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CONFORMAL η-RICCI SOLITONS IN δ- LORENTZIAN TRANS SASAKIAN MANIFOLDS

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Abstract. The object of the present paper is to study the δ-Lorentzian Trans Sasakian manifolds admitting the conformal η-Ricci Solitons and gradient conformal Ricci soliton. It is shown that a symmetric second order covariant tensor in a δ-Lorentzian Trans Sasakian manifold is a constant multiple of metric tensor. Also an example of conformal η-Ricci soliton in 3-dimensional δ-Lorentzian Trans Sasakian manifold is provided in the region where δ-Lorentzian Trans-Sasakian manifold expanding.

1. Introduction

In recent years the pioneering works of R. Hamilton [22] and G. Perelman [34] towards the solution of the Poincare conjecture in dimension 3 have produced a flourishing activity in the research of self similar solutions, or solitons, of the Ricci flow. The study of the geometry of solitons, in particular their classification in dimension 3, has been essential in providing a positive answer to the conjecture; however in higher dimension and in the complete, possibly noncompact case, the understanding of the geometry and the classification of solitons seems to remain a desired goal for a not too proximate future.

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In the generic case a soliton structure on the Riemannian manifold (M, g) is the choice of a smooth vector field X on M and a real constant λ satisfying the structural requirement

$$Ric + \frac{1}{2}\mathcal{L}_X g = \lambda g, \tag{1.1}$$

where Ric is the Ricci tensor of the metric g and $\mathcal{L}_X g$ is the Lie derivative of this latter in the direction of X. In what follows we shall refer to λ as to the soliton constant. The soliton is called expanding, steady or shrinking if, respectively, $\lambda > 0$, $\lambda = 0$ or $\lambda > 0$. When X is the gradient of a potential $\psi \in C^\infty(M)$, the soliton is called a gradient Ricci soliton [13] and the previous equation (1.1) takes the form

$$\nabla\nabla\psi = S + \lambda g. \tag{1.2}$$

Both equations (1.1) and (1.2) can be considered as perturbations of the Einstein equation

$$Ric = \lambda g. \tag{1.3}$$

and reduce to this latter in case X or $\nabla\psi$ are Killing vector fields. When $X = 0$ or ψ is constant we call the underlying Einstein manifold a trivial Ricci soliton.

Definition 1.1. *A Ricci soliton (g, V, λ) on a Riemannian manifold is defined by*

$$\mathcal{L}_V g + 2S + 2\lambda = 0, \tag{1.4}$$

where S is the Ricci tensor, $\mathcal{L}_V$ is the Lie derivative along the vector field V on M and λ is a real scalar. Ricci soliton is said to be shrinking, steady or expanding according as $\lambda < 0, \lambda = 0$ and $\lambda > 0$, respectively.

It is well know fact that, if the potential vector filed ψ is zero or Killing then the Ricci soliton is an Einstein real hypersurfaces on non-flat complex soace forms [11]. Motivated by this in 2009, J.T. Cho and M. Kimura [12] introduced the notion of η-Ricci solitons and gave a classification of real hypersurfaces in non-flat complex space forms admitting η-Ricci solitons.

Definition 1.2. *An η-Ricci soliton (g, V, λ, μ) on a Riemannian manifold is defined by*

$$\mathcal{L}_X g + 2S + 2\lambda g + 2\mu\eta \otimes \eta = 0, \tag{1.5}$$

where S is the Ricci tensor, $\mathcal{L}_X$ is the Lie derivative along the vector field X on M and λ is a real scalar. In particular $\mu = 0$ then the data (g, ξ, λ) is a Ricci soliton.

In [19], A.E. Fischer introduced a new concept called conformal Ricci flow which is a variation of the classical Ricci flow equation that modifies the unit volume constraint of that equation to a scalar curvature constraint. Since the conformal geometry plays an important role to constrain the scalar curvature and the equations are the vector field sum of a conformal flow equation and a Ricci flow equation, the resulting equations are named as the conformal Ricci flow equations. These new equations are given by

$$\frac{\partial t}{\partial t} = -2S - \left(p + \frac{2}{n}\right) g, \tag{1.6}$$

where $R(g) = -1$ and p is a non-dynamical scalar field(time dependent scalar field), $R(g)$ is the scalar curvature of the manifold and n is the dimension of the manifold M.

The conformal Ricci flow equations are analogous to the Navier-Stokes equations of fluid mechanics and because of this analogy the time dependent scalar field p is called a conformal pressure and, as for the real physical pressure in fluid mechanics that serves to maintain the incompressibility of the fluid, the conformal pressure serves as a Lagrange multiplier to conformally deform the metric flow so as to maintain the scalar curvature constraint. The equilibrium points of the conformal Ricci flow equations are Einstein metrics with Einstein constant $\frac{-1}{n}$. Thus the conformal pressure p is zero at an equilibrium point and positive otherwise.

In 2015, N. Basu and A. Bhattacharyya [1] introduced the notion of conformal Ricci soliton and the equation is as follows

$$\mathcal{L}_V g + 2S + \left[2\lambda - \left(p + \frac{2}{n}\right)\right] g = 0, \tag{1.7}$$

where λ is a constant.

Therefore, It is an interesting and natural to see the condition in case of conformal η-Ricci soliton. From equations (1.5) and (1.7) we are introducing the notion of conformal η-Ricci soliton by the following equation

$$\mathcal{L}_V g + 2S + \left[2\lambda - \left(p + \frac{2}{n}\right)\right] g + 2\mu\eta \otimes \eta = 0, \tag{1.8}$$

where S is the Ricci tensor, $\mathcal{L}_X$ is the Lie derivative along the vector field X on M and λ is a real scalar. In particular $\mu = 0$ then the data (g, ξ, λ) is a conformal-Ricci soliton [1]. The theory of differentiable manifolds with Lorentizain metric is a natural and interesting topic in differential geometry. In [24], T. Ikawa and M. Erdogan studied Lorentzian Sasakian manifold. Lorentzian Kenmotsu manifold introduced by Mihai et al. [29] and K. Kenmotsu [25]. Also Lorentzian para contact manifolds were introduced by K. Matsumoto [28]. Trans Lorentzian para Sasakian manifolds have been used by H. Gill and K. K. Dube [21]. In ([48]

[49]) A. Yıldız et al. studied Lorentzian α- Sasakian also Lorentzian-Sasakian manifolds and Lorentzian β-Kenmotsu manifold studied by Funda et al. in [47]. After that in 2011 S. S Pujar and V. J. Khairnar [35] have initiated the study of Lorentzian Trans-Sasakian manifolds and studied the some basic results with some of its properties. Earlier to this , S. S. Pujar [36] has initiated the study of δ-Lorentzian α Sasakian manifolds. In [16] U. C. De also studied properties of curvatures in Lorentzian Trans Sasakian manifolds.

