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QUASI HEMI-SLANT CONFORMAL SUBMERSIONS FROM KENMOTSU MANIFOLD

MOHAMMAD SHUAIB * AND TANVEER FATIMA

Abstract. We take into account the quasi hemi-slant conformal submersion from Kenmotsu manifold onto the Riemannian manifold as a generalization of anti-invariant submersions, semi-slant submersions, and hemi-slant submersions. We discussed the integrability and totally geodesicness of the different distributions. Moreover, we have obtained a condition under which the conformal hemi-slant submersions become a homothetic map.

Keywords: Kenmotsu manifolds, slant submersion, hemi-slant submersion, quasi-hemi slant submersion

2010 Mathematics Subject Classification: 53D15, 53D10.

1. Introduction

The concept discussed by B. O'Neill [26] and A Gray [16] is known as Riemannian submersions. In 1976, B. Watson [42], considered the submersion between almost Hermitian manifolds with name as almost Hermitian submersions. He established that, if the whole manifold is a Kaehler manifold, then the base manifold is also a Kaehler manifold. The Riemannian submersions consist many applications in mathematics and in physics, specially in Yang-Mills theory ([8],[44]), Kaluza-Klein theory ([9],[22]). The Riemannian submersions are very interesting tools in geometry to study Riemannian manifolds having differentiable structures. B. Sahin, in ([37], [39]), respectively, presented the idea of anti-invariant Riemannian

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* Corresponding author

Mohammad Shuaib; shuaibyousuf6@gmail.com; https://orcid.org/0000-0003-0022-893X

Tanveer Fatima; tansari@taibahu.edu.sa; https://orcid.org/0000-0001-9905-2107

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submersions and slant-submersion from virtually Hermitian manifold as a generalisation of Riemannian submersions.

The notion of almost contact Riemannian submersions from almost contact manifold was introduced by Chinea in [11]. He also studied the fibre space, base space and total space with differential geometric point of view. As a generalization of Riemannian submersions, Fuglede [15] and Ishihara [23] separately, studied horizontally conformal submersions. Later on, many authors investigated different kinds of Riemannian submersions like anti-invariant submersions ([5], [37]), slant submersions ([4], [39]), semi-slant submersions ([2], [19], [28]) and hemi-slant submersions ([43], [1]) between almost Hermitian manifolds as well as almost contact manifolds. R Prasad et al. ([31], [32], [33], [34]) studied Quasi-bi-slant submersion from Kenmotsu manifold onto Riemannian manifolds and they also studied Riemannian submersion from Kenmotsu manifolds with different aspect whereas Sezin [41] studied bi-slant submersions from contact manifold with taking ξ as horizontal vector field.

In this paper, we study quasi hemi-slant conformal submersions from Kenmotsu manifold onto a Riemannian manifold taking 4 mutually orthogonal complementary distributions. This paper contains 4 sections. Section 2 consists some definitions of almost contact metric manifold and specially kenmotsu manifold, In section 3, we study some basic results for quasi hemi-slant conformal submersion from Kenmotsu manifold which are needed for our main sections. Section 4 contains the results of integrability and totally geodesicness of distributions.

2. Preliminaries

Let M be a $(2n+1)$-dimensional almost contact manifold with almost contact structures (ϕ, ξ, η), where ϕ is a $(1,1)$ tensor field ξ, a vector field and η, a 1- form satisfying

$$\phi^2 = -I + \eta \otimes \xi, \ \eta(\xi) = 1, \ \eta \circ \phi = 0. \tag{2.1}$$

On an almost contact manifold, there exists a Riemannian metric g which is compatible with the almost contact structure (M, ϕ, ξ, η) in the sense that

$$g(\phi U, \phi V) = g(U, V) - \eta(U)\eta(V), \tag{2.2}$$

from which it can be observed that

$$g(U, \xi) = \eta(U), \tag{2.3}$$

for any $U,\ V \in \Gamma(TM)$ and the manifold (M, ϕ, ξ, η, g) is called an *almost contact metric manifold*. If $[\phi, \phi]$ denotes the Nijenhuis tensor of ϕ, then the almost contact structure is normal if and only if the torsion tensor $[\phi, \phi] + 2d\eta \otimes \xi$ vanishes. An almost contact metric structure is called a *contact metric structure* if $d\eta = \Phi$, where Φ is the fundamental 2-form defined by $\Phi(U, V) = g(U, \phi V)$. Almost contact metric structure (ϕ, ξ, η, g) are said to define a *Kenmotsu structure* on M if the following characterizing tensorial equation is satisfied

$$(\bar{\nabla}_U \phi)V = g(\phi U, V)\xi - \eta(V)\phi U. \tag{2.4}$$

One can deduce from the above relations that

$$\bar{\nabla}_U \xi = U - \eta(U)\xi. \tag{2.5}$$

It is also seen that

$$g(\phi U, V) = -g(U, \phi V). \tag{2.6}$$

The covariant derivative of ϕ is defined by

$$(\nabla_U \phi)V = \nabla_U \phi V - \phi \nabla_U V. \tag{2.7}$$

Now, we recall the notion of Riemannian submersion and horizontally conformal submersion followed by some basic results those will be useful throughout the text.

Definition 2.1. *Let (M, g) and (N, g') be two Riemannian manifolds and $F : M \to N$ be a smooth Riemannian submersion. Then F is called a horizontally conformal submersion, with a positive function λ such that*

$$g(X, Y) = \frac{1}{\lambda^2} g'(F_*X, F_*Y), \tag{2.8}$$

for any $X, Y \in \Gamma(ker F_*)^{\perp}$. It is clear that a horizontally conformal submersion with $\lambda = 1$ is Riemannian submersions.

Let $F : M \to N$ be a conformal submersion. A vector field E on M is called projectable if there exists a vector field $\bar{E}$ on N such that $F_*(E_p) = \bar{E}$ for any $p \in \Gamma(TM)$.

