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A NOTE ON $CSI-$SUBMERSIONS FROM COSYMPLECTIC MANIFOLDS

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ABSTRACT. In this paper, our main objective is to study the notion of Clairaut semi-invariant submersions ($CSI-$ submersions, in short) from Cosymplectic manifolds onto Riemannian manifolds. We investigate some fundamental results pertaining to the geometry of such submersions. We also obtain totally geodesicness conditions for the distributions. Moreover, we provide a non-trivial example of such Riemannian submersion.

Keywords: Riemannian submersions, Clairaut semi-invariant submersions, Almost contact metric manifolds.

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1. Introduction

Firstly, O' Neill [16] and Gray [9] separately studied the concept of Riemannian submersions between Riemannian manifolds in the 1960s. Using the notion of Riemannian submersions between almost complex manifolds, Watson [34] studied almost Hermitian submersions. Further, the concept of anti-invariant submersion was first defined by Sahin [23] from almost Hermitian manifolds onto Riemannian manifolds. Later, he introduced semi-invariant submersion [25] from almost Hermitian manifolds onto Riemannian manifolds as a generalization of holomorphic submersions and anti-invariant submersion. Further, different kinds of Riemannian submersions on different structures have been studied, such as: slant submersions

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[24], semi-slant submersions [17], conformal semi-slant submersion ([12],[20]), hemi-slant Riemannian submersions [31], conformal hemi-slant submersion [11], quasi-bi-slant submersions [18] (see also [13], [19], [21], [26], [28], [29]) etc.

Presently, the Riemannian submersions have abundant applications in pure mathematics and physics, for example, Kaluza-Klein theory [7], Yang-Mills theory [8], Supergravity and superstring theories [10] etc. C. Altafini [2] commenced using the notion of Riemannian submersions for the modeling and control of redundant robotic chain and proved that Riemannian submersion gives a close relationship between inverse kinematic in robotics and the pull back vectors.

In the theory of surfaces created by rotating the curves, we note that, for any geodesic $c(c : I_1 \subset R \rightarrow \mathcal{N}_1$ on $\mathcal{N}_1)$ on the rotating surface $\mathcal{N}_1$, the product $r \sin \Theta$ is constant along geodesic c, where $\Theta(s)$ is the angle between $c(s)$ and the meridian curve through $c(s)$, $s \in I_1$, called Clairaut's theorem [5]. It means that it is independent of s. In 1972, Bishop [5] applied this idea to the Riemannian submersions and introduced the concept of Clairaut submersion. Afterwards, Clairaut submersions have been studied in Spacelike spaces, Timelike and Lorentzian spaces ([15], [32], [33]) and its applications in Static spacetimes [1]. Later on this notion has been generalized in [3] and [15]. Kumar et al., in [14], introduce the notion of Clairaut semi-invariant Riemannian map and Gupta and Singh in [22] initiate the notion of Clairaut semi-invariant submersion from Kähler manifold and investigate some interesting geometric properties of these submersions.

In the present paper, our focus is on investigating the notion of the $CSI-$submersions from Cosymplectic manifolds onto Riemannian manifolds. The paper is organized as follows: In the second section, we gather main notions and formulae for other sections. In the third section, we give the definition of the $CSI-$submersions from Cosymplectic manifolds onto Riemannian manifolds. We investigate differential geometric properties of such submersions. In the last section, we illustrate a non-trivial example of the $CSI-$submersions from Cosymplectic manifolds onto Riemannian manifolds.

2. Preliminaries

A $(2n+1)$-dimensional smooth manifold $\mathcal{N}_1$ is said to have an almost contact structure [26] if there exist on $\mathcal{N}_1$, a tensor field ϕ of type $(1,1)$, a vector field ξ and 1-form η such that

$$\phi^2 = -I + \eta \otimes \xi, \; \eta \circ \phi = 0, \quad \phi\xi = 0, \tag{2.1}$$

$$g_1(\xi,\xi)=\eta(\xi)=1. \tag{2.2}$$

If there exists a Riemannian metric g_1 on an almost contact manifold $\mathcal{N}_1$ satisfying:

$$g_1(\phi Z_1,\phi Z_2) = g_1(Z_1,Z_2)-\eta(Z_1)\eta(Z_2), \tag{2.3}$$
$$g_1(Z_1,\phi Z_2) = -g_1(\phi Z_1,Z_2),$$

$$g_1(Z_1,\xi)=\eta(Z_1), \tag{2.4}$$

where Z_1, Z_2 are any vector fields on $\mathcal{N}_1$. Then $\mathcal{N}_1$ is called almost contact metric manifold [6] with almost contact structure (ϕ,ξ,η) and is represented by $(\mathcal{N}_1,\phi,\xi,\eta,g_1)$.

An almost contact structure (ϕ,ξ,η) is said to be normal if the almost complex structure J on the product manifold $\mathcal{N}_1\times R$ is given by

$$J(Z_1,\mathcal{F}\frac{d}{dt})=(\phi Z_1-\mathcal{F}\xi,\eta(Z_1)\frac{d}{dt}), \tag{2.5}$$

where $J^2=-I$ and $\mathcal{F}$ is a differentiable function on $\mathcal{N}_1\times R$ that has no torsion, i.e., J is integrable. The form for normality in terms of ϕ, ξ and η is given by $[\phi,\phi]+2d\eta\otimes\xi=0$ on $\mathcal{N}_1$, where $[\phi,\phi]$ is the Nijenhuis tensor of ϕ . Further, the fundamental 2-form Φ is defined by $\Phi(Z_1,Z_2)=g_1(Z_1,\phi Z_2)$.

A manifold $\mathcal{N}_1$ with the structure (ϕ,ξ,η,g_1) is said to be Cosymplectic [26] if

$$(\nabla_{Z_1}\phi)Z_2=0 \tag{2.6}$$

for any vector fields Z_1, Z_2 on $\mathcal{N}_1$, where ∇ stands for the Riemannian connection of the metric g_1 on $\mathcal{N}_1$. For a Cosymplectic manifold, we have

$$\nabla_{Z_1}\xi=0 \tag{2.7}$$

for any vector field Z_1 on $\mathcal{N}_1$.