The study of manifolds with indefinite metrics is of interest from the standpoint of physics and relatively. In 1969, Takahashi [42] has introduced the notion of almost contact metric manifolds equipped with pseudo Riemannian metric. These indefinite almost conatct metric manifolds and indefinite Sasakian manifolds are known as (ε)-almost contact metric manifolds [46]. The concept of (ε)-Sasakian manifolds was initiated by Bejancu and Duggal [4]. U. C. De and A. Sarkar [14] studied the notion of (ε)-Kenmotsu manifolds. S.S. Shukla and D. D. Singh [38] extended the study to (ε)-Trans-Sasakian manifolds with indefnite metric. Siddiqi et al. [39] also studied some properties of Indefinite trans-Sasakian manifolds which is closely related to this topic. The semi-Riemannian manifolds has the index 1 and the structure vector field ξ is always a time like. This motivated the Thripathi and others [43] to introduced (ε)-almost para contact structure where the vector filed ξ is space like or time like according as $(\varepsilon) = 1$ or $(\varepsilon) = -1$.

When M has a Lorentzian metric g, that is, a symmetric non degenerate $(0, 2)$ tensor field of index 1, then M is called a Lorentzian manifold. Since the Lorentzian metric is of index 1, Lorentzian manifold M has not only spacelike vector fields but also timelike and lightlike vector fields. This difference with the Riemannian case give interesting properties on the Lorentzian manifold. A differentiable manifold M has a Lorentzian metric if and only if M has a 1- dimensional distribution. Hence odd dimensional manifold is able to have a Lorentzian metric. Inspired by the above results In 2014, S. M Bhati [2] introduced the notion of δ-Lorentzian Trans Sasakian manifolds.

In 1925, Levy [26] proved that a second order parallel symmetric non-sigular tensor in real space forms is proportional the metric tensor. Later, R. Sharma [37] initiated the study of Ricci solitons in contact Riemannian geometry . After that, many authors extensively studied Ricci soliton (see [8], [9], [10], [23], [30], [31], [40], [41]). The study of η-Ricci solitons in (ε)-almost paracontact metric manifolds have been studied by A. M. Blaga et al. [7]. Recently, A. M. Blaga and various others authors also have been studied η-Ricci solitons in manifolds with different structures (see [5], [6], [37]). Recently K. Venu et al. [45] study the η-Ricci solitns in trans-Sasakian maanifold. In 2016, T. Dutta et al. [17] studied conformal Ricci soliton in Lorentzian α-Sasakian manifols. It is natural and interesting to study Conformal η-Ricci soliton in δ-Lorentzian Trans-Sasakian manifolds. In this paper we derive the condition for a 3 dimensional δ-Lorentzian Trans-Sasakian manifold as a confromal η-Ricci soliton and derive expression for the scalar curvature. Moreover, in the last section studied the gradient conformal Ricci soliton for a 3 dimensional δ-Lorentzian Trans-Sasakian manifolds.

2. Preliminaries

Let M be an δ-almost contact metric manifold equipped with δ-almost contact metric structure $(\phi, \xi, \eta, g, \delta)$ consisting of a $(1,1)$ tensor field ϕ, a vector field ξ, a 1-form η and an indefinite metric g such that

$$\phi^2 = X + \eta(X)\xi, \qquad \eta(\xi) = -1, \qquad \eta \circ \phi = 0, \qquad \phi\xi = 0, \tag{2.9}$$

$$g(\phi X, \phi Y) = g(X,Y) + \delta\eta(X)\eta(Y), \quad \eta(X) = \delta g(X,\xi), \quad g(\xi,\xi) = -\delta, \tag{2.10}$$

for all $X, Y \in M$, where δ is such that $\delta^2 = 1$ so that $\delta = \pm 1$. The above structure $(\phi, \xi, \eta, g, \delta)$ on M is called the δ Lorentzian structure on M. If $\delta = 1$ and this is usual Lorentzian structure [35] on M, the vector field ξ is the time like [43], that is M contains a time like vector field. In [44], Tanno classified the connected almost contact metric manifold.

In [20], Grey and Harvella was introduced the classification of almost Hermitian manifolds, there appears a class W_4 of Hermitian manifolds which are closely related to the conformal Kaehler manifolds. The class $C_6 \oplus C_5$ [32] coincides with the class of trans-Sasakian structure of type (α, β). In fact, the local nature of the two sub classes, namely C_6 and C_5 of trans-Sasakian structures are characterized completely [27].

An almost contact metric structure on M is called a trans-Sasakian (see [3], [32]) if $(M \times R, J, G)$ belongs to the class W_4, where J is the almost complex structure on $M \times R$ defined by

$$J\left(X, f\frac{d}{dt}\right) = \left(\phi(X) - f\xi, \eta(X)\frac{d}{dt}\right)$$

for all vector fields X on M and smooth functions f on $M \times R$ and G is the product metric on $M \times R$. This may be expressed by the condition

$$(\nabla_X \phi)Y = \alpha(g(X,Y)\xi - \eta(Y)X) + \beta(g(\phi X, Y)\xi - \eta(Y)\phi X) \tag{2.11}$$

for any vector fields X and Y on M, ∇ denotes the Levi-Civita connection with respect to g, α and β are smooth functions on M. The existence of condition (2.3) is ensure by the above discussion.

With the above literature now we define the δ-Lorentzian trans-Sasakian manifolds [2] as follows.

Definition 2.1. *A δ-Lorentzian manifold with structure $(\phi, \xi, \eta, g, \delta)$ is said to be δ-Lorentzian trans-Sasakian manifold of type (α, β) if it satisfies the condition*

$$(\nabla_X \phi)Y = \alpha(g(X, Y)\xi - \delta\eta(Y)X) + \beta(g(\phi X, Y)\xi - \delta\eta(Y)\phi X). \tag{2.12}$$

for any vector fields X and Y on M.