B. O' Neill defined the tensors $\mathcal{T}$ and $\mathcal{A}$ called fundamental tensors and defined by for vector fields E_1 and E_2 on M such that

$$\mathcal{A}_{E_1} E_2 = \mathcal{H}\nabla_{\mathcal{H}E_1} \mathcal{V}E_2 + \mathcal{V}\nabla_{\mathcal{H}E_1} \mathcal{H}E_2 \tag{2.9}$$

$$\mathcal{T}_{E_1} E_2 = \mathcal{H}\nabla_{\mathcal{V}E_1} \mathcal{V}E_2 + \mathcal{V}\nabla_{\mathcal{V}E_1} \mathcal{H}E_2 \tag{2.10}$$

where the vertical and horizontal projections are $\mathcal{V}$ and $\mathcal{H}$ respectively. Considering the equations (2.9) and (2.10), we have

$$\nabla_{U_1} V_1 = \mathcal{T}_{U_1} V_1 + \mathcal{V}\nabla_{U_1} V_1 \tag{2.11}$$

$$\nabla_{U_1} X_1 = \mathcal{T}_{U_1} X_1 + \mathcal{H}\nabla_{U_1} X_1 \tag{2.12}$$

$$\nabla_{X_1} U_1 = \mathcal{A}_{X_1} U_1 + \mathcal{V}\nabla_{X_1} U_1 \tag{2.13}$$

$$\nabla_{X_1} Y_1 = \mathcal{H}\nabla_{X_1} Y_1 + \mathcal{A}_{X_1} Y_1 \tag{2.14}$$

for any $U_1, V_1 \in \Gamma(kerF_*)$ and $X_1, Y_1 \in \Gamma(kerF_*)^\perp$.

For $q \in M$, $V \in \mathcal{V}_q$ and $X \in \mathcal{H}_q$, the linear operators $\mathcal{T}_V, \mathcal{A}_X : T_pM \to T_pM$ are skew-symmetric, that is

$$g(\mathcal{A}_X E_1, E_2) = -g(E_1, \mathcal{A}_X E_2) \tag{2.15}$$

$$g(\mathcal{T}_V E_1, E_2) = -g(E_1, \mathcal{T}_V E_2) \tag{2.16}$$

for any $E_1, E_2 \in \Gamma(T_pM)$.

Let (M, g) and (N, g') be two Riemannian manifolds. Let $\varphi : M \to N$ be a smooth map. Then, the second fundamental form of φ is given by

$$(\nabla\varphi_*)(X, Y) = \nabla^\varphi_X \varphi_* Y - \varphi_*(\nabla_X Y), \tag{2.17}$$

for all $X, Y \in \Gamma(T_pM)$, where ∇ the Levi-Civita connection of the metrics g and g' and ∇^φ is the pullback connection. The map φ is said to be totally geodesic map if $(\nabla\varphi_*)(U, V) = 0$ for any $U, V \in \Gamma(T_pM)$.

Lemma 2.1. *Let $F : M \to N$ be a horizontal conformal submersion. Then, for any horizontal vector fields X_1, Y_1 and vertical vector fields U_1, V_1*

(i) $(\nabla F_*)(X_1, Y_1) = X_1(ln\lambda)F_*(Y_1) + Y_1(ln\lambda)F_*(X_1) - g(X_1, Y_1)F_*(grad\ ln\lambda)$,
(ii) $(\nabla F_*)(U_1, V_1) = -F_*(\mathcal{T}_{U_1} V_1)$,
(iii) $(\nabla F_*)(X_1, U_1) = -F_*(\nabla_{X_1} U_1) = -F_*(\mathcal{A}_{X_1} U_1)$.

3. Quasi Hemi-slant Conformal Submersions

Definition 3.1. *A conformal submersion F from almost contact metric manifold (M, ϕ, ξ, η, g) onto a Riemannian manifold (N, g') is said to be a quasi hemi-slant conformal submersion (QHSC submersions) if its vertical distribution* kerF_* *of F admits four orthogonal complementary distributions D_T, D_θ, $D_\perp$ and $<\xi>$ such that*

(i) $\mathrm{kerF}_* = D_T \oplus D_\theta \oplus D_\perp \oplus <\xi>$

(ii) *D_T is invariant, i.e., $\phi D_T = D_T$*

(iii) *$D_\perp$ is anti-invariant, i.e., $\phi D_\perp \subseteq (\,ker F_*{}^\perp)$*

(iv) *for any non-zero vector field $X \in (D_\theta)_p, p \in M$, the angle θ between ϕX and $(D_\theta)_p$ is constant and independent of the choice of point p and X in $(D_\theta)_p$,*

where $<\xi>$ is 1-dimensional distribution spanned by ξ. Then, we say that F is QHSC submersion where angle θ is called the quasi hemi-slant angle of submersion. Here we have some particular cases which are stated as :

(i) If the distribution $D_T = 0$ then the map F is a conformal hemi-slant submersion.

(ii) If the distribution $D_\theta = 0$ then the map F is a conformal semi-invariant submersion.

(iii) If the distribution $D_\perp = 0$ then the map F is a conformal semi-slant submersion.

Hence, it is clear that the QHSC submersions are generalized version of conformal hemi-slant submersions, conformal semi-invariant submersions and conformal semi-slant submersions.