O'Neill's tensors [16] $\mathcal{T}$ and $\mathcal{A}$ are given by

$$\mathcal{A}_{X_1}X_2=\mathcal{H}\nabla_{\mathcal{H}X_1}\mathcal{V}X_2+\mathcal{V}\nabla_{\mathcal{H}X_1}\mathcal{H}X_2, \tag{2.8}$$

$$\mathcal{T}_{X_1}X_2=\mathcal{H}\nabla_{\mathcal{V}X_1}\mathcal{V}X_2+\mathcal{V}\nabla_{\mathcal{V}X_1}\mathcal{H}X_2 \tag{2.9}$$

for any X_1, X_2 on $\mathcal{N}_1$. For vertical vector fields Y_1, Y_2, the tensor field $\mathcal{T}$ has the symmetry property, that is,

$$\mathcal{T}_{Y_1}Y_2=\mathcal{T}_{Y_2}Y_1, \tag{2.10}$$

while for horizontal vector fields X_1, X_2, the tensor field $\mathcal{A}$ has alternation property, that is

$$\mathcal{A}_{X_1}X_2=-\mathcal{A}_{X_2}X_1. \tag{2.11}$$

From the equations (2.8) and (2.9), we have

$$\nabla_{Y_1} Y_2 = \mathcal{T}_{Y_1} Y_2 + \mathcal{V}\nabla_{Y_1} Y_2, \tag{2.12}$$

$$\nabla_{Y_1} Z_1 = \mathcal{T}_{Y_1} Z_1 + \mathcal{H}\nabla_{Y_1} Z_1, \tag{2.13}$$

$$\nabla_{Z_1} Y_1 = \mathcal{A}_{Z_1} Y_1 + \mathcal{V}\nabla_{Z_1} Y_1, \tag{2.14}$$

$$\nabla_{Z_1} Z_2 = \mathcal{H}\nabla_{Z_1} Z_2 + \mathcal{A}_{Z_1} Z_2 \tag{2.15}$$

for all $Y_1, Y_2 \in \Gamma(\ker F_*)$ and $Z_1, Z_2 \in \Gamma(\ker F_*)^{\perp}$, where $\mathcal{H}\nabla_{Y_1} Z_1 = \mathcal{A}_{Z_1} Y_1$, if Z_1 is basic. It can be easily seen that $\mathcal{T}$ acts on the fibers as the second fundamental form, while $\mathcal{A}$ acts on the horizontal distribution and measures the obstruction to the integrability of this distribution.

The Riemannian submersion F between two Riemannian manifolds is totally geodesic if

$$(\nabla F_*)(U_1, U_2) = 0 \ \ \forall \ \ U_1, U_2 \in \Gamma(T\mathcal{N}_1).$$

Totally umbilical Riemannian submersion is a Riemannian submersion with totally umbilical fibers ([4], [5]) if

$$\mathcal{T}_{Z_1} Z_2 = g_1(Z_1, Z_2) H \tag{2.16}$$

for all $Z_1, Z_2 \in \Gamma(\ker F_*)$, where H denotes the mean curvature vector field of fibers.

Let $F : (\mathcal{N}_1, g_1) \to (\mathcal{N}_2, g_2)$ be a Riemannian submersion between Riemannian manifolds. The differential map F_* of F can be viewed as a section of the bundle $Hom(T\mathcal{N}_1, F^{-1}T\mathcal{N}_2) \to \mathcal{N}_1$, where $F^{-1}T\mathcal{N}_2$ is the pullback bundle whose fibers at $q \in \mathcal{N}_1$ is $(F^{-1}T\mathcal{N}_2)_q = T_F(q)\mathcal{N}_2, q \in \mathcal{N}_1$. The bundle $Hom(T\mathcal{N}_1, F^{-1}T\mathcal{N}_2)$ has a connection ∇ induced from the Levi-Civita connection $\nabla^{\mathcal{N}_1}$ and the pullback connection ∇^F. Then the second fundamental form of F is given by

$$(\nabla F_*)(V_1, V_2) = \nabla^F_{V_1} F_*(V_2) - F_*(\nabla^{\mathcal{N}_1}_{V_1} V_2) \tag{2.17}$$

for the vector fields $V_1, V_2 \in \Gamma(T\mathcal{N}_1)$.

3. The $CSI-$submersions from Cosymplectic manifolds

In the theory of Riemannian submersions, Bishop [5] initiated the concept of Clairaut submersion as: a submersion $F : (\mathcal{N}_1, g_1) \to (\mathcal{N}_2, g_2)$ is called a Clairaut submersion if there exist a function $r : \mathcal{N}_1 \to R^+$ in such a way that any geodesic that makes an angle Θ with a horizontal subspace, $r \sin\Theta$ is constant.

On the other side, Sahin [27] generalized the concept of Clairaut submersion and initiated the study of Clairaut Riemannian maps and investigated its geometric properties.

Theorem 3.1. [5] *Let $F : (\mathcal{N}_1, g_1) \to (\mathcal{N}_2, g_2)$ be a Riemannian submersion with connected fibers. Then, F is a Clairaut Riemannian submersion with $r = e^h$ if each fiber is totally umbilical and has the mean curvature vector field $H = -\nabla h$, where ∇h is the gradient of the function h with respect to g_1.*

Definition 3.1. [26] *Let F be a Riemannian submersion from an almost contact metric manifold $(\mathcal{N}_1, \phi, \xi, \eta, g_1)$ onto a Riemannian manifold $(\mathcal{N}_2, g_2)$. Then we say that F is a semi-invariant submersion if there is a distribution $\mathfrak{D}_1 \subseteq \ker F_*$ such that*

$$\ker F_* = \mathfrak{D}_1 \oplus \mathfrak{D}_2, \quad \phi(\mathfrak{D}_1) = \mathfrak{D}_1, \phi(\mathfrak{D}_2) \subseteq (\ker F_*)^{\perp},$$

where $\mathfrak{D}_1$ and $\mathfrak{D}_2$ mutually orthogonal distributions in $(\ker F_*)$.

Let μ denotes the complementary orthogonal subbundle to $\phi(\mathfrak{D}_2)$ in $(\ker F_*)^{\perp}$. Then we have

$$(\ker F_*)^{\perp} = \phi(\mathfrak{D}_2) \oplus \mu.$$

Obviously μ is an invariant subbundle of $(\ker F_*)^{\perp}$ with respect to the contact structure ϕ.

We say that a semi-invariant submersion $F : \mathcal{N}_1 \to \mathcal{N}_2$ admits a vertical Reeb vector field ξ if it is tangent to $(\ker F_*)$ and it admits horizontal Reeb vector field ξ it is normal to $(\ker F_*)$. One can easily observe that μ contains the Reeb vector field in case of the Riemannian submersion admits horizontal Reeb vector field.