If $\delta = 1$, then the δ-Lorentzian trans Sasakian manifold is the usual Lorentzian trans Sasakian manifold of type (α, β) [32]. δ-Lorentzian trans Sasakian manifold of type $(0, 0)$, $(0, \beta)$ $(\alpha, 0)$ are the Lorentzian cosymplectic, Lorentzian β-Kenmotsu and Lorentzian α-Sasakian manifolds respectively. In particular if $\alpha = 1$, $\beta = 0$ and $\alpha = 0$, $\beta = 1$, the δ-Lorentzian trans Sasakian manifolds reduces to δ-Lorentzian Sasakian and δ-Lorentzian Kenmotsu manifolds respectively. Form (2.12), we have

$$\nabla_X \xi = \delta \left\{-\alpha\phi(X) - \beta(X + \eta(X)\xi\right\}, \tag{2.13}$$

and

$$(\nabla_X \eta)Y = \alpha g(\phi X, Y) + \beta[g(X, Y) + \delta\eta(X)\eta(Y)]. \tag{2.14}$$

In a δ-Lorentzian trans Sasakian manifold M, we have the following relations:

$$R(X, Y)\xi = (\alpha^2 + \beta^2)[\eta(Y)X - \eta(X)Y] + 2\alpha\beta[\eta(Y)\phi X - \eta(X)\phi Y] \tag{2.15}$$

$$+\delta[(Y\alpha)\phi X - (X\alpha)\phi Y + (Y\beta)\phi^2 X - (X\beta)\phi^2 Y]$$

$$S(X, \xi) = [((n-1)(\alpha^2 + \beta^2) - (\xi\beta)]\eta(X) + \delta((\phi X)\alpha) + (n-2)\delta(X\beta), \tag{2.16}$$

$$Q\xi = \delta(n-1)(\alpha^2 + \beta^2) - (\xi\beta))\xi + \delta\phi(grad\alpha) - \delta(n-2)(grad\beta), \tag{2.17}$$

where R is curvature tensor, while Q is the Ricci operator given by $S(X, Y) = g(QX, Y)$. Further in an δ-Lorentzian trans Sasakian manifold , we have

$$\delta\phi(grad\alpha) = \delta(n-2)(grad\beta), \tag{2.18}$$

$$2\alpha\beta - \delta(\xi\alpha) = 0. \tag{2.19}$$

Using (2.15) and (2.18), for constants α and β , we have

$$R(\xi, X)Y = (\alpha^2 + \beta^2)[\delta g(X, Y)\xi - \eta(Y)X], \tag{2.20}$$

$$R(X, Y)\xi = (\alpha^2 + \beta^2)[\eta(Y)X - \eta(X)Y], \tag{2.21}$$

$$\eta(R(X, Y)Z) = \delta(\alpha^2 + \beta^2)[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)], \tag{2.22}$$

$$S(X, \xi) = [((n-1)(\alpha^2 + \beta^2) - \delta(\xi\beta)]\eta(X), \tag{2.23}$$

$$Q\xi = [(n-1)(\alpha^2 + \beta^2) - (\xi\beta)]\xi. \tag{2.24}$$

An important consequence of (2.21) is that ξ is a geodesic vector field.

$$\nabla_\xi \xi = 0. \tag{2.25}$$

For arbitrary vector field X, we have that

$$d\eta(\xi, X) = 0. \tag{2.26}$$

The ξ-sectional curvature K_ξ of M is the sectional curvature of the plane spanned by ξ and a unit vector field X. From (2.21), we have

$$K_\xi = g(R(\xi, X), \xi, X) = (\alpha^2 + \beta^2) - \delta(\xi\beta). \tag{2.27}$$

It follows from (2.27) that ξ-sectional curvature does not depend on X.

3. CONFORMAL η-SOLITONS ON $(M, \phi, \xi, \eta, g, \delta)$

In the study of the conformal η-Ricci soliton equation we will consider certain assumptions, one essential condition being $\nabla\xi = I_\xi(M) + \eta \otimes \xi$ which naturally arises in different geometry of δ-Lorentzian trans-Sasakian manifolds.

An important geometrical object in studying Ricci solitons is a symmetric $(0, 2)$- tensor field which is parallel with respect to the Levi-Civita connection
Fix h a symmetric tensor field of $(0, 2)$-type which we suppose to be parallel with respect to the Levi-Civita connection ∇ that is $\nabla h = 0$. Applying the Ricci commutation identity [18].

$$\nabla^2 h(X, Y; Z, W) - \nabla^2 h(X, Y; W, Z) = 0, \tag{3.28}$$

we obtain the relation

$$h(R(X, Y)Z, W) + h(Z, R(X, Y)W) = 0. \tag{3.29}$$

Replacing $Z = W = \xi$ in (3.29) and using (2.15) and also use the symmetry of h, we have

$$2(\alpha^2 + \beta^2)[\eta(Y)h(X, \xi) - \eta(X)h(Y, \xi)] + 2\delta[(Y\alpha)h(\phi X, \xi) - (X\alpha)h(\phi Y, \xi)] \tag{3.30}$$

$$+2\delta[(Y\beta)h(\phi^2X,\xi)-(X\beta)h(\phi^2Y,\xi)]+4\alpha\beta[\eta(Y)h(\phi X,\xi)-\eta(X)h(\phi Y,\xi)]$$

Putting $X=\xi$ in (3.30) and by virtue of (2.9), we obtain

$$-2[(\delta\xi\alpha-2\alpha\beta]h(\phi Y,\xi)+2[(\alpha^2+\beta^2)-\delta(\xi\beta)][\eta(Y)h(\xi,\xi)-h(Y,\xi)]=0. \tag{3.31}$$

By using (2.19) in (3.31), we have

$$[(\alpha^2+\beta^2)-\delta(\xi\beta)][\eta(Y)h(\xi,\xi)-h(Y,\xi)]=0. \tag{3.32}$$

Suppose $(\alpha^2+\beta^2)-\delta(\xi\beta)\neq 0$, it results

$$h(Y,\xi)=\eta(Y)h(\xi,\xi). \tag{3.33}$$

Now, we can call a regular δ-Lorentzian trans Sasakian manifold with $(\alpha^2+\beta^2)-\delta(\xi\beta)\neq 0$, where regularity, means the non-vanishing of the Ricci curvature with respect to the generator of δ-Lorentzian trans Sasakian manifolds. Differentiating (3.33) covariantly with respect to X, we have

$$(\nabla_X h)(Y,\xi)+h(\nabla_X Y,\xi)+h(Y,\nabla_X\xi)=[\delta g(\nabla_X Y,\xi)+\delta g(Y,\nabla_X\xi)]h(\xi,\xi) \tag{3.34}$$

$$+\eta(Y)[(\nabla_X h)(Y,\xi)+2h((\nabla_X\xi,\xi)].$$

By using the parallel condition $\nabla h=0$, $\eta(\nabla_X\xi)=0$ and by the virtue of (3.33) in (3.34), we get

$$h(Y,\nabla_X\xi)=\delta g(Y,\nabla_X\xi)h(\xi,\xi).$$

Now using (2.13) in the above equation, we get

$$-\alpha h(Y,\phi X)+\beta\delta h(Y,X)=-\alpha g(Y,\phi X)h(\xi,\xi)+\beta\delta g(Y,X)h(\xi,\xi). \tag{3.35}$$

