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CHEN'S BASIC INEQUALITIES FOR HYPERSURFACES OF STATISTICAL RIEMANNIAN MANIFOLDS

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Abstract. Some basic equalities and inequalities involving the Riemannian curvature invariants for hypersurfaces of statistical Riemannian manifolds are presented. With the help of these relations, the necessary conditions for these hypersurfaces to be total geodesic, total umbilical, or minimal have been obtained.

Keywords: Curvature, Statistical Riemannian manifold, Hypersurface.

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1. Introduction

With the J. F. Nash's embedding theorem, which concludes that every Riemannian manifold can be isometrically embedded into some Euclidean space, the question arose how to characterize a Riemannian manifold with the help of its intrinsic and extrinsic invariants. Riemann curvature invariants are utilized to solve this problem since these invariants are widely convenient tools to characterize Riemannian manifolds and the basic properties of the shape operator of a Riemannian manifold can be shown by the relations obtained on the section curvature, Ricci curvature, and scalar curvature.

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During the 1990s, B.-Y. Chen established some inequalities involving the intrinsic invariants and the extrinsic invariants. Some of the important inequalities and their results are given as follows:

In [8], B.-Y. Chen proved the following relation between the sectional curvature K and the shape operator A_N for an n-dimensional submanifold M in Riemannian space form $R^m(\bar{c})$:

$$A_N > \frac{n-1}{n}(c-\bar{c})I_n, \tag{1.1}$$

where $c = \inf K \neq \bar{c}$ and I_n is the identity map. The equality case of (1.1) holds for all $p \in M$ if and only if M is totally geodesic.

In [9], B.-Y. Chen established the following inequality between the squared mean curvature and Ricci curvature for a submanifold in a real space form $R^m(\bar{c})$:

For each unit tangent vector $X \in T_pM^n$, the following inequality is satisfied

$$\|H\|^2 \geq \frac{4}{n^2}\{Ric(X) - (n-1)\bar{c}\}, \tag{1.2}$$

where $\|H\|^2$ is the squared mean curvature and $Ric(X)$ is the Ricci curvature of M^n at X.

The equality case of (1.2) holds for all unit tangent vectors at p if and only if either p is a totally geodesic point or $n = 2$ and p is a totally umbilical point.

In literature, these types of inequalities are known as Chen-like inequalities.

In addition to these facts, the theory of statistical manifolds has substantial physical and geometrical aspects. It has applications in neural networks, machine learning, artificial intelligence, and black holes [2, 7, 14, 27]. Statistical manifolds were firstly introduced by S. Amari [1] in his book. Later, the basic geometrical properties of hypersurfaces of statistical manifolds were exposed by H. Furuhata in [15, 16]. Recently, Chen-type inequalities for submanifolds of statistical manifolds have been studied by various authors in [3, 4, 5, 6, 11, 12, 13, 18, 19, 21, 22, 23, 24], etc.

The main purpose of the present paper is to establish Chen-like inequalities on hypersurfaces of statistical manifolds. Although it is clear that hypersurfaces are a special case of submanifolds and there are various studies related to Chen-like inequalities on the submanifolds of statistical manifolds in the literature, many exclusive and different results on the hypersurfaces of these manifolds have been obtained with the help of the Riemannian curvature invariants in this paper.

2. Preliminaries

Let $(\widetilde{M}, \widetilde{g})$ be an $n-$dimensional Riemannian manifold equipped with a Riemannian metric $\widetilde{g}$ and $\{e_1, \ldots, e_n\}$ be any orthonormal frame field of $\Gamma(T\widetilde{M})$. The Ricci tensor $\widetilde{S}^0$ is defined by

$$\widetilde{S}^0(X, Y) = \sum_{j=1}^{n} \widetilde{g}(\widetilde{R}^0(e_j, X)Y, e_j)$$

for any $X, Y \in \Gamma(T\widetilde{M})$, where $\widetilde{R}^0$ is the Riemannian curvature tensor field of $\widetilde{M}$. The Ricci curvature $\widetilde{Ric}^0(X)$ of any vector field X is defined by

$$\widetilde{Ric}^0(X) = \widetilde{S}^0(X, X).$$

For a fixed $i \in \{1, \cdots, n\}$, we write

$$\widetilde{Ric}^0(e_i) \equiv \widetilde{S}^0(e_i, e_i) = \sum_{j=1}^{n} \widetilde{g}(\widetilde{R}^0(e_i, e_j)e_j, e_i), \tag{2.3}$$

which is equal to

$$\widetilde{Ric}^0(e_i) = \sum_{j \neq i}^{n} \widetilde{K}^0(e_i, e_j). \tag{2.4}$$

Here, $\widetilde{K}^0(e_i, e_j)$ denotes the sectional curvature of a plane section spanned by e_i and e_j for $i \neq j \in \{1, \ldots, n\}$.