Let F be a QHSC submersion from an almost contact metric manifold (M, ϕ, ξ, η, g) onto a Riemannian manifold (N, g'). Then, for any $U \in \Gamma(ker F_*)$, we have

$$U = PU + QU + RU + \eta(U)\xi \tag{3.18}$$

where P, Q and R are the projections morphism onto D_T, D_θ and $D_\perp$. Now, For any $U \in \Gamma(ker F_*)$

$$\phi U = \beta U + \delta U \tag{3.19}$$

where $\beta U \in \Gamma(ker F_*)$ and $\delta U \in \Gamma((ker F_*)^\perp)$. From equations (3.18), (3.19) and definition 3.1, we have

$$\begin{aligned}\phi U &= \phi(PU) + \phi(QU) + \phi(RU)\\ &= \beta(PU) + \delta(PU) + \beta(QU) + \delta(QU) + \beta(RU) + \delta(RU)\end{aligned}$$

We obtain $\delta \bar{P} U = 0$ and $\beta \bar{R} U = 0$, we have

$$\phi U = \beta(PU) + \beta(QU) + \delta(QU) + \delta(RU).$$

Hence, we have the decomposition as :

$$\text{kerF}_*{}^{\perp} = \delta D_\theta \oplus \delta D_\perp \oplus \mu, \tag{3.20}$$

where μ is the orthogonal complementary distribution to $\delta D_\theta \oplus \delta D_\perp$ in $((\text{kerF}_*)^\perp)$ and μ is invariant with respect to ϕ. Now, for any $X \in (\Gamma(kerF_*)^\perp)$, we have

$$\phi X = BX + CX \tag{3.21}$$

where $BX \in \Gamma(kerF_*)$ and $CX \in \Gamma(\mu)$.

Lemma 3.1. *Let (M, ϕ, ξ, η, g) be a Kenmotsu manifold and (N, g') be a Riemannian manifold. If $F : M \to N$ is a QHSC submersion, then we have*

$$\delta BX + C^2X = X, \;\; \beta BX + BCX = 0$$

$$\beta^2U + B\delta U = U - \eta(U)\xi, \;\; \delta\beta U + C\delta U = 0$$

for $U \in \Gamma(\text{kerF}_*)$ and $X \in \Gamma((kerF_*)^\perp)$.

Proof. On using equations (2.1), (3.19) and (3.21), we get the desired results.

Lemma 3.2. [31] *Let F be a QHSC submersion from an almost contact metric manifold (M, ϕ, ξ, η, g) onto a Riemannian manifold (N, g'), then we have*

(i) $\beta^2U = -\cos^2\theta\, U$

(ii) $g(\beta U, \beta V) = \cos^2\theta\, g(U, V)$

(iii) $g(\delta U, \delta V) = \sin^2\theta\, g(U, V)$,

$U, V \in \Gamma(D_\theta)$.

Proof. The proof of above Lemma is similar to the proof of the Theorem (3.5) of [35].

Lemma 3.3. *Let (M, ϕ, ξ, η, g) be a Kenmotsu manifold and (N, g') be a Riemannian manifold. If $F : M \to N$ is a QHSC submersion, then we have*

$$\mathcal{A}_X BY + \mathcal{H}\nabla_X CY = \beta\mathcal{H}\nabla_X Y + B\mathcal{A}_X Y - g(\phi X, Y)\xi \tag{3.22}$$

$$\mathcal{V}\nabla_X BY + \mathcal{A}_X CY = \delta\mathcal{H}\nabla_X Y + C\mathcal{A}_X Y. \tag{3.23}$$

$$\mathcal{V}\nabla_X \beta V + \mathcal{A}_X \delta V = B\mathcal{A}_X V + \beta\mathcal{V}\nabla_X V + g(BX, V)\xi - \eta(V)BX \tag{3.24}$$

$$\mathcal{A}_X \beta V + \mathcal{H}\nabla_X \delta V = C\mathcal{A}_X V + \delta\mathcal{V}\nabla_X V + \eta(V)CX. \tag{3.25}$$

$$\mathcal{V}\nabla_V BX + \mathcal{T}_V CX = \beta\mathcal{T}_V CX + B\mathcal{H}\nabla_V X + g(\delta V, X)\xi \tag{3.26}$$

$$\mathcal{T}_V BX + \mathcal{H}\nabla_V CX = \delta \mathcal{T}_V X + C\mathcal{H}\nabla_V X. \tag{3.27}$$

$$\mathcal{V}\nabla_U \beta V + \mathcal{T}_U \delta V + \eta(V)\beta U = B\mathcal{T}_U V + \beta\mathcal{V}\nabla_U V + g(\phi U, V)\xi \tag{3.28}$$

$$\mathcal{T}_U \beta V + \mathcal{H}\nabla_U \delta V + \eta(V)\delta U = C\mathcal{T}_U V + \delta\mathcal{V}\nabla_U V. \tag{3.29}$$

for $U, V \in \Gamma(\ker F_*)$ *and* $X, Y \in \Gamma((\mathrm{kerF}_*)^{\perp})$.

Now we define the following :

$$(\nabla_U \beta)V = \mathcal{V}\nabla_U \beta V - \beta\mathcal{V}\nabla_U V \tag{3.30}$$

$$(\nabla_U \delta)V = \mathcal{H}\nabla_U \delta V - \delta\mathcal{V}\nabla_U V \tag{3.31}$$

$$(\nabla_X B)Y = \mathcal{V}\nabla_X BY - B\mathcal{H}\nabla_X Y \tag{3.32}$$

$$(\nabla_X C)Y = \mathcal{H}\nabla_X CY - C\mathcal{H}\nabla_X Y \tag{3.33}$$

for $U, V \in \Gamma(\ker F_*)$ and $X, Y \in \Gamma((\ker F_*)^{\perp})$.