Replacing $X=\phi X$ in (3.35) and after simplification, we get

$$h(X,Y)=\delta g(X,Y)h(\xi,\xi), \tag{3.36}$$

which together with the standard fact that the parallelism of h implies that $h(\xi,\xi)$ is a constant, via (3.33). Now by considering the above equations, we can gives the conclusion:

Theorem 3.1. *Let $(M,\phi,\xi,\eta,g,\delta)$ be an δ-Lorentzian trans Sasakian manifold with non-vanishing ξ-sectional curvature and endowed with a tensor field $h\in\Gamma(T_2^0(M))$ which is symmetric and ϕ-skew-symmetric. If h is parallel with respect to ∇ then it is a constant multiple of the metric tensor g.*

Definition 3.1. *Let $(M, \phi, \xi, \eta, g, \delta)$ be an δ-almost contact metric manifold. consider the equation*

$$\mathcal{L}_\xi g + 2S + \left[2\lambda - \left(p + \frac{2}{n}\right)\right] g + 2\mu\eta \otimes \eta = 0, \tag{3.37}$$

where $\mathcal{L}_\xi$ is the Lie derivative operator along the vector field ξ, S is the Ricci curvature tensor field of the metric g and λ and μ are real constants. For $\mu \neq 0$, the data (g, ξ, λ, μ) will be called conformal Ricci-soliton.

Remark 3.1. *If the scalar curvature $-\frac{1}{2}(p + \frac{2}{n})$ of the manifold is constant, then the conformal η-Ricci soliton $(g, \xi, \left\{\lambda - \frac{1}{2}(p + \frac{2}{n})\right\}, \mu)$ reduces to an η-Ricci soliton and, moreover, if $\mu = 0$, to a Ricci soliton $(g, \xi, \left\{\lambda - \frac{1}{2}(p + \frac{2}{n})\right\})$. Therefore, the two concepts of Conformal η-Ricci soliton and η-Ricci soliton are distinct on manifolds of non constant scalar curvature.*

Writing $\mathcal{L}_\xi g$ in terms of the Levi-Civita connection ∇, we obtain [13]:

$$2S(X,Y) = -g(\nabla_X \xi, Y) - g(X, \nabla_X \xi) - \frac{1}{2}\left[2\lambda - \left(p + \frac{2}{n}\right)\right] g(X,Y) - 2\mu\eta(X)\eta(Y), \tag{3.38}$$

for any $X, Y \in \chi(M)$.

The data (g, ξ, λ, μ) which satisfy the equation (3.37) is said to be conformal η- Ricci soliton on M [12] and its called shrinking, steady or expanding according as $\lambda < 0$, $\lambda = 0$ or $\lambda > 0$ respectively [12]. Now, from (2.13) , the equation (3.37) becomes:

$$S(X,Y) = -\frac{1}{2}\left[2\lambda - \left(p + \frac{2}{n}\right) + \delta\beta\right] g(X,Y) + (\beta\delta - \mu)\eta(X)\eta(Y). \tag{3.39}$$

The above equations yields

$$S(X,\xi) = -\frac{1}{2}\left[2\lambda - \left(p + \frac{2}{n}\right) + \mu\right]\eta(X) \tag{3.40}$$

$$QX = -\frac{1}{2}\left[2\lambda - \left(p + \frac{2}{n}\right) + \delta\beta\right] X + (\beta\delta - \mu)\xi \tag{3.41}$$

$$Q\xi = -\frac{1}{2}\left[2\lambda - \left(p + \frac{2}{n}\right) + \mu\right]\xi \tag{3.42}$$

$$r = -\frac{1}{2}\left[2\lambda - \left(p + \frac{2}{n}\right) + \delta\beta\right] n - (n-1)\beta\delta - \mu, \tag{3.43}$$

where r is the scalar curvature. Of the two natural situations regrading the vector field V: $V \in Span\,\{\xi\}$ and $V \perp \xi$, we investigate only the case $V = \xi$.

Our interest is in the expression for $\mathcal{L}_\xi g + 2S + 2\mu\eta \otimes \eta$. A direct computation gives

$$\mathcal{L}_\xi g(X,Y) = 2\beta\delta[g(X,Y) + \eta(X)\eta(Y)]. \tag{3.44}$$

In 3-dimensional δ-Lorentzian trans Sasakian manifold the Riemannian curvature tensor is given by

$$R(X,Y)Z = g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X - S(X,Z)Y \\ - \frac{r}{2}[g(Y,Z)X - g(X,Z)Y], \tag{3.45}$$

Putting $Z = \xi$ in (3.45) and using (2.15) and (2.16) for 3-dimensional δ-Lorentzian trans-Sasakian manifold, we get

$$(\alpha^2+\beta^2)[\eta(Y)X - \eta(X)Y] + 2\alpha\beta[\eta(Y)\phi X - \eta(X)\phi Y] \\ +\delta[(Y\alpha)\phi X - (X\alpha)\phi Y] + \delta[(Y\beta)\phi^2 X - (X\beta)\phi^2 Y] \\ = [(\alpha^2+\beta^2) - (\xi\beta)][\eta(Y)X - \eta(X)Y] \\ + \delta\eta(Y)QX - \delta\eta(X)QY - \delta[((\phi Y)\alpha)X + (Y\beta)X] \\ +\delta[((\phi X)\alpha)Y + (X\beta)Y]. \tag{3.46}$$

Again, putting $Y = \xi$ in the (3.46) and using (2.9) and (2.19), we obtain

$$QX = \left[\frac{r}{2} + (\xi\beta) - (\alpha^2+\beta^2)\right] X + \left[\frac{r}{2} + (\xi\beta) - 3(\alpha^2+\beta^2)\right] \eta(X)\xi. \tag{3.47}$$

From (3.47), we have

$$S(X,Y) = \left[\frac{r}{2} + (\xi\beta) - (\alpha^2+\beta^2)\right] g(X,Y) \\ + \left[\frac{r}{2} + (\xi\beta) - 3(\alpha^2+\beta^2)\right] \delta\eta(X)\eta(Y). \tag{3.48}$$