We now define the notion of $CSI-$ submersion in contact manifolds as follows:

Definition 3.2. *A semi-invariant submersions from a Cosymplectic manifold $(\mathcal{N}_1, \phi, \xi, \eta, g_1)$ onto a Riemannian manifold $(\mathcal{N}_2, g_2)$ is called $CSI-$ submersion if it satisfies the condition of Clairaut Riemannian submersion.*

For any vector field $W_1 \in \Gamma(\ker F_*)$, we put

$$W_1 = PW_1 + QW_1, \tag{3.18}$$

where P and Q are projection morphisms [4] of $\ker F_*$ onto $\mathfrak{D}_1$ and $\mathfrak{D}_2$, respectively.

For $U_1 \in (\ker F_*)$, we get

$$\phi U_1 = \psi U_1 + \omega U_1, \tag{3.19}$$

where $\psi U_1 \in \Gamma(\mathfrak{D}_1)$ and $\omega U_1 \in \Gamma(\phi \mathfrak{D}_2)$. Also for $V_2 \in \Gamma(\ker F_*)^{\perp}$, we get

$$\phi V_2 = BV_2 + CV_2, \tag{3.20}$$

where $BV_2 \in \Gamma(\mathfrak{D}_2)$ and $CV_2 \in \Gamma(\mu)$.

Definition 3.3. [30] *Let F be a $CSI-$ submersion from an almost contact metric manifold $(\mathcal{N}_1, \phi, \xi, \eta, g_1)$ onto a Riemannian manifold $(\mathcal{N}_2, g_2)$. If $\mu = \{0\}$ or $\mu =< \xi >$, i.e., $(\ker F_*)^\perp = \phi(\mathfrak{D}_2)$ or $(\ker F_*)^\perp = \phi(\mathfrak{D}_2) \oplus < \xi >$ respectively, then we call ϕ a Lagrangian Riemannian submersion. In this case, for any horizontal vector field Z_1, we have*

$$BZ_1 = \phi Z_1 \text{ and } CZ_1 = 0. \tag{3.21}$$

Lemma 3.1. *Let F be a $CSI-$ submersion from a Cosymplectic manifold $(\mathcal{N}_1, \phi, \xi, \eta, g_1)$ onto a Riemannian manifold $(\mathcal{N}_2, g_2)$ admitting vertical or horizontal Reeb vector field. Then, we get*

$$\mathcal{V}\nabla_{W_1}\psi W_2 + \mathcal{T}_{W_1}\omega W_2 = B\mathcal{T}_{W_1}W_2 + \psi\mathcal{V}\nabla_{W_1}W_2, \tag{3.22}$$

$$\mathcal{T}_{W_1}\psi W_2 + \mathcal{H}\nabla_{W_1}\omega W_2 = C\mathcal{T}_{W_1}W_2 + \omega\mathcal{V}\nabla_{W_1}W_2, \tag{3.23}$$

$$\mathcal{V}\nabla_{U_1}BU_2 + \mathcal{A}_{U_1}CU_2 = B\mathcal{H}\nabla_{U_1}U_2 + \psi\mathcal{A}_{U_1}U_2, \tag{3.24}$$

$$\mathcal{A}_{U_1}BU_2 + \mathcal{H}\nabla_{U_1}CU_2 = C\mathcal{H}\nabla_{U_1}U_2 + \omega\mathcal{A}_{U_1}U_2, \tag{3.25}$$

$$\mathcal{V}\nabla_{W_1}BU_1 + \mathcal{T}_{W_1}CU_1 = \psi\mathcal{T}_{W_1}U_1 + B\mathcal{H}\nabla_{W_1}U_1, \tag{3.26}$$

$$\mathcal{T}_{W_1}BU_1 + \mathcal{H}\nabla_{W_1}CU_1 = \omega\mathcal{T}_{W_1}U_1 + C\mathcal{H}\nabla_{W_1}U_1, \tag{3.27}$$

$$\mathcal{V}\nabla_{U_1}\psi W_1 + \mathcal{A}_{U_1}\omega W_1 = B\mathcal{A}_{U_1}W_1 + \psi\mathcal{V}\nabla_{U_1}W_1, \tag{3.28}$$

$$\mathcal{A}_{U_1}\psi W_1 + \mathcal{H}\nabla_{U_1}\omega W_1 = C\mathcal{A}_{U_1}W_1 + \omega\mathcal{V}\nabla_{U_1}W_1, \tag{3.29}$$

where $W_1, W_2 \in \Gamma(\ker F_*)$ and $U_1, U_2 \in \Gamma(\ker F_*)^\perp$.

Proof. Using (2.12)−(2.15),(3.19) and (3.20), we get Lemma 3.1.

Corollary 3.1. *Let F be a Lagrangian submersion from a Cosymplectic manifold $(\mathcal{N}_1, \phi, \xi, \eta, g_1)$ onto a Riemannian manifold $(\mathcal{N}_2, g_2)$ admitting vertical or horizontal Reeb vector field. Then we get*

$$\mathcal{V}\nabla_{V_1}\psi V_2 + \mathcal{T}_{V_1}\omega V_2 = B\mathcal{T}_{V_1}V_2 + \psi\mathcal{V}\nabla_{V_1}V_2, \mathcal{T}_{V_1}\psi V_2 + \mathcal{H}\nabla_{V_1}\omega V_2 = \omega\mathcal{V}\nabla_{V_1}V_2,$$

$$\mathcal{V}\nabla_{Y_1}BY_2 = B\mathcal{H}\nabla_{Y_1}Y_2 + \psi\mathcal{A}_{Y_1}Y_2, \mathcal{A}_{Y_1}BY_2 = \omega\mathcal{A}_{Y_1}Y_2,$$

$$\mathcal{V}\nabla_{V_1}BY_1 = \psi\mathcal{T}_{V_1}Y_1 + B\mathcal{H}\nabla_{V_1}Y_1, \mathcal{T}_{V_1}BY_1 = \omega\mathcal{T}_{V_1}Y_1,$$

$$\mathcal{V}\nabla_{Y_1}\psi V_1 + \mathcal{A}_{Y_1}\omega V_1 = B\mathcal{A}_{Y_1}V_1 + \psi\mathcal{V}\nabla_{Y_1}V_1, \mathcal{A}_{Y_1}\psi V_1 + \mathcal{H}\nabla_{Y_1}\omega V_1 = \omega\mathcal{V}\nabla_{Y_1}V_1,$$

where $V_1, V_2 \in \Gamma(\ker F_*)$ and $Y_1, Y_2 \in \Gamma(\ker F_*)^\perp$.