Lemma 3.4. *Let* (M, ϕ, ξ, η, g) *be Kenmotsu manifold and* (N, g') *be a Riemannian manifold. If* $F : M \to N$ *is a QHSC submersion, then we have*

$$(\nabla_U \beta)V = B\mathcal{T}_U V - \mathcal{T}_U \delta V + g(\phi U, V)\xi - \eta(V)\beta U$$

$$(\nabla_U \delta)V = C\mathcal{T}_U V - \mathcal{T}_U \beta V - \eta(V)\delta U$$

$$(\nabla_X B)Y = \beta\mathcal{A}_X Y - \mathcal{A}_X CY + g(\phi X, Y)\xi - \eta(Y)BX$$

$$(\nabla_X C)Y = \delta\mathcal{A}_X Y - \mathcal{A}_X BY - \eta(Y)CX,$$

for $U, V \in \Gamma(\mathrm{kerF}_*)$ *and* $X, Y \in \Gamma((\,ker F_*{}^{\perp}))$.

Proof. On using equations (2.7), (2.11)- (2.14), equations (3.19)-(3.21) and equations (3.30)-(3.32), we get the proof of the lemma.

If the tenors β and δ are parallel with respect to the connection ∇ of M, then we have

$$B\mathcal{T}_U V = \mathcal{T}_U \delta V - g(\phi U, V)\xi + \eta(V)\beta U$$

$$C\mathcal{T}_U V = \mathcal{T}_U \delta V + \eta(V)\delta U$$

for $X, Y \in \Gamma(TM)$.

4. Integrability and totally geodesicness of distributions

Now, we start the discussion of the integrability of distributions and firstly we finding out the integrability of slant distribution as follows:

Theorem 4.1. *Let $F : (M, \phi, \xi, \eta, g) \to (N, g')$ be QHSC submersion from a Kenmotsu manifold onto a Riemannian manifold N. Then D_θ is integrable if and only if*

$$g'((\nabla F_*)(V_1, \beta P\alpha), F_*(\delta V_2)) + g'((\nabla F_*)(V_2, \beta P\alpha), F_*(\delta V_1))$$
$$= \lambda^2 g(\mathcal{V}\nabla_{V_1}\beta\mathcal{P}\alpha - \mathcal{T}_{V_1}\delta\mathcal{R}\alpha, \beta V_2)$$
$$+ \lambda^2 g(\mathcal{V}\nabla_{V_2}\beta\mathcal{P}\alpha - \mathcal{T}_{V_2}\delta\mathcal{R}\alpha, \beta V_1)$$
$$+ g(\mathcal{H}\nabla_{V_2}\delta\mathcal{R}\alpha, \delta V_1) - g(\mathcal{H}\nabla_{V_1}\delta\mathcal{R}\alpha, \delta V_2)$$

for any $V_1, V_2 \in \Gamma(D_\theta)$ and $\alpha \in \Gamma(D_T \oplus D_\perp \oplus <\xi>)$.

Proof. For any $V_1, V_2 \in \Gamma(D_\theta)$ and $\alpha \in \Gamma(D_T \oplus D_\perp \oplus <\xi>)$ with using (2.2).(2.7) and (2.4), we get

$$g([V_1, V_2], \alpha) = g(\nabla_{V_2}\phi\alpha, \phi V_1) - g(\nabla_{V_1}\phi\alpha, \phi V_2).$$

Taking equation (3.18), we have

$$g([V_1, V_2], \alpha) = g(\nabla_{V_2}\beta P\alpha, \beta V_1) + g(\nabla_{V_2}\delta R\alpha, \phi V_1)$$
$$- g(\nabla_{V_1}\beta P\alpha, \phi V_2) - g(\nabla_{V_1}\delta R\alpha, \phi V_2).$$

From (2.11) and (2.12), we can write

$$g([V_1, V_2], \alpha) = g(\mathcal{T}_{V_1}\beta P\alpha - \mathcal{H}\nabla_{V_1}\delta R\alpha, \delta V_2)$$
$$+ g(\mathcal{V}\nabla_{V_1}\beta P\alpha - \mathcal{T}_{V_1}\delta R\alpha, \beta V_2)$$
$$+ g(\mathcal{T}_{V_2}\beta P\alpha - \mathcal{H}\nabla_{V_2}\delta R\alpha, \delta V_1)$$
$$+ g(\mathcal{V}\nabla_{V_2}\beta P\alpha - \mathcal{T}_{V_2}\delta R\alpha, \beta V_1).$$

Considering equation (2.17), we may write

$$g([V_1, V_2], \alpha) = g(\mathcal{V}\nabla_{V_2}\beta P\alpha - \mathcal{T}_{V_2}\delta R\alpha, \beta V_1)$$
$$+ g(\mathcal{V}\nabla_{V_1}\beta P\alpha - \mathcal{T}_{V_1}\delta R\alpha, \beta V_2)$$
$$- g(\mathcal{H}\nabla_{V_1}\delta R\alpha, \delta V_2) + g(\mathcal{H}\nabla_{V_2}\delta R\alpha, \delta V_1)$$
$$- \frac{1}{\lambda^2} g'((\nabla F_*)(V_!, \beta P\alpha), F_*(\delta V_2))$$
$$- \frac{1}{\lambda^2} g'((\nabla F_*)(V_2, \beta P\alpha), F_*(\delta V_1))$$

from which we get the desired result.

Theorem 4.2. *Let $F : (M, \phi, \xi, \eta, g) \to (N, g')$ be QHSC submersion from a Kenmotsu manifold onto a Riemannian manifold N. Then invariant distribution D_T is integrable if and only if*

$$P(\mathcal{V}\nabla_{U_1}\beta Q\alpha + \mathcal{T}_{U_1}\delta\alpha) = 0 \tag{4.34}$$

for $U_1 \in \Gamma(D_T)$ and $\alpha \in \Gamma(D_\theta \oplus D_\perp \oplus <\xi>)$.