Equation (3.48) shows that a 3-dimensional (ϵ,δ)-trans-Sasakian manifold is η-*Einstein*. Next, we consider the equation

$$h(X,Y) = (\mathcal{L}_\xi g)(X,Y) + 2S(X,Y) + 2\mu\eta(X)\eta(Y). \tag{3.49}$$

By Using (3.44) and (3.48) in (3.49), we have

$$h(X,Y) = \left[r - 4(\alpha^2+\beta^2) + 2\beta\delta\right] g(X,Y) \\ + \left[8(\alpha^2+\beta^2) - 2\beta\delta - r\right] \delta\eta(X)\eta(Y) + 2\mu\eta(X)\eta(Y). \tag{3.50}$$

Putting $X = Y = \xi$ in (2.11), we get

$$h(\xi,\xi) = 2[2\delta(\alpha^2+\beta^2) - 2\mu] \tag{3.51}$$

Now, (3.36) becomes

$$h(X,Y) = 2[2\delta(\alpha^2+\beta^2) - 2\mu]\delta g(X,Y). \tag{3.52}$$

From (3.49) and (3.52), it follows that g is conformal η-Ricci soliton.

Therefore, we can state as:

Theorem 3.2. *Let $(M, \phi, \xi, \eta, g, \delta)$ be a 3-dimensional δ-Lorentzian trans-Sasakian manifold, then $(g, \xi, \left\{\lambda - \frac{1}{2}(p + \frac{2}{n})\right\}, \mu)$ yields a conformal η-Ricci soliton on M.*

Let V be pointwise collinear with ξ. i.e., $V = b\xi$, where b is a function on the 3-dimensional δ-Lorentzian trans-Sasakian manifold. Then

$$g(\nabla_X b\xi, Y) + g(\nabla_Y b\xi, X) + 2S(X,Y) + \left[2\lambda - \left(p + \frac{2}{n}\right)\right] g(X,Y) + 2\mu\eta(X)\eta(Y) = 0.$$

or

$$\begin{gathered} bg((\nabla_X \xi, Y) + (Xb)\eta(Y) + bg(\nabla_Y \xi, X) + (Yb)\eta(X) \\ +2S(X,Y) + \left[2\lambda - \left(p + \frac{2}{n}\right)\right] g(X,Y) + 2\mu\eta(X)\eta(Y) = 0. \end{gathered}$$

Using (2.13), we obtain

$$\begin{gathered} bg(-\delta\alpha\phi X - \beta\delta(X + \eta(X)\xi, Y) + (Xb)\eta(Y) + bg(-\delta\alpha\phi Y - \beta\delta(Y + \eta(Y)\xi, X) \\ +(Yb)\eta(X) + 2S(X,Y) + \left[2\lambda - \left(p + \frac{2}{n}\right)\right] g(X,Y) + 2\mu\eta(X)\eta(Y) = 0. \end{gathered}$$

which yields

$$\begin{gathered} -2b\beta\delta g(X,Y) - 2b\beta\delta\eta(X)\eta(Y) + (Xb)\eta(Y) \\ +(Yb)\eta(X) + 2S(X,Y) + \left[2\lambda - \left(p + \frac{2}{n}\right)\right] g(X,Y) + 2\mu\eta(X)\eta(Y) = 0. \end{gathered} \tag{3.53}$$

Replacing Y by ξ in (3.53), we obtain

$$(Xb) + (\xi b)\eta(X) + 2\left[2(\alpha^2 + \beta^2) - (\xi\beta) + \left[2\lambda - \left(p + \frac{2}{n}\right)\right] + \mu - 2b\beta\delta\right]\eta(X). \tag{3.54}$$

Again putting $X = \xi$ in (3.54), we obtain

$$\xi b = -2(\alpha^2 + \beta^2) + (\xi\beta) - \frac{1}{2}\left[2\lambda - \left(p + \frac{2}{n}\right)\right] - \mu + 2b\beta\delta.$$

Plugging this in (3.54), we get

$$(Xb) + 2[2(\alpha^2 + \beta^2) - (\xi\beta) - \frac{1}{2}\left[2\lambda - \left(p + \frac{2}{n}\right)\right] + \mu - 2b\beta\delta]\eta(X) = 0,$$

or

$$db = -\frac{1}{2}\left[2\lambda - \left(p + \frac{2}{n}\right)\right] + \mu - (\xi\beta) + 2((\alpha^2 + \beta^2) - 2b\beta\delta)\eta. \tag{3.55}$$

Applying d on (3.55), we get $\left\{-\frac{1}{2}\left[2\lambda - \left(p + \frac{2}{n}\right)\right] + \mu - (\xi\beta) + 2(\alpha^2 + \beta^2) - 2b\beta\delta\right\} d\eta$. Since $d\eta \neq 0$ we have

$$-\frac{1}{2}\left[2\lambda - \left(p + \frac{2}{n}\right)\right] + \mu - (\xi\beta) + 2(\alpha^2 + \beta^2) - 2b\beta\delta = 0. \tag{3.56}$$

Equation(3.56) in (3.55) yields b as a constant. Therefore from (3.53), it follows that

$$S(X,Y) = \left(-\frac{1}{2}\left[2\lambda - \left(p + \frac{2}{n}\right)\right] + 2b\beta\delta\right) g(X,Y) + (2b\beta\delta - \mu)\eta(X)\eta(Y), \tag{3.57}$$

which implies that M is of constant scalar curvature for constant $2b\beta\delta$. This leads to the following:

Theorem 3.3. *If in a 3-dimensional δ-Lorentzian trans-Sasakian manifold the metric g is a conformal η-Ricci soliton and V is positive collinear with ξ, then V is a constant multiple of ξ and g is of constant scalar curvature provided $b\beta\delta$ is a constant.*

Tanking $X = Y = \xi$ in (3.36) and (3.48) and comparing, we get

$$\lambda = \frac{1}{2}\left(p + \frac{2}{n}\right) - 2(\alpha^2 + \beta^2) - \delta(\xi\beta) + \mu - 2b\beta\delta = -2K_\xi + \frac{1}{2}\left(p + \frac{2}{n}\right) - \mu. \tag{3.58}$$

From (3.43) and (3.57) also put $n = 3$, we obtain

$$r = \left(\frac{p}{2} + \frac{1}{3}\right) + 6(\alpha^2 + \beta^2) - 3\delta(\xi\beta) - 2\beta\delta + 2\mu. \tag{3.59}$$

Now for conformal Ricci soliton $r = -1$, so putting this value in the above equation we get

$$\mu = -\left(p + \frac{2}{3}\right) - (\alpha^2 + \beta^2) + \frac{3}{2}\delta(\xi\beta) + \beta\delta.$$

Since λ is a constant, it follows from (3.57) that K_ξ is a constant.

