α, we observe that $\sigma + k \leq \alpha \leq \lambda$ and choose $y \in N \cap H_\alpha(M)$. Since $H_\alpha(M) = H_{\alpha+k}(M) = H_1(M)$, we have $y \in H_\alpha(N)$. Hence N is a λ-pure submodule of M, as required.

We now settle the example to constructing extensions of (σ, λ)-module, which is parallel as assertion due to Moore and Hewett [14].

Example: Let M be a (σ, λ)-module with N a γ-pure submodule of M, for $\sigma \leq \gamma < \lambda$. One can easily constructs a submodule L such that L is σ-pure. Applying Lemma 3.2, L is λ-pure and, hence, δ-pure for $\sigma \leq \gamma < \lambda$. Thus, in view of Theorem 3.3, N is δ-pure, as required.

4. Open Problems

We close the work by formulating the following problems.

Problem 4.1. *Suppose M is a $QTAG$-module such that $M/H_\sigma(M)$ is a direct sum of uniserial modules and $1 \leq \sigma < \lambda \leq \infty$. Is then M (σ, λ)-module if and only if $H_\sigma(M)$ is?*

Problem 4.2. *If $1 \leq \sigma < \lambda \leq \infty$ and M is a (σ, λ)-module such that $M = \Sigma_{\lambda \in I} M_\lambda$, and N_λ is a λ-pure submodule of M_λ, then is it true that $\Sigma_{\lambda \in I} N_\lambda$ is a λ-pure submodule of M?*

Problem 4.3. *If $\omega \leq \lambda \leq \infty$. Can M is a (ω, λ)-module if and only if $M = M_1 \oplus M_2$, where M_1 is an h-divisible and M_2 is a direct sum of separable modules?*

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Department of Applied Mathematics, Faculty of Engineering and Technology, Aligarh Muslim University, Aligarh-202002, India,

College of Applied Industrial Technology, Jazan University, Jazan, P.O. Box 2097, Kingdom of Saudi Arabia,

Department of Applied Mathematics, Faculty of Engineering and Technology, Aligarh Muslim University, Aligarh-202002, India,