Proof. On using (2.2), (2.4) and (3.18), we have

$$g(\nabla_{U_1}U_2, \alpha) = -g(\nabla_{U_1}(\phi Q\alpha + \phi R\alpha), \phi U_2) - \eta(\alpha)g(\phi U_1, \phi U_2),$$

for $U_1 \in \Gamma(D_T)$ and $\alpha \in \Gamma(D_\theta \oplus D_\perp \oplus <\xi>)$. Since $\delta(Q\alpha + R\alpha) = \delta\alpha$ and from (2.11), (2.12), we can write

$$\begin{aligned} g(\nabla_{U_1}U_2, \alpha) &= -g(\mathcal{V}\nabla_{U_1}\beta Q\alpha, \phi U_2) - g(\mathcal{T}_{U_1}\delta\alpha, \phi U_2) \\ &- \eta(\alpha)g(\phi U_1, \phi U_2) \end{aligned}$$

Change the role of U_1 and U_2, we have

$$\begin{aligned} g([U_1, U_2], \alpha) &= -g(\mathcal{V}\nabla_{U_1}\beta Q\alpha + \mathcal{T}_{U_1}\delta\alpha, \phi U_2) \\ &+ g(\mathcal{V}\nabla_{U_2}\beta Q\alpha + \mathcal{T}_{U_2}\delta\alpha, \phi U_1). \end{aligned}$$

We obtain the proof of the theorem from above equation.

Theorem 4.3. *Let $F : (M, \phi, \xi, \eta, g) \to (N, g')$ be QHSC submersion from a Kenmotsu manifold onto a Riemannian manifold N. Then anti-invariant distribution $D_\perp$ is integrable if and only if*

$$\begin{aligned} &\frac{1}{\lambda^2}[g'(\nabla_{Z_2}F_*\delta Q\alpha, F_*(\delta Z_1)) - g'(\nabla_{Z_1}F_*\delta Q\alpha, F_*(\delta Z_2))] \\ &= g(grad(\ln\lambda), Z_1)g(\delta Q\alpha, \delta Z_2) \\ &- g(grad(\ln\lambda), Z_2)g(\delta Q\alpha, \delta Z_1) \\ &- g(\mathcal{T}_{Z_2}\beta\alpha, \delta Z_1) + g(\mathcal{T}_{Z_1}\beta\alpha, \delta Z_2) \end{aligned} \tag{4.35}$$

for $Z_1, Z_2 \in \Gamma(D_\perp)$ and $\alpha \in \Gamma(D_T \oplus D_\theta \oplus <\xi>)$.

Proof. From (2.2), (2.3), (2.4) and (3.18), we have

$$g(\nabla_{Z_1}Z_2, \alpha) = -\eta(\alpha)g(Z_1, \phi Z_2) - g(\nabla_{Z_1}(\beta P\alpha + \beta Q\alpha + \delta R\alpha), \phi Z_2).$$

Since $\beta(P\alpha + Q\alpha) = \beta\alpha$, we can write

$$g(\nabla_{Z_1}Z_2, \alpha) = -\eta(\alpha)g(Z_1, \phi Z_2) - g(\nabla_{Z_1}\beta\alpha + \nabla_{Z_1}\delta Q\alpha, \delta Z_2).$$

Now, change the roles of Z_1 and Z_2, we can write

$$g([Z_1, Z_2], \alpha) = g(\nabla_{Z_2}\beta\alpha + \nabla_{Z_1}\delta Q\alpha, \delta Z_1) - g(\nabla_{Z_1}\beta\alpha + \nabla_{Z_2}\delta Q\alpha, \delta Z_2).$$

Considering equations (2.11) and (2.12), we get

$$\begin{aligned} g([Z_1, Z_2], \alpha) &= g(\nabla_{Z_2}\beta\alpha, \delta Z_1) + g(\mathcal{H}\nabla_{Z_2}\delta Q\alpha, \delta Z_2) \\ &\quad - g(\mathcal{T}_{Z_1}\beta\alpha, \delta Z_2) + g(\mathcal{H}\nabla_{Z_1}\delta Q\alpha, \delta Z_2). \end{aligned}$$

From (2.8), (2.17) and lemma 2.1, we have

$$\begin{aligned} g([Z_1, Z_2], \alpha) = &\frac{1}{\lambda^2}[g'(\nabla_{Z_2}F_*\delta Q\alpha, F_*(\delta Z_1)) - g'(\nabla_{Z_1}F_*\delta Q\alpha, F_*(\delta Z_2))] \\ &+ g(\mathcal{T}_{Z_2}\beta\alpha, \delta Z_1) - g(\mathcal{T}_{Z_1}\beta\alpha, \delta Z_2) \\ &+ g(grad(\ln\lambda), Z_2)g(\delta Q\alpha, \delta Z_1) \\ &- g(grad(\ln\lambda), Z_1)g(\delta Q\alpha, \delta Z_2) \end{aligned}$$

which completes the proof of the theorem.

Now, we will discussed the totally geodesicness of fibers of the distributions. Firstly, we will start with the totally geodesicness of the invariant distribution D_T.

Theorem 4.4. *Let $F : (M, \phi, \xi, \eta, g) \to (N, g')$ be QHSC submersion from a Kenmotsu manifold onto a Riemannian manifold N. Then D_T is not totally geodesic.*

Proof. On considering $U, V \in \Gamma(D_T)$ and since V and ξ are orthogonal, we have

$$g(\nabla_U V, \xi) = -g(V, \nabla_U \xi)$$

Taking account the fact of equation (2.5), we have

$$g(\nabla_U V, \xi) = -g(U, V).$$

For $U, V \in \Gamma(D_T), -g(U, V) \neq 0$, that is $g(\nabla_U V, \xi) \neq 0$. Hence, the distribution is not totally geodesic.