Theorem 3.4. *Let (g, ξ, μ) be a conformal η-Ricci soliton in $(M, \phi, \xi, \eta, g, \delta)$ a 3-dimensional δ-Lorentzian trans-Sasakian manifold. Then the scalar $\lambda - \left(\frac{p}{2} + \frac{1}{3}\right) + \mu = -2K_\xi$, $r = 6K_\xi + 2\mu - 3(\xi\beta) - 2b\beta\delta + \left(\frac{p}{2} + \frac{1}{3}\right)$.*

Remark 3.2. *For $\mu = 0$, (3.57) reduces to $\lambda = -2K_\xi + \left(\frac{p}{2} + \frac{1}{3}\right)$, so confromal Ricci soliton in 3-dimensional δ-Lorentzian trans-Sasakian manifold is shrinking.*

Example 3.1. *Consider the 3-dimensional manifold $M = \left\{(x, y, z) \in \mathbb{R}^3 : z \neq 0\right\}$, where (x, y, z) are the Cartesian coordinates in $\mathbb{R}^3$ and let the vector fields are*

$$e_1 = \frac{e^x}{z^2}\frac{\partial}{\partial x}, \quad e_2 = \frac{e^y}{z^2}\frac{\partial}{\partial y}, \quad e_3 = \frac{-(\delta)}{2}\frac{\partial}{\partial z},$$

where e_1, e_2, e_3 are linearly independent at each point of M. Let g be the Riemannain metric defined by

$g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = -\delta$, $g(e_1, e_3) = g(e_2, e_3) = g(e_1, e_2) = 0$,

where δ is such that $\delta^2 = 1$ so that $\delta = \pm 1$.

Let η be the 1-form defined by $\eta(X) = \delta g(X, \xi)$ for any vector field X on M, and ϕ be the (1,1) tensor field defined by $\phi(e_1) = e_2, \quad \phi(e_2) = -e_1, \quad \phi(e_3) = 0$. Then by using the linearity of ϕ and g, we have $\phi^2 X = X + \eta(X)\xi$, with $\xi = e_3$. Further $g(\phi X, \phi Y) = g(X, Y) + \delta\eta(X)\eta(Y)$ for any vector fields X and Y on M. Hence for $e_3 = \xi$, the structure defines an (δ)-almost contact structure in $\mathbb{R}^3$.

Let ∇ be the Levi-Civita connection with respect to the metric g, then we have

$$2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) - g(X, [Y, Z])$$

$$-g(Y, [X, Z]) + g(Z, [X, Y]),$$

which is know as Koszul's formula.

$\nabla_{e_1} e_3 = -\frac{\delta}{z} e_1, \quad \nabla_{e_2} e_3 = -\frac{\delta}{z} e_2, \quad \nabla_{e_1} e_2 = 0,$
using the above relation, for any vector X on M, we have $\nabla_X \xi = \delta[-\alpha\phi X - \beta(X + \eta(X)\xi)]$, where $\alpha = \frac{1}{z}$ and $\beta = -\frac{1}{z}$. Hence (ϕ, ξ, η, g) structure defines the δ-Lorentzian trans-Sasakian structure in $\mathbb{R}^3$.

Here ∇ be the Levi-Civita connection with respect to the metric g , then we have

$$[e_1, e_2] = 0, \qquad [e_1, e_3] = -\frac{(\delta)}{z} e_1, \quad [e_2, e_3] = -\frac{(\delta)}{z} e_2.$$

Since $g(e_1, e_2) = 0$. Thus we have

$$\nabla_{e_1} e_3 = -\frac{(\delta)}{z} e_1 + e_2, \quad \nabla_{e_1} e_2 = 0$$

$$\nabla_{e_2} e_1 = 0, \quad \nabla_{e_2} e_2 = -\frac{(\delta)}{z} e_2, \quad \nabla_{e_2} e_3 = -\frac{(\delta)}{z} e_2 - e_1$$

$$\nabla_{e_3} e_1 = 0, \quad \nabla_{e_3} e_2 = 0, \quad \nabla_{e_3} e_3 = -\frac{(\delta)}{z} e_1 + e_2.$$

The manifold M satisfies (2.5) with $\alpha = \frac{1}{z}$ and $\beta = -\frac{1}{z}$. Hence M is an δ-Lorentzian trans-Sasakian manifolds. Then the non-vanishing components of the curvature tensor fields are computed as follows:

$$R(e_1, e_3)e_3 = \frac{(\delta)}{z^2} e_1, \quad R(e_3, e_1)e_3 = -\frac{(\delta)}{z^2} e_1,$$

$$R(e_2, e_3)e_3 = \frac{(\delta)}{z^2} e_1, \quad R(e_3, e_2)e_3 = -\frac{(\delta)}{z^2} e_1.$$

From the above expression of the curvature tensor we can also obtain the Ricci tensor

$$S(e_1, e_1) = S(e_2, e_2) = S(e_3, e_3) = \frac{(\delta^2)}{z^2}$$

since $g(e_1,e_3)=g(e_1,e_2)=0$.

Therefore, we have

$$S(e_i,e_i)=\frac{(\delta)}{z^2}g(e_i,e_i),$$

and the scalar curvature $scal=3\frac{(\delta^2)}{z^2}$. for $i=1,2,3$, and $\alpha=\frac{1}{z}$, $\beta=-\frac{1}{z}$. Hence M is also an *Einstein* manifold. In this case, from (3.11), computed (e_i,e_i) as follows

$$2[g(e_i,e_i)-\eta(e_i)\eta(e_i)]+2S(e_i,e_i)+\left[2\lambda-\left(p+\frac{2}{3}\right)\right]g(e_i,e_i)+2\mu\eta(e_i)\eta(e_i)=0$$

for all $i\in\{1,2,3\}$, and we have

$$2(1-\delta_{i3})+2\frac{\delta}{z^2}+(2\lambda-3\frac{\delta}{z^2})+2\mu\delta_{i3}=0$$

for all $i\in\{1,2,3\}$

Therefore $\lambda=2\left(\frac{p}{4}-\frac{1}{3}-\frac{(\delta)}{z^2}\right)$ and $\mu=-\frac{(\delta)}{z^2}+1$, the data (g,ξ,λ,μ) is an conformal η-Ricci soliton on $(M,\phi,\xi,\eta,g,\delta)$.