Lemma 3.2. *Let F be a $CSI-$ submersion from a Cosymplectic manifold $(\mathcal{N}_1, \phi, \xi, \eta, g_1)$ onto a Riemannian manifold $(\mathcal{N}_2, g_2)$ admitting vertical or horizontal Reeb vector field. Then we have*

$$\mathcal{T}_{Z_1}\xi = 0, \mathcal{A}_{Z_2}\xi = 0 \tag{3.30}$$

for $Z_1 \in \Gamma(\ker F_*)^{\perp}$ and $Z_2 \in \Gamma(\ker F_*)^{\perp}$.

Proof. Using (2.12)−(2.15) and (2.7), we get Lemma 3.2.

Lemma 3.3. *Let F be a $CSI-$ submersion from a Cosymplectic manifold $(\mathcal{N}_1, \phi, \xi, \eta, g_1)$ onto a Riemannian manifold $(\mathcal{N}_2, g_2)$. If $\gamma : I_2 \subset R \to \mathcal{N}_1$ is a regular curve and $Z_1(t)$ and $Z_2(t)$ are the vertical and horizontal components of the tangent vector field $\dot{\gamma} = E$ of $\gamma(t)$, respectively, then γ is a geodesic if and only if along γ the following equations hold:*

$$\mathcal{V}\nabla_{\dot{\gamma}}BZ_2 + \mathcal{V}\nabla_{\dot{\gamma}}\psi Z_1 + (\mathcal{T}_{Z_1} + \mathcal{A}_{Z_2})CZ_2 + (\mathcal{T}_{Z_1} + \mathcal{A}_{Z_2})\omega Z_1 = 0, \tag{3.31}$$

$$\mathcal{H}\nabla_{\dot{\gamma}}CZ_2 + \mathcal{H}\nabla_{\dot{\gamma}}\omega Z_1 + (\mathcal{T}_{Z_1} + \mathcal{A}_{Z_2})BZ_2 + (\mathcal{A}_{Z_2} + \mathcal{T}_{Z_1})\psi Z_1 = 0. \tag{3.32}$$

Proof. Let $\gamma : I_2 \to \mathcal{N}_1$ be a regular curve on $\mathcal{N}_1$. Since $\dot{\gamma}(t) = Z_1(t) + Z_2(t)$, where $Z_1(t)$ and $Z_2(t)$ are the vertical and horizontal components of $\dot{\gamma}(t)$. Using (2.6),(2.12)−(2.15), (3.19) and (3.20), we have

$$\begin{aligned}
\phi\nabla_{\dot{\gamma}}\dot{\gamma} &= \nabla_{\dot{\gamma}}\phi\dot{\gamma} \\
&= \nabla_{Z_1}\psi Z_1 + \nabla_{Z_1}\omega Z_1 + \nabla_{Z_2}\psi Z_1 + \nabla_{Z_2}\omega Z_1 + \\
&\quad \nabla_{Z_1}BZ_2 + \nabla_{Z_1}CZ_2 + \nabla_{Z_2}BZ_2 + \nabla_{Z_2}CZ_2, \\
&= \mathcal{V}\nabla_{\dot{\gamma}}BZ_2 + \mathcal{V}\nabla_{\dot{\gamma}}\psi Z_1 + (\mathcal{T}_{Z_1} + \mathcal{A}_{Z_2})CZ_2 + (\mathcal{T}_{Z_1} + \mathcal{A}_{Z_2})\omega Z_1 + \\
&\quad \mathcal{H}\nabla_{\dot{\gamma}}CZ_2 + \mathcal{H}\nabla_{\dot{\gamma}}\omega Z_1 + (\mathcal{T}_{Z_1} + \mathcal{A}_{Z_2})BZ_2 + (\mathcal{A}_{Z_2} + \mathcal{T}_{Z_1})\psi Z_1.
\end{aligned}$$

From above, vertical and horizontal components are:

$$\mathcal{V}\phi\nabla_{\dot{\gamma}}\dot{\gamma} = \mathcal{V}\nabla_{\dot{\gamma}}BZ_2 + \mathcal{V}\nabla_{\dot{\gamma}}\psi Z_1 + (\mathcal{T}_{Z_1} + \mathcal{A}_{Z_2})CZ_2 + (\mathcal{T}_{Z_1} + \mathcal{A}_{Z_2})\omega Z_1,$$

$$\mathcal{H}\phi\nabla_{\dot{\gamma}}\dot{\gamma} = \mathcal{H}\nabla_{\dot{\gamma}}CZ_2 + \mathcal{H}\nabla_{\dot{\gamma}}\omega Z_1 + (\mathcal{T}_{Z_1} + \mathcal{A}_{Z_2})BZ_2 + (\mathcal{A}_{Z_2} + \mathcal{T}_{Z_1})\psi Z_1.$$

Thus γ is a geodesic on $\mathcal{N}_1$ if and only if $\mathcal{V}\phi\nabla_{\dot{\gamma}}\dot{\gamma} = 0$ and $\mathcal{H}\phi\nabla_{\dot{\gamma}}\dot{\gamma} = 0$.

Theorem 3.2. *Let F be a Clairaut semi-invariant submersion from a Cosymplectic manifold $(\mathcal{N}_1, \phi, \xi, \eta, g_1)$ onto a Riemannian manifold $(\mathcal{N}_2, g_2)$. Then F is a $CSI-$ submersion with $r = e^h$ if and only if*

$$\begin{aligned} g_1(\nabla h, Z_2)||Z_1||^2 &= g_1(\mathcal{V}\nabla_{\dot{\gamma}} BZ_2, \psi Z_1) + g_1((\mathcal{T}_{Z_1} + \mathcal{A}_{Z_2})CZ_2, \psi Z_1) + \\ &\quad g_1(\mathcal{H}\nabla_{\dot{\gamma}} CZ_2, \omega Z_1) + g_1((\mathcal{T}_{Z_1} + \mathcal{A}_{Z_2})BZ_2, \omega Z_1), \end{aligned}$$

where $\gamma : I_2 \to \mathcal{N}_1$ is a geodesic on $\mathcal{N}_1$, $Z_1(t)$ and $Z_2(t)$ are vertical and horizontal components of $\dot{\gamma}(t)$, respectively.