Theorem 4.5. *Let $F : (M, \phi, \xi, \eta, g) \to (N, g')$ be QHSC submersion from a Kenmotsu manifold onto a Riemannian manifold N. Then $(D_T \oplus \xi)$ defines totally geodesic foliation on M if and only if*

(i) $g(\mathcal{V}\nabla_{U_1}\phi U_2, \beta\alpha) = \frac{1}{\lambda^2}[g'(\nabla F_*)(U_1, \phi U_2), F_*(\delta\alpha)]$

(ii) $g(\mathcal{V}\nabla_{U_1}\phi U_2, BX) = \frac{1}{\lambda^2}[g'((\nabla F_*)(U_1, \phi U_2), F*(CX))]$

for $U_1, U_2 \in \Gamma(D_T \oplus <\xi>), X \in \Gamma((kerF_*)^\perp)$ *and* $\alpha \in \Gamma(D_\theta \oplus D_\perp)$.

Proof. On using (2.2), (2.4) and (2.7), we get

$$g(\nabla_{U_1}U_2, \alpha) = g(\nabla_{U_1}\phi U_2, \phi\alpha),$$

for any $U_1, U_2 \in \Gamma(D_T \oplus <\xi>)$ and $\alpha \in \Gamma(D_\theta \oplus D_\perp)$. Now, from (2.11) and decomposition (3.19), we can write

$$g(\nabla_{U_1}U_2, \alpha) = g(\nabla_{U_1}\phi U_2, \delta\alpha) + g(\mathcal{V}\nabla_{U_1}\phi U_2, \beta\alpha).$$

Considering (2.8) and (2.17), we may have

$$g(\nabla_{U_1}U_2, \alpha) = -\frac{1}{\lambda^2}g'((\nabla F_*)(U_1, \phi U_2), F_*(\delta\alpha)) + g(\mathcal{V}\nabla_{U_1}\phi U_2, \beta\alpha) \tag{4.36}$$

On the other hand, for $U_1, U_2 \in \Gamma(D_T)$ and $X \in \Gamma((kerF_*)^\perp)$ with using (2.2), (2.4), (2.7) and decomposition (3.21), we get

$$g(\nabla_{U_1}U_2, X) = g(\nabla_{U_1}\phi U_2, BX) + g(\nabla_{U_1}\phi U_2, CX).$$

Considering equation (2.11), we may write

$$g(\nabla_{U_1}U_2, X) = g(\mathcal{V}\nabla_{U_1}\phi U_2, BX) + g(\mathcal{T}_{U_1}\phi U_2, CX).$$

From (2.17) and (2.17), we have

$$g(\nabla_{U_1}U_2, X) = g(\mathcal{V}\nabla_{U_1}\phi U_2, BX) + \frac{1}{\lambda^2}g'((\nabla F_*)(U_1, \phi U_2), F_*(CX)). \tag{4.37}$$

From equations (4.36) and (4.37), we get (i) and (ii) part of theorem 4.5.

Theorem 4.6. *Let* $F : (M, \phi, \xi, \eta, g) \to (N, g')$ *be QHSC submersion from a Kenmotsu manifold onto a Riemannian manifold* N. *Then* D_θ *is not totally geodesic on* M.

Proof. On considering $Z, W \in \Gamma(D_\theta)$ and since W and ξ are orthogonal, we have

$$g(\nabla_Z W, \xi) = -g(W, \nabla_Z \xi)$$

Taking account the fact of equation (2.5), we have

$$g(\nabla_Z W, \xi) = -g(Z, W).$$

For $Z, W \in \Gamma(D_\theta), -g(Z, W) \neq 0$, that is $g(\nabla_Z W, \xi) \neq 0$. Hence, the distribution is not totally geodesic.

Theorem 4.7. *Let $F : (M, \phi, \xi, \eta, g) \to (N, g')$ be QHSC submersion from a Kenmotsu manifold onto a Riemannian manifold N. Then $\Gamma(D_\theta \oplus <\xi>)$ defines totally geodesic foliation if and only if*

(i) $$\lambda^2[g(\mathcal{H}\nabla_{V_1}\delta QV_2, \phi R\alpha) - \cos^2\theta g(\nabla_{V_1}QV_2, \alpha) = g'((\nabla F_*)(V_1, \alpha), F_*(\delta\beta QV_2)) - g'((\nabla F_*)(V_1, \phi P\alpha), F_*(\delta QV_2)) - \eta(\beta QV_2)g(\phi V_1, \alpha)]$$

(ii) $$\lambda^2[g(\mathcal{H}\nabla_{V_1}\delta\beta QV_2, X) - g(\mathcal{H}\nabla_{V_1}\delta QV_2, CX) - \eta(\beta QV_2)g(V_1, BX)] = g'(\nabla F_*)(V_1, BX), F_*(\delta QV_2) - g'((\nabla F_*)(V_1, QV_2), F_*(X))$$

for any $V_1, V_2 \in \Gamma(D_\theta \oplus <\xi>)$, $X \in \Gamma((kerF_*)^\perp)$ and $\alpha \in \Gamma(D_T \oplus D_\perp)$.

Proof. From equations (2.2), (2.4), (3.18) and decomposition (3.19), we get

$$g(\nabla_{V_1}V_2, \alpha) = g(\nabla_{V_1}\beta QV_2, \phi\alpha) + g(\nabla_{V_1}\delta QV_2, \phi\alpha)$$

for any $V_1, V_2 \in \Gamma(D_\theta \oplus <\xi>)$ and $\alpha \in \Gamma(D_T \oplus D_\perp)$. Again on using (2.4) and (2.7), we can write

$$g(\nabla_{V_1}V_2, \alpha) = g(\nabla_{V_1}\delta QV_2, \phi P\alpha + \phi R\alpha) - g(\nabla_{V_1}\phi\beta QV_2, \alpha) - \eta(\beta QV_2)g(\phi V_1, \alpha)$$