Here in this example if $\mu=0$, then (g,ξ,λ,μ) reduce to conformal Ricci soliton for $\lambda=2\left(\frac{p}{4}-\frac{1}{3}-\frac{(\delta)}{z^2}\right)$ which is positive. Therefore conformal Ricci soliton is expanding for $\lambda>0$.

4. Gradient Conformal Ricci Solitons in 3-dimensional δ-Lorentzian trans-Sasakian

Definition 4.1. *A Riemannian manifold (M,g) is said to be conformal gradient Ricci soliton if there exist a confromal change of the metric $\bar{g}=e^u g$, $u\in C^\infty(M)$, a function $\psi\in C^\infty(M)$ and a constant $\lambda\in\mathbb{R}$ such that*

$$Ric+Hess(\psi)=\lambda\bar{g} \tag{4.60}$$

If the vector field V is the gradient of a potential function -ψ then $\bar{g}$ is called a conformal gradient Ricci soliton and (1.2) assume the form

$$\nabla\nabla\psi=S+\frac{1}{2}\left[2\lambda-\left(p+\frac{2}{n}\right)\right]e^u g. \tag{4.61}$$

This reduces to

$$\nabla_Y D\psi=QY+\frac{1}{2}\left[2\lambda-\left(p+\frac{2}{n}\right)\right]Y, \tag{4.62}$$

where D denoted the gradient operator of g. From (4.61) it follows

$$R(X,Y)D\psi=(\bar{\nabla}_X Q)Y-(\bar{\nabla}_Y Q)X. \tag{4.63}$$

Differentiating (3.47) we get

$$(\nabla_W Q)X = \frac{dr(W)}{2}(X - \eta(X)\xi)) - (\frac{r}{2} - 3(\alpha^2+\beta^2))(\alpha(g(\phi W, X) \tag{4.64}$$

$$+\beta\delta g(W,X) - \delta\beta\eta(X)\eta(W)) + \eta(X)\nabla_W\xi.$$

In (4.63) replacing $W = \xi$, we obtain

$$(\nabla_\xi Q)X = \frac{dr(\xi)}{2}(X - \eta(X)\xi)). \tag{4.65}$$

Then we have

$$g(\nabla_\xi Q)X - (\bar{\nabla}_X Q)(\xi,\xi) \tag{4.66}$$

$$= g(\frac{dr(\xi)}{2}(X - \eta(X)\xi,\xi)) = \frac{dr(\xi)}{2}(g(X,\xi) - \eta(X))) = 0.$$

Using (4.65) and (4.64), we obtain

$$g(R(\xi,X)D\psi,\xi) = 0. \tag{4.67}$$

From (2.20)

$$g(\bar{R}(\xi,Y)D\psi,\xi) = (\alpha^2+\beta^2)(g(Y,D\psi) - \eta(Y)\eta(D\psi)).$$

Using (4.66), we get

$$(\alpha^2+\beta^2)(g(Y,D\psi) - \eta(Y)\eta(D\psi)) = 0$$

$$(\alpha^2+\beta^2)(g(Y,D\psi) - \eta(Y)g(D\psi,\xi)) = 0,$$

or

$$(g(Y,D\psi) - g(Y,\xi)g(D\psi,\xi)) = 0,$$

which implies

$$(g(Y,D\psi) - g(Y,\xi)g(D\psi,\xi)) = 0,$$

which implies

$$D\psi = (\xi\psi)\xi, \quad \text{since} \quad \alpha^2+\beta^2 \neq -\delta(\xi\beta). \tag{4.68}$$

Using (4.67) and (4.61)

$$S(X,Y) + \frac{1}{2}\left[2\lambda - \left(p + \frac{2}{n}\right)\right]e^u g(X,Y) = g(\nabla_Y D\psi, X) = g(\nabla_Y(\xi\psi)\xi, X)$$

$$= (\xi\psi)g(\bar{\nabla}_Y\xi, X) + Y(\xi\psi)\eta(X)$$

$$= (\xi\psi)g(-\delta\alpha\phi Y - \delta\beta Y - \delta\beta\eta(Y)\xi, X) + Y(\xi\psi)\eta(X)$$

$$S(X,Y) + \frac{1}{2}\left[2\lambda - \left(p + \frac{2}{n}\right)\right]\bar{g}(X,Y) = -\delta\alpha(\xi\psi)g(\phi Y, X) - \delta\beta(\xi\psi)\bar{g}(Y,X) \tag{4.69}$$

$$-\delta\beta(\xi\psi)\eta(Y)\eta(X) + Y(\xi\psi)\eta(X).$$

Putting $X=\xi$ in (4.68) and using (2.23) we get

$$\bar{S}(Y,\xi)+\frac{1}{2}\left[2\lambda-\left(p+\frac{2}{n}\right)\right]e^{u}\eta(Y)=Y(\xi\psi)=[\lambda+2\delta\beta+2(\alpha^2+\beta^2-\delta(\xi\beta))]e^{u}\eta(Y). \tag{4.70}$$

Interchanging X and Y in (4.68), we get

$$S(X,Y)+\frac{1}{2}\left[2\lambda-\left(p+\frac{2}{n}\right)\right]\bar{g}(X,Y)=-\delta\alpha(\xi\psi)g(Y,\phi X) \tag{4.71}$$

$$-\delta\beta(\xi\psi)\bar{g}(X,Y)-\delta\beta(\xi\psi)\eta(Y)\eta(X)+X(\xi\psi)\eta(Y).$$

Adding (4.68) and (4.70) we get

$$2S(X,Y)+\left[2\lambda-\left(p+\frac{2}{n}\right)\right]\bar{g}(X,Y)=-2\delta\beta(\xi\psi)\bar{g}(X,Y)+Y(\xi\psi)\eta(X) \tag{4.72}$$

$$-2\delta\beta(\xi\psi)\eta(X)\eta(Y)+X(\xi\psi)\eta(Y).$$

Using (4.69) in (4.71) we have

$$S(X,Y)+\frac{1}{2}\left[2\lambda-\left(p+\frac{2}{n}\right)\right]\bar{g}(X,Y)=-\delta\beta(\xi\psi)[g(X,Y)-\eta(X)\eta(Y)] \tag{4.73}$$

$$+\frac{1}{2}\left[2\lambda-\left(p+\frac{2}{n}\right)\right]+\delta\beta+2(\alpha^2+\beta^2-\delta(\xi\beta))]\eta(X)\eta(Y).$$