Proof. Let $\gamma : I_2 \to \mathcal{N}_1$ be a geodesic on $\mathcal{N}_1$ with $Z_1(t) = \mathcal{V}\dot{\gamma}(t)$ and $Z_2(t) = \mathcal{H}\dot{\gamma}(t)$. Let $\Theta(t)$ denotes the angle in $[0, \pi]$ between $\dot{\gamma}(t)$ and $Z_2(t)$. Assuming $\upsilon = ||\dot{\gamma}(t)||,^2$ then we get

$$g_1(Z_1(t), Z_1(t)) = \upsilon \sin^2 \Theta(t), \tag{3.33}$$

$$g_1(Z_2(t), Z_2(t)) = \upsilon \cos^2 \Theta(t). \tag{3.34}$$

Now, differentiating (3.33), we get

$$\begin{aligned} \frac{d}{dt} g_1(Z_1(t), Z_1(t)) &= 2\upsilon \sin \Theta(t) \cos \Theta(t) \frac{d\Theta}{dt}, \\ g_1(\nabla_{\dot{\gamma}} Z_1(t), Z_1(t)) &= \upsilon \cos \Theta(t) \cos \Theta(t) \frac{d\Theta}{dt}. \end{aligned}$$

Using equations (2.3) and (2.6) in above equation, we get

$$g_1(\nabla_{\dot{\gamma}} \phi Z_1(t), \phi Z_1(t)) = \upsilon \sin \Theta(t) \cos \Theta(t) \frac{d\Theta}{dt}. \tag{3.35}$$

Now we obtain

$$\begin{aligned} g_1(\nabla_{\dot{\gamma}} \phi Z_1, \phi Z_1) &= g_1(\mathcal{V}\nabla_{\dot{\gamma}} \psi Z_1, \psi Z_1) + g_1(\mathcal{H}\nabla_{\dot{\gamma}} \omega Z_1, \omega Z_1) + \\ &\quad g_1((\mathcal{T}_{Z_1} + \mathcal{A}_{Z_2})\psi Z_1, \omega Z_1) + g_1((\mathcal{T}_{Z_1} + \mathcal{A}_{Z_2})\omega Z_1, \psi Z_1). \end{aligned} \tag{3.36}$$

Using equations (3.31) and (3.32) in (3.37), we have

$$\begin{aligned} g_1(\nabla_{\dot{\gamma}} \phi Z_1, \phi Z_1) &= -g_1(\mathcal{V}\nabla_{\dot{\gamma}} BZ_2, \psi Z_1) - g_1((\mathcal{T}_{Z_1} + \mathcal{A}_{Z_2})CZ_2, \psi Z_1) - \\ &\quad g_1(\mathcal{H}\nabla_{\dot{\gamma}} CZ_2, \omega Z_1) - g_1((\mathcal{T}_{Z_1} + \mathcal{A}_{Z_2})BZ_2, \omega Z_1). \end{aligned} \tag{3.37}$$

From (3.35) and (3.38), we have

$$\begin{aligned} \upsilon \cos \Theta(t) \cos \Theta(t) \frac{d\Theta}{dt} &= -g_1(\mathcal{V}\nabla_{\dot{\gamma}} BZ_2, \psi Z_1) - g_1((\mathcal{T}_{Z_1} + \mathcal{A}_{Z_2})CZ_2, \psi Z_1) - \\ &\quad g_1(\mathcal{H}\nabla_{\dot{\gamma}} CZ_2, \omega Z_1) - g_1((\mathcal{T}_{Z_1} + \mathcal{A}_{Z_2})BZ_2, \omega Z_1). \end{aligned} \tag{3.38}$$

Moreover, π is a $CSI-$ Riemannian submersion with $r = e^h$ if and only if $\frac{d}{dt}(e^{h\circ\gamma}\sin\Theta) = 0$, i.e., $e^{h\circ\gamma}(\cos\Theta\frac{d\Theta}{dt} + \sin\Theta\frac{dh}{dt}) = 0$. By multiplying this with non-zero factor $\upsilon\sin\Theta$, we have

$$\begin{aligned} -\upsilon\cos\Theta\sin\Theta\frac{d\Theta}{dt} &= \upsilon\sin^2\Theta\frac{dh}{dt}, \\ \upsilon\cos\Theta\sin\Theta\frac{d\Theta}{dt} &= -g_1(Z_1, Z_1)\frac{dh}{dt}, \\ \upsilon\cos\Theta\sin\Theta\frac{d\Theta}{dt} &= -g_1(\nabla h, \dot{\gamma})||Z_1||^2, \\ \upsilon\cos\Theta\sin\Theta\frac{d\Theta}{dt} &= -g_1(\nabla h, Z_2)||Y_1||^2. \end{aligned} \tag{3.39}$$

Thus, from equations (3.39) and (3.39), we have

$$\begin{aligned} g_1(\nabla h, Z_2)||Z_1||^2 &= g_1(\mathcal{V}\nabla_{\dot{\gamma}}BZ_2, \psi Z_1) + g_1((\mathcal{T}_{Z_1} + \mathcal{A}_{Z_2})CZ_2, \psi Z_1) + \\ &\quad g_1(\mathcal{H}\nabla_{\dot{\gamma}}CZ_2, \omega Z_1) + g_1((\mathcal{T}_{Z_1} + \mathcal{A}_{Z_2})BZ_2, \omega Z_1). \end{aligned}$$

Hence Theorem 3.2 is proved.

Corollary 3.2. *Let F be a $CSI-$ submersion from a Cosymplectic manifold $(\mathcal{N}_1, \phi, \xi, \eta, g_1)$ to a Riemannian manifold $(\mathcal{N}_2, g_2)$ admitting horizontal Reeb vector field. Then we get*

$$g_1(\nabla h, \xi) = 0.$$

Theorem 3.3. *Let F be a $CSI-$ submersion from a Cosymplectic manifold $(\mathcal{N}_1, \phi, \xi, \eta, g_1)$ onto a Riemannian manifold $(\mathcal{N}_2, g_2)$ with $r = e^h$, then at least one of the following statement is true:*

(i) h is constant on $\phi(\mathfrak{D}_2)$,

(ii) the fibers are one-dimensional,

(iii) $\overset{F}{\nabla}_{\phi X_1} F_*(W_1) = -W_1(h)F_*(\phi X_1)$, for all $X_1 \in \Gamma(\mathfrak{D}_2), W_1 \in \Gamma(\mu)$ and $\xi \neq W_1$.

Proof. Let F be $CSI-$ submersion from a Cosymplectic manifold onto a Riemannian manifold. For $Y_1, Y_2 \in \Gamma(\mathfrak{D}_2)$, using (2.16) and Theorem 3.1, we get

$$\mathcal{T}_{Y_1}Y_2 = -g_1(Y_1, Y_2)gradh. \tag{3.40}$$

Taking the inner product in (3.40) with ϕX_1, we get

$$g_1(\mathcal{T}_{Y_1}Y_2, \phi X_1) = -g_1(Y_1, Y_2)g_1(gradh, \phi X_1) \tag{3.41}$$

for all $X_1 \in \Gamma(\mathfrak{D}_2)$.