Considering lemma 3.2, equation (2.12) and skew symmetry property of $\mathcal{T}$, we have

$$g(\nabla_{V_1}V_2, \alpha) = -\cos^2\theta g(\nabla_{V_1}QV_2, \alpha) + g(\mathcal{H}\nabla_{V_1}\delta QV_2, \phi R\alpha) + g(\mathcal{T}_{V_1}\alpha, \delta\beta QV_2) - g(\mathcal{T}_{V_1}\phi P\alpha, \delta QV_2) - \eta(\beta QV_2)g(\phi V_1, \alpha)$$

Finally, from equations (2.8) and (2.17), we yield

$$\begin{aligned} g(\nabla_{V_1}V_2, \alpha) = & -\cos^2\theta g(\nabla_{V_1}QV_2, \alpha) + g(\mathcal{H}\nabla_{V_1}\delta QV_2, \phi R\alpha) \\ & - \frac{1}{\lambda^2}g'((\nabla F_*)(V_1, \alpha), F_*(\delta\beta QV_2)) \\ & + \frac{1}{\lambda^2}g'((\nabla F_*)(V_1, \phi P\alpha), F_*(\delta QV_2)) \\ & - \eta(\beta QV_2)g(\phi V_1, \alpha). \end{aligned} \tag{4.38}$$

In similar way, for any $V_1, V_2 \in \Gamma(D_\theta)$ and $X \in \Gamma((kerF_*)^\perp)$ with using (2.2), (2.4), (2.7) and (3.19), we get

$$g(\nabla_{V_1}V_2, X) = g(\nabla_{V_1}\beta QV_2, \phi X) - g(\nabla_{V_1}\delta QV_2, \phi X).$$

From equation (2.2), (2.4), (2.7) and (3.19), (2.11) we can write

$$\begin{aligned} g(\nabla_{V_1}V_2, X) &= -g(\nabla_{V_1}\beta^2 QV_2, X) - g(\nabla_{V_1}\delta\beta QV_2, X) \\ &\quad + g(\mathcal{T}_{V_1}\delta QV_2, BX) + g(\mathcal{H}\nabla_{V_1}\delta QV_2, CX) \\ &\quad + \eta(\beta QV_2)g(V_1, BX) \end{aligned}$$

At last, considering equation (2.8), (2.17), (2.12), and lemma 3.2, we have

$$\begin{aligned} g(\nabla_{V_1}V_2, X) &= \frac{1}{\lambda^2}g'((\nabla F_*)(V_1, BX), F_*(\delta QV_2)) \\ &\quad - \cos^2\theta\frac{1}{\lambda^2}g'((\nabla F_*)(V_1, QV_2), F_*(X)) \\ &\quad - g(\mathcal{H}\nabla_{V_1}\delta\beta QV_2, X) + g(\mathcal{H}\nabla_{V_1}\delta QV_2, CX) \\ &\quad + \eta(\beta QV_2)g(V_1, BX). \end{aligned} \tag{4.39}$$

Finally, from equation (4.38) and (4.39), we get the results (i) and (ii) of theorem 4.7. This completes the proof of theorem.

Theorem 4.8. *Let $F : (M, \phi, \xi, \eta, g) \to (N, g')$ be QHSC submersion from a Kenmotsu manifold onto a Riemannian manifold N. Then $D_\perp$ is not defined totally geodesic foliation on M.*

Proof. On considering $Z, W \in \Gamma(D_\perp)$ and since W and ξ are orthogonal, we have

$$g(\nabla_Z W, \xi) = -g(W, \nabla_Z \xi)$$

Taking account the fact of equation (2.5), we have

$$g(\nabla_Z W, \xi) = -g(Z, W).$$

For $Z, W \in \Gamma(D_\perp), -g(Z, W) \neq 0$, that is $g(\nabla_Z W, \xi) \neq 0$. Hence, the distribution is not totally geodesic.

Theorem 4.9. *Let $F : (M, \phi, \xi, \eta, g) \to (N, g')$ be QHSC submersion from a Kenmotsu manifold onto a Riemannian manifold N. Then $D_\perp \oplus < \xi >$ defines totally geodesic foliation if and only if*

(i) $\frac{1}{\lambda^2}g'((\nabla F_*)(Z_1, \beta\alpha), F_*(\phi Z_2)) = g(\mathcal{H}\nabla_{Z_1}\phi Z_2, \delta Q\alpha)$

(ii) $\frac{1}{\lambda^2}g'((\nabla F_*)(Z_1, BX), F_*(\phi Z_2)) = g(\mathcal{H}\nabla_{Z_1}CX, \phi Z_2)$

for any $Z_1, Z_2 \in \Gamma(D_\perp \oplus < \xi >), X \in \Gamma((ker F_*)^\perp)$ and $\alpha \in \Gamma(D_T \oplus D_\theta)$.

Proof. On using equations (2.2), (2.4), (2.7), we can write

$$g(\nabla_{Z_1}Z_2,\alpha)=g(\nabla_{Z_1}\phi Z_2,\phi\alpha),$$

for any $Z_1, Z_2 \in \Gamma(D_\perp \oplus <\xi>)$ and $\alpha \in \Gamma(D_T \oplus D_\theta)$. On using the fact that $\beta P\alpha + \beta Q\alpha = \beta\alpha$ with equations (3.18), (2.11), we get

$$g(\nabla_{Z_1}Z_2,\alpha)=g(\mathcal{T}_{Z_1}\phi Z_2,\beta\alpha)+g(\mathcal{H}\nabla_{Z_1}\phi Z_2,\delta Q\alpha).$$

Considering equation (2.8) and (2.17) and use anti-symmetric property of $\mathcal{T}$, we have

$$g(\nabla_{Z_1}Z_2,\alpha)=\frac{1}{\lambda^2}g'((\nabla F_*)(Z_1,\beta\alpha),F_*(\phi Z_2))+g(\mathcal{H}\nabla_{Z_1}\phi Z_2,\delta Q\alpha). \tag{4.40}$$