Then using (4.61) we have

$$\nabla_Y D\psi=-\delta\beta(\xi\psi)(Y-\eta(Y)\xi) \tag{4.74}$$

$$+\left[\frac{1}{2}\left[2\lambda-\left(p+\frac{2}{n}\right)\right]+\delta\beta+2(\alpha^2+\beta^2-\delta(\xi\beta))\right]\eta(Y)\xi.$$

Using (4.73) we calculate

$$R(X,Y)D\psi=\nabla_X\nabla_Y D\psi-\nabla_Y\nabla_X D\psi-\nabla_{[X,Y]}D\psi$$

$$=-\delta\beta X(\xi\psi)Y+\delta\beta Y(\xi\psi)X \tag{4.75}$$

$$-\delta\beta Y(\xi\psi)\eta(X)\xi+\delta\beta X(\xi\psi)\eta(Y)\xi$$

$$+\left[\frac{1}{2}\left[2\lambda-\left(p+\frac{2}{n}\right)\right]+\delta\beta+2(\alpha^2+\beta^2-\delta(\xi\beta))\right]((\nabla_X\eta)(Y)\xi-(\nabla_Y\eta)(X)\xi)$$

$$+\left[\frac{1}{2}\left[2\lambda-\left(p+\frac{2}{n}\right)\right]+\delta\beta+2(\alpha^2+\beta^2-\delta(\xi\beta))\right]((\nabla_X\xi)\eta(Y)\xi-(\nabla_Y\xi)\eta(X)).$$

Taking inner product with ξ in (4.74), we get

$$0=g((X,Y)D\psi,\xi)=2\delta\alpha+\left[\frac{1}{2}\left[2\lambda-\left(p+\frac{2}{n}\right)\right]+\delta\beta+2(\alpha^2+\beta^2-\delta(\xi\beta))\right]g(\phi Y,X). \tag{4.76}$$

Thus we have $2\delta\alpha+\left[\frac{1}{2}\left[2\lambda-\left(p+\frac{2}{n}\right)\right]+\delta\beta+2(\alpha^2+\beta^2-\delta(\xi\beta))\right]=0.$

Now we consider the following cases:

Case (i) $\delta\alpha = 0$, or

Case (ii) $[\left[\lambda - \left(\frac{p}{2} + \frac{1}{n}\right)\right] + \delta\beta + 2(\alpha^2 + \beta^2 - \delta(\xi\beta))] = 0$,

Case (iii) $\alpha = 0$ and $[\left[\lambda - \left(\frac{p}{2} + \frac{1}{n}\right)\right] + \delta\beta + 2(\alpha^2 + \beta^2 - \delta(\xi\beta))] = 0$.

Case (i) If $\alpha = 0$, the manifold reduces to a δ-Lorentzian β-Kenmotsu manifold.

Case (ii) Let $[\left[\lambda - \left(\frac{p}{2} + \frac{1}{n}\right)\right] + \delta\beta + 2(\alpha^2 + \beta^2 - \delta(\xi\beta))] = 0$. If we use this in (4.69) we get $Y(\xi\psi) = -\delta\beta(\xi\psi)\eta(Y)$. Substitute this value in (4.71) we obtain

$$S(X,Y) + \frac{1}{2}\left[2\lambda - \left(p + \frac{2}{n}\right)\right] g(X,Y) = -\delta\beta(\xi\psi)g(X,Y) - 2\delta\beta\eta(X)\eta(Y). \tag{4.77}$$

Now, contracting (4.76), we get

$$r + \frac{3}{2}\left[2\lambda - \left(p + \frac{2}{n}\right)\right] = -3\delta\beta(\xi\psi) - 2\delta\beta. \tag{4.78}$$

Putting $n = 3$ and for conformal Ricci soliton $r = -1$ in (4.78) which implies

$$(\xi\psi) = -\frac{1}{-\delta\beta}\left(\lambda + \frac{p}{2}\right) - \frac{2}{3}. \tag{4.79}$$

If $r = -1$, then $(\xi\psi) = constant = k(say)$. Therefore from (4.67) we have $D\psi = (\xi\psi)\xi = k\xi$. This we can write this equation as

$$g(D\psi, X) = k\eta(X), \tag{4.80}$$

which means that $d\psi(X) = k\eta(X)$. Applying d this, we get $kd\eta = 0$. Since $d\eta \neq 0$, we have $k = 0$. Hence we get $D\psi = 0$. This means that $\psi = constant$ Therefore equation (4.60) reduces to

$$S(X,Y) = 2(\alpha^2 + \beta^2 - \delta(\xi\beta))g(X,Y),$$

that is M is an *Einstein* manifold.

Case (iii) Using $\alpha = 0$ and $\left[\frac{1}{2}\left[2\lambda - \left(p + \frac{2}{n}\right)\right] + \delta\beta + 2(\alpha^2 + \beta^2 - \delta(\xi\beta))\right] = 0$. in (4.69) we obtain $Y(\xi\psi) = -\delta\beta(\xi\psi)\eta(Y)$. Now as in *Case (ii)* we conclude that the manifold is an *Einstein* manifold.

Thus we have the following :

Theorem 4.1. *If a 3-dimensional δ-Lorentzian trans Sasakian manifold with constant scalar curvature admits gradient Einstein soliton, then the manifold is either a δ-Lorentzian β-Kenmotsu manifold or an Einstein manifold provided $\alpha, \beta = constant$.*

In [15] it was proved that if a 3-dimensional compact connected trans-Sasakian manifold is of constant curvature, then it is either α-Sasakian or β-Kenmotsu. Since for a 3-dimensional Riemannian manifold constant curvature and Einstein manifold are equivalent, therefore from the Theorem 3 (see [15]) we state the following:

Corollary 4.1. *If a compact 3-dimensional δ-Lorentzian trans-Sasakian manifold with constant scalar curvature admits Ricci soliton, then the manifold is either δ-Lorentzian α-Sasakian or δ-Lorentzian β-Kenmotsu.*

Also in [15], authors proved that a 3-dimensional connected trans-Sasakian manifold is locally ϕ-symmetric if and only if the scalar curvature is constant provided α and β are constants. Hence from Theorem 3 in [15], we obtain the following:

Corollary 4.2. *If a locally ϕ-symmetric 3-dimensional connected δ-Lorentzian trans-Sasakian manifold its admits gradient conformal soliton, then manifold is either δ-Lorentzian β-Kenmotsu or Einstein manifold provided $\alpha, \beta = constant$.*

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