From $(2.3), (2.6)$ and (3.41), we obtain

$$g_1(\nabla_{Y_1}\phi Y_2, X_1) = g_1(Y_1, Y_2)g_1(gradh, \phi X_1).$$

By using (2.3) and (2.16) in above equation, we have

$$g_1(Y_1, X_1)g_1(gradh, \phi Y_2) = g_1(Y_1, Y_2)g_1(gradh, \phi X_1). \tag{3.42}$$

Taking $X_1 = Y_2$ and interchanging the role of Y_1 and Y_2, we get

$$g_1(Y_2, Y_2)g_1(gradh, \phi Y_1) = g_1(Y_1, Y_2)g_1(gradh, \phi Y_2). \tag{3.43}$$

Using (3.42) with $X_1 = Y_1$ in (3.43), we have

$$g_1(gradh, \phi Y_1) = \frac{(g_1(Y_1, Y_2))^2}{||Y_1||^2||Y_2||^2} g_1(gradh, \phi Y_1). \tag{3.44}$$

If $gradh \in \Gamma(\phi(\mathfrak{D}_2))$, then (3.44) and the equality condition of Schwarz inequality implies that either h is constant on $\phi(\mathfrak{D}_2)$ or the fibers are 1-dimensional. This implies the proof of (i) and (ii).

Now, from (2.15) and (2.16), we get

$$g_1(\nabla_{Y_1} X_1, W_1) = -g_1(Y_1, X_1)g_1(gradh, W_1), \tag{3.45}$$

for all $W_1 \in \Gamma(\mu)$ and $\xi \neq W_1$. Using (2.3), (2.6) and (3.45), we have

$$g_1(\nabla_{Y_1} \phi X_1, \phi W_1) = -g_1(Y_1, X_1)g_1(gradh, W_1),$$

which implies

$$g_1(\nabla_{\phi X_1} Y_1, \phi W_1) = -g_1(Y_1, X_1)g_1(gradh, W_1). \tag{3.46}$$

By using (2.14) and (3.46), we have

$$g_1(\mathcal{H}\nabla_{\phi X_1} W_1, \phi Y_1) = -g_1(\phi Y_1, \phi X_1)g_1(gradh, W_1).$$

Also for Riemannian submersion F, we have

$$g_2(F_*(\nabla^{\mathcal{N}_1}_{\phi X_1} W_1), F_*(\phi Y_1)) = -g_2(F_*(\phi Y_1), F_*(\phi X_1))g_1(gradh, W_1). \tag{3.47}$$

Again, using (2.17) and (3.47), we get

$$g_2(\overset{F}{\nabla}_{\phi X_1} F_*(W_1), F_*(\phi Y_1)) = -g_2(F_*(\phi Y_1), F_*(\phi X_1))g_1(gradh, W_1),$$

which implies.

$$\overset{F}{\nabla}_{\phi X_1} F_*(W_1) = -W_1(h)F_*(\phi X_1). \tag{3.48}$$

If $gradh \in \Gamma(\mu)\backslash\{\xi\}$, then (3.48) implies (iii).

Corollary 3.3. *Let F be a $CSI-$ submersion from a Cosymplectic manifold $(\mathcal{N}_1, \phi, \xi, \eta, g_1)$ onto a Riemannian manifold $(\mathcal{N}_2, g_2)$ with $r = e^h$ and $\dim(D_2) > 1$. Then the fibers of F are totally geodesic if and only if $\overset{F}{\nabla}_{\phi X_1} F_*(W_1) = 0\ \forall X_1 \in \Gamma(\mathfrak{D}_2)$ and $W_1 \in \Gamma(\mu)$.*

Lemma 3.4. *Let F be a $CSI-$ submersion from a Cosymplectic manifold $(\mathcal{N}_1, \phi, \xi, \eta, g_1)$ onto a Riemannian manifold $(\mathcal{N}_2, g_2)$ with $r = e^h$ and $\dim(D_2) > 1$. Then $\overset{F}{\nabla}_{W_1} F_*(\phi Y_1) = W_1(h) F_*(\phi Y_1)$ for $Y_1 \in \Gamma(\mathfrak{D}_2)$ and $W_1 \in \Gamma(\ker F_*)^{\perp} \backslash \{\xi\}$.*

Proof. Let F be a $CSI-$ submersion from a Cosymplectic manifold onto a Riemannian manifold. From Theorem 3.1, fibers are totally umbilical with mean curvature vector field $H = -gradh$, then we get

$$\begin{aligned} -g_1(\nabla_{Y_1} W_1, Y_2) &= g_1(\nabla_{Y_1} Y_2, W_1), \\ -g_1(\nabla_{Y_1} W_1, Y_2) &= -g_1(Y_1, Y_2) g_1(gradh, W_1) \end{aligned}$$

for $Y_1, Y_2 \in \Gamma(\mathfrak{D}_2)$ and $W_1 \in \Gamma(\ker F_*)^{\perp} \backslash \{\xi\}$.

Using (2.3) and (2.6) in above equation, we get

$$g_1(\nabla_{W_1} \phi Y_1, \phi Y_2) = g_1(\phi Y_1, \phi Y_2) g_1(gradh, W_1). \tag{3.49}$$

Since F is $CSI-$ submersion and using (3.49), we have

$$g_2(F_*(\nabla^F_{W_1} \phi Y_1), F_*(\phi Y_2)) = g_2(F_*(\phi Y_1), F_*(\phi Y_2)) g_1(gradh, W_1). \tag{3.50}$$

From (2.17) in (3.50), we obtain

$$g_2(\overset{F}{\nabla}_{W_1} F_*(\phi Y_1), F_*(\phi Y_2)) = g_2(F_*(\phi Y_1), F_*(\phi Y_2)) g_1(gradh, W_1), \tag{3.51}$$

which implies $\overset{F}{\nabla}_{W_1} F_*(\phi Y_1) = W_1(h) F_*(\phi Y_1)$ for $Y_1 \in \Gamma(\mathfrak{D}_2)$ and $W_1 \in \Gamma(\ker F_*)^{\perp} \backslash \{\xi\}$.