On the other hand, for any $Z_1, Z_2 \in \Gamma(D_\perp)$ and $X \in \Gamma((kerF_*)^\perp)$ with using equations (2.2), (2.4), (2.7) and (3.21), we have

$$g(\nabla_{Z_1}Z_2,X)=-g(\nabla_{Z_1}BX,\phi Z_2)-g(\nabla_{Z_1}CX,\phi Z_2).$$

Considering equations (2.8), (2.11), (2.12) and (2.17), we can write

$$g(\nabla_{Z_1}Z_2,X)=\frac{1}{\lambda^2}g'((\nabla F_*)(Z_1,BX),F_*(\phi Z_2))-g(\mathcal{H}\nabla_{Z_1}CX,\phi Z_2). \tag{4.41}$$

From equations (4.40) and (4.41), the proof of the theorem is complete.

Theorem 4.10. *Let $F:(M,\phi,\xi,\eta,g)\to(N,g')$ be QHSC submersion from a Kenmotsu manifold onto a Riemannian manifold N. Then the vertical distribution $(kerF_*)$ defines totally geodesic foliation if and only if*

$$\begin{aligned}&\cos^2\theta\frac{1}{\lambda^2}g'((\nabla F_*)(Y_1,QY_2),F_*(X))+\frac{1}{\lambda^2}g'((\nabla F_*)(Y_1,\beta PY_2),F_*(CX))\\&=g(\mathcal{V}\nabla_{Y_1}\beta PY_2+\mathcal{T}_{Y_1}\delta QY_2+\mathcal{T}_{Y_1}\delta RY_2,BX)\\&\ +g(\mathcal{H}\nabla_{Y_1}\delta QY_2+\mathcal{H}\nabla_{Y_1}\delta RY_2,CX)-g(\mathcal{H}\nabla_{Y_1}\delta\beta QY_2,X),\end{aligned}$$

for any $Y_1, Y_2 \in \Gamma(kerF_*)$ and $X \in \Gamma((kerF_*)^\perp)$.

Proof. On using (2.2), (2.4) and (2.7) with decomposition (3.18), we have

$$g(\nabla_{Y_1}Y_2,X)=g(\nabla_{Y_1}\beta PY_2+\beta QY_2+\delta QY_2+\delta RY_2,\phi X),$$

for any $Y_1, Y_2 \in \Gamma(kerF_*)$ and $X \in \Gamma((kerF_*)^\perp)$. From equations (2.11), (2.12) and (3.21), we yield

$$\begin{aligned} g(\nabla_{Y_1}Y_2, X) &= g(\mathcal{V}\nabla_{Y_1}\beta PY_2 + \mathcal{T}_{Y_1}\delta QY_2 + \mathcal{T}_{Y_1}\delta RY_2, BX) \\ &+ g(\mathcal{T}_{Y_1}\beta PY_2 + \mathcal{H}\nabla_{Y_1}\delta QY_2 + \mathcal{H}\nabla_{Y_1}\delta RY_2, CX) \\ &+ g(\nabla_{Y_1}\beta QY_2, \phi X). \end{aligned}$$

Taking with equations (2.4), (2.7) and (3.18), we may have

$$\begin{aligned} g(\nabla_{Y_1}Y_2, X) &= g(\mathcal{V}\nabla_{Y_1}\beta PY_2 + \mathcal{T}_{Y_1}\delta QY_2 + \mathcal{T}_{Y_1}\delta RY_2, BX) \\ &+ g(\mathcal{T}_{Y_1}\beta PY_2 + \mathcal{H}\nabla_{Y_1}\delta QY_2 + \mathcal{H}\nabla_{Y_1}\delta RY_2, CX) \\ &- g(\nabla_{Y_1}\beta^2 QY_2, X) - g(\nabla_{Y_1}\delta\beta QY_2, X). \end{aligned}$$

Consider lemma 3.2 with equations (2.8) and (2.17), we get

$$\begin{aligned} g(\nabla_{Y_1}Y_2, X) &= g(\mathcal{V}\nabla_{Y_1}\beta PY_2 + \mathcal{T}_{Y_1}\delta QY_2 + \mathcal{T}_{Y_1}\delta RY_2, BX) \\ &+ g(\mathcal{H}\nabla_{Y_1}\delta QY_2 + \mathcal{H}\nabla_{Y_1}\delta RY_2, CX) \\ &+ \cos^2\theta g(\nabla_{Y_1}QY_2, X) - g(\nabla_{Y_1}\delta\beta QY_2, X) \\ &- \frac{1}{\lambda^2}g'((\nabla F_*)(Y_1, \beta PY_2), F_*(CX)). \end{aligned}$$

Again using (2.8) and (2.17), we finally have

$$\begin{aligned} g(\nabla_{Y_1}Y_2, X) &= g(\mathcal{V}\nabla_{Y_1}\beta PY_2 + \mathcal{T}_{Y_1}\delta QY_2 + \mathcal{T}_{Y_1}\delta RY_2, BX) \\ &+ g(\mathcal{H}\nabla_{Y_1}\delta QY_2 + \mathcal{H}\nabla_{Y_1}\delta RY_2, CX) \\ &+ \cos^2\theta \frac{1}{\lambda^2}g'((\nabla F_*)(Y_1, QY_2), F_*X) - g(\nabla_{Y_1}\delta\beta QY_2, X) \\ &- \frac{1}{\lambda^2}g'((\nabla F_*)(Y_1, \beta PY_2), F_*(CX)). \end{aligned}$$

This completes the proof of the theorem.

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Department of Mathematics, School of Chemical Engineering and Physical Sciences, Lovely Professional University, Punjab-India.

Department of Mathmatics and Statics, Taibah University-KSA.