Theorem 3.4. *Let F be a $CSI-$ submersion with $r = e^h$ from a Cosymplectic manifold $(\mathcal{N}_1, \phi, \xi, \eta, g_1)$ onto a Riemannian manifold $(\mathcal{N}_2, g_2)$. If $\mathcal{T}$ is not equal to zero identically, then the invariant distribution $\mathfrak{D}_1$ cannot defined a totally geodesic foliation on $\mathcal{N}_1$.*

Proof. For $Y_1, Y_2 \in \Gamma(\mathfrak{D}_1)$ and $U_1 \in \Gamma(\mathfrak{D}_2)$, using (2.3), (2.6), (2.13) and (2.16), we get

$$\begin{aligned} g_1(\nabla_{Y_1} Y_2, U_1) &= g_1(\nabla_{Y_1} \phi Y_2, \phi U_1), \\ &= g_1(\mathcal{T}_{Y_1} \phi Y_2, \phi U_1), \\ &= -g_1(Y_1, \phi Y_2) g_1(gradh, \phi U_1). \end{aligned}$$

Thus, one can easily obtain the assertion from above equation and the fact that $gradh \in \phi(\mathfrak{D}_2)$.

Theorem 3.5. *The $CSI-$ submersion F with $r = e^h$ from a Cosymplectic manifold $(\mathcal{N}_1, \phi, \xi, \eta, g_1)$ onto a Riemannian manifold $(\mathcal{N}_2, g_2)$. Then the fibers of F are totally geodesic or the anti-invariant distribution $\mathfrak{D}_2$ is one-dimensional.*

Proof. The result is quite obvious when we take the fibers of F are totally geodesic. For second one, since F is a $CSI-$ submersion, then either $\dim(\mathfrak{D}_2) = 1$ or $\dim(\mathcal{D}_2) > 1$. If $\dim(\mathcal{D}_2) > 1$, then we can choose $U_1, U_2 \in \Gamma(\mathfrak{D}_2)$ such that $\{U_1, U_2\}$ is orthonormal. From (2.13), (3.19) and (3.20), we get

$$\begin{aligned} \mathcal{T}_{U_1}\phi U_2 + \mathcal{H}\nabla_{U_1}\phi U_2 &= \nabla_{U_1}\phi U_2, \\ \mathcal{T}_{U_1}\phi U_2 + \mathcal{H}\nabla_{U_1}\phi U_2 &= B\mathcal{T}_{U_1}U_2 + C\mathcal{T}_{U_1}U_2 + \psi\mathcal{V}\nabla_{U_1}U_2 + \omega\mathcal{V}\nabla_{U_1}U_2. \end{aligned}$$

Taking the inner product above equation with U_1, we obtain

$$g_1(\mathcal{T}_{U_1}\phi U_2, U_1) = g_1(B\mathcal{T}_{U_1}U_2, U_1) + g_1(\psi\mathcal{V}\nabla_{U_1}U_2, U_1). \tag{3.52}$$

From (2.3), (2.6) and (2.13), we have

$$g_1(\mathcal{T}_{U_1}U_1, \phi U_2) = -g_1(\mathcal{T}_{U_1}\phi U_2, U_1) = g_1(\mathcal{T}_{U_1}U_2, \phi U_1). \tag{3.53}$$

Now, using (2.16) and (3.53), we get

$$g_1(\mathcal{T}_{U_1}U_1, \phi U_2) = -g_1(gradh, \phi U_2). \tag{3.54}$$

From equations (2.16) and (3.54), we obtain

$$-g_1(gradh, \phi U_2) = g_1(\mathcal{T}_{U_1}U_1, \phi U_2) = -g_1(\mathcal{T}_{U_1}\phi U_2, U_1) = g_1(\mathcal{T}_{U_1}U_2, \phi U_1). \tag{3.55}$$

From above equation, we get

$$\begin{aligned} g_1(gradh, \phi U_2) &= -g_1(\mathcal{T}_{U_1}U_2, \phi U_1), \\ g_1(gradh, \phi U_2) &= g_1(U_1, U_2)g_1(gradh, \phi U_1), \\ g_1(gradh, \phi U_2) &= 0. \end{aligned}$$

Thus, we get $gradh \perp \phi(\mathfrak{D}_2)$.

Therefore, the dimension of $\mathfrak{D}_2$ must be one.

4. Example

Example 4.1. *Taking an Euclidean space $\mathcal{N}_1$, given by $\mathcal{N}_1 = \{(x_1, x_2, y_{1,}y_2, z) \in R^5 : (x_1, x_2, y_{1,}y_2) \neq (0,0,0,0)$ and $z \neq 0\}$. We define the Riemannian metric g_1 on $\mathcal{N}_1$ defined as $g_1 = e^{2z}dx_1^2 + e^{2z}dx_2^2 + e^{2z}dy_1^2 + e^{2z}dy_2^2 + dz^2$ and the Cosymplectic structure on ϕ and $\mathcal{N}_1$ defined as $\phi(x_1, x_2, y_{1,}y_2, z) = (y_{1,}y_2, -x_1, -x_2, z)$.*

Let $\mathcal{N}_2 = \{(v_1, v_2) \in R^2\}$ be a Riemannian manifold with Riemannian metric g_2, given by $g_2 = e^{2z}dv_1^2 + dv_2^2$. Define a map $F : R^5 \to R^2$ by

$$F(x_1, x_2, y_{1,}y_2, z) = (\frac{x_2 - y_2}{\sqrt{2}}, z).$$

Then, we have

$$\ker F_* =< X_1 = e_1, X_2 = e_2 + e_4, X_3 = e_3 >,$$

$$\mathfrak{D}_1 =< X_1 = e_1, X_3 = e_3 >, \mathfrak{D}_2 =< X_2 = e_2 + e_4 >,$$

$$(\ker F_*)^\perp =< H_1 = e_2 - e_4, H_2 = e_5 >,$$

where $\{e_1 = e^{-z}\frac{\partial}{\partial x_1}, e_2 = e^{-z}\frac{\partial}{\partial x_2}, e_3 = e^{-z}\frac{\partial}{\partial y_1}, e_4 = e^{-z}\frac{\partial}{\partial y_2}, e_7 = \frac{\partial}{\partial z}\}$, $\{e_1^* = \frac{\partial}{\partial v_1}, e_2^* = \frac{\partial}{\partial v_2}\}$ are bases on $T_p\mathcal{N}_1$ and $T_{F(p)}\mathcal{N}_2$, respectively, for all $p \in \mathcal{N}_1$. By direct computations, we can see that $F_*(H_1) = \sqrt{2}e^{-z}e_1^*, F_*(H_2) = e_2^*$, and $g_1(H_i, H_j) = g_2(F_*H_i, F_*H_j)$ for all $H_i, H_j \in \Gamma(\ker F_*)^\perp$, $i, j = 1, 2$. Thus, F is submersion. Moreover, it is easy to see that $\phi X_1 = -X_3, \phi X_2 = -H_1$ and $\phi X_3 = X_1$. Therefore F is a $CSI-$ submersion.

Now, using the Cosymplectic structure, we see that

$$\begin{aligned}
[e_1, e_1] &= [e_2, e_2] = [e_3, e_3] = [e_4, e_4] = [e_5, e_5] = 0, \qquad (4.56)\\
[e_1, e_2] &= 0, [e_1, e_3] = 0, [e_1, e_4] = 0, [e_1, e_5] = e_1,\\
[e_2, e_3] &= 0, [e_2, e_4] = 0, [e_2, e_5] = e_2, [e_3, e_4] = 0,\\
[e_3, e_5] &= e_3, [e_4, e_5] = e_4.
\end{aligned}$$

The Levi-Civita connection ∇ of the metric g_1 is given by the Koszul$'$s formula which is

$$\begin{aligned}
&2g_1(\nabla_X Y, Z) \qquad (4.57)\\
&= Xg_1(Y, Z) + Yg_1(Z, X) - Zg_1(X, Y) + g_1([X, Y], Z) - g_1([Y, Z], X) + g_1([Z, X], Y).
\end{aligned}$$

Using equations (4.56) and (4.57), we obtain

$$\begin{aligned} \nabla_{e_1} e_1 &= \nabla_{e_2} e_2 = \nabla_{e_3} e_3 = \nabla_{e_4} e_4 = -\frac{\partial}{\partial z}, \\ \nabla_{e_1} e_2 &= \nabla_{e_1} e_3 = \nabla_{e_1} e_4 = \nabla_{e_2} e_1 = \nabla_{e_2} e_3 = \nabla_{e_2} e_4 = 0, \\ \nabla_{e_3} e_1 &= \nabla_{e_3} e_2 = \nabla_{e_3} e_4 = \nabla_{e_4} e_1 = \nabla_{e_4} e_2 = \nabla_{e_4} e_3 = 0. \\ \nabla_{e_1} e_5 &= e_1, \nabla_{e_2} e_5 = e_2, \nabla_{e_3} e_5 = e_3, \nabla_{e_4} e_5 = e_4, \nabla_{e_5} e_5 = 0. \end{aligned} \tag{4.58}$$

Therefore, we have

$$\begin{aligned} \nabla_{X_1} X_1 &= \nabla_{e_1} e_1 = -\frac{\partial}{\partial z}, \nabla_{X_2} X_2 = \nabla_{e_2+e_4} e_2 + e_4 = -2\frac{\partial}{\partial z}, \\ \nabla_{X_3} X_3 &= \nabla_{e_3} e_3 = -2\frac{\partial}{\partial z}, \nabla_{X_1} X_2 = \nabla_{e_1} e_2 = \nabla_{X_1} X_3 = \nabla_{e_1} e_3 = 0, \\ \nabla_{X_2} X_3 &= \nabla_{e_2} e_3 = 0, \nabla_{X_2} X_1 = \nabla_{e_2} e_1 = 0, \nabla_{X_3} X_1 = \nabla_{e_3} e_1 = 0, \\ \nabla_{X_3} X_2 &= \nabla_{e_3} e_2 + e_4 = 0. \end{aligned} \tag{4.59}$$

Thus, we have

$$\mathcal{T}_V V = \mathcal{T}_{\lambda_1 X_1 + \lambda_2 X_2 + \lambda_3 X_3} \lambda_1 V_1 + \lambda_2 V_2 + \lambda_3 V_3, \lambda_1, \lambda_2, \lambda_3 \in R,$$

$$\begin{aligned} \mathcal{T}_V V &= \lambda_1^2 \mathcal{T}_{X_1} X_1 + \lambda_2^2 \mathcal{T}_{X_2} X_2 + \lambda_3^2 \mathcal{T}_{X_3} X_3 + \\ &\quad \lambda_1 \lambda_2 \mathcal{T}_{X_1} X_2 + \lambda_1 \lambda_3 \mathcal{T}_{X_1} X_3 + \lambda_2 \lambda_3 \mathcal{T}_{X_2} X_3 + \\ &\quad \lambda_1 \lambda_2 \mathcal{T}_{X_2} X_1 + \lambda_1 \lambda_3 \mathcal{T}_{X_3} X_1 + \lambda_2 \lambda_3 \mathcal{T}_{X_3} X_2. \end{aligned} \tag{4.60}$$

Using equations (2.12) and (4.59), we obtain

$$\begin{aligned} \mathcal{T}_{X_1} X_1 &= -\frac{\partial}{\partial z}, \mathcal{T}_{X_2} X_2 = -2\frac{\partial}{\partial z}, \mathcal{T}_{X_3} X_3 = -\frac{\partial}{\partial z}, \\ \mathcal{T}_{X_1} X_2 &= 0, \mathcal{T}_{X_1} X_3 = 0, \mathcal{T}_{X_2} X_3 = 0, \mathcal{T}_{X_2} X_1 = 0, \\ \mathcal{T}_{X_2} X_3 &= 0, \mathcal{T}_{X_3} X_1 = 0. \end{aligned} \tag{4.61}$$

Now using equations (4.60) and (4.61), we get

$$\mathcal{T}_V V = -(\lambda_1^2 + 2\lambda_2^2 + \lambda_3^2)\frac{\partial}{\partial z}. \tag{4.62}$$

Since $X = \lambda_1 X_1 + \lambda_2 X_2 + \lambda_3 X_3$, so $g_1(\lambda_1 V_1 + \lambda_2 V_2 + \lambda_3 V_3, \lambda_1 V_1 + \lambda_2 V_2 + \lambda_3 V_3) = \lambda_1^2 + 2\lambda_2^2 + \lambda_3^2$. For a smooth function h on R^5, the ∇h w. r. t. the metric g_1 is given by $\nabla h = e^{-2z}\frac{\partial h}{\partial x_1}\frac{\partial}{\partial x_1} + e^{-2z}\frac{\partial h}{\partial x_2}\frac{\partial}{\partial x_2} + e^{-2z}\frac{\partial h}{\partial y_1}\frac{\partial}{\partial y_1} + e^{-2z}\frac{\partial h}{\partial y_2}\frac{\partial}{\partial y_2} + \frac{\partial h}{\partial z}\frac{\partial}{\partial z}$. Hence $\nabla h = \frac{\partial}{\partial z}$ for the function $h = z$. Then one can easily find that $\mathcal{T}_V V = -g_1(V, V)\nabla h$, thus by Theorem 3.1, the map F is a $CSI-$ submersion from Cosymplectic manifold onto Riemannian manifold.

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