In [9], B.-Y. Chen extended the notion of Ricci curvature to k-Ricci curvature, $2 \leq k \leq n$, in an n-dimensional Riemannian manifold. Let π_k be a k-plane section of $T_p\widetilde{M}$ and X be a unit vector field in π_k. If $k = n$ then $\pi_n = T_pM$; and if $k = 2$ then π_2 is a plane section of $T_p\widetilde{M}$. Let us choose an orthonormal basis $\{e_1, \ldots, e_k\}$ of π_k such that $e_1 = X$. The k-Ricci curvature of π_k at X, denoted by $\widetilde{Ric}^0_{\pi_k}(X)$, is defined by

$$\widetilde{Ric}^0_{\pi_k}(X) = \sum_{j \neq i}^{k} \widetilde{K}^0(e_1, e_j).$$

For $k = n$, the n-Ricci curvature of X is denoted by $\widetilde{Ric}^0_{T_p\widetilde{M}}(X)$.

The scalar curvature is one of the most studied classical curvature invariants. The scalar curvature $\widetilde{\tau}^0(p)$ at a point p is defined by

$$\begin{aligned} \widetilde{\tau}^0(p) &= \sum_{1 \leqslant j \leqslant n} \widetilde{K}^0(e_i, e_j) \\ &= \frac{1}{2} \sum_{i=1}^{n} \sum_{j \neq i}^{n} \widetilde{g}(\widetilde{R}^0(e_i, e_j)e_j, e_i). \end{aligned} \tag{2.5}$$

The scalar curvature $\widetilde{\tau}(\pi_k)$ of the k-plane section π_k is given by

$$\widetilde{\tau}(\pi_k) = \frac{1}{2}\sum_{i=1}^{k}\sum_{j\neq i}^{k}\widetilde{K}^0(e_i, e_j).$$

In particular, for $k = n$, the n-scalar curvature at a point p is denoted by $\widetilde{\tau}_{T_p\widetilde{M}}(p)$.

Let (M, g) be a hypersurface of $(\widetilde{M}, \widetilde{g})$ and N be the unit normal vector field of (M, g). Denote by the Levi-Civita connection of $(\widetilde{M}, \widetilde{g})$ by $\widetilde{\nabla}^0$. The Gauss and Weingarten formulas are, respectively, given by

$$\widetilde{\nabla}^0_X Y = \nabla^0_X Y + g(A^0_N X, Y)N \tag{2.6}$$

and

$$\widetilde{\nabla}^0_X N = -A^0_N X \tag{2.7}$$

for any $X, Y \in \Gamma(TM)$, where ∇^0 is the induced linear connection and A^0_N is the shape operator of (M, g).

Denote the Riemannian curvature tensor of (M, g) by R^0. The equation of Gauss is given by

$$R^0(X, Y)Z = \widetilde{R}^0(X, Y)Z + g(A^0_N Y, Z)A^0_N X - g(A^0_N X, Z)A^0_N Y \tag{2.8}$$

for any $X, Y, Z \in \Gamma(TM)$.

The hypersurface (M, g) is called totally geodesic if $A^0_N = 0$, minimal if $trace A^0_N = 0$. If $A^0_N(X) = \lambda X$, where λ is a smooth function on M, then (M, g) is called totally umbilical [10].

3. Statistical Manifolds and Their Hypersurfaces

Let $(\widetilde{M}, \widetilde{g})$ be a Riemannian manifold and $\widetilde{\nabla}$ be a torsion-free connection on $(\widetilde{M}, \widetilde{g})$. The manifold is called a statistical manifold if the following relation is satisfied for any $X, Y, Z \in \Gamma(T\widetilde{M})$:

$$\widetilde{g}(\widetilde{\nabla}_Z X, Y) = Z\widetilde{g}(X, Y) - \widetilde{g}(X, \widetilde{\nabla}^*_Z Y), \tag{3.9}$$

where

$$\widetilde{\nabla}^0_X Y = \frac{1}{2}(\widetilde{\nabla}_X Y + \widetilde{\nabla}^*_X Y). \tag{3.10}$$

Here, $\widetilde{\nabla}^*$ is called the dual connection of $\widetilde{\nabla}^*$, the pair $(\widetilde{\nabla}, g)$ is called a statistical structure on $(\widetilde{M}, \widetilde{g})$. A statistical manifold with a torsion-free connection $\widetilde{\nabla}$ is usually denoted by $(\widetilde{M}, \widetilde{g}, \widetilde{\nabla})$ [1].

Now, let us denote the Riemannian curvature tensor fields with respect to $\widetilde{\nabla}$ and $\widetilde{\nabla}^*$ by $\widetilde{R}$ and $\widetilde{R}^*$. Then we have

$$\widetilde{g}(\widetilde{R}^*(X,Y)Z,W) = -\widetilde{g}(Z,\widetilde{R}(X,Y)W) \tag{3.11}$$

for any $X, Y, Z \in \Gamma(T\widetilde{M})$.

A statistical manifold is said to be of constant curvature c, if the equation

$$\widetilde{R}(X,Y)Z = \frac{c}{4}\{\widetilde{g}(Y,Z)X - \widetilde{g}(X,Z)Y\} \tag{3.12}$$

holds for any $X, Y, Z \in \Gamma(TM)$ [15].

Considering the eq. (3.11), we see that $(\widetilde{M}, \widetilde{g}, \widetilde{\nabla})$ is of constant curvature with respect to $\widetilde{\nabla}$ if and only if it is of constant curvature with respect to $\widetilde{R}^*$.

Let (M, g) be a hypersurface of $(\widetilde{M}, \widetilde{g}, \widetilde{\nabla})$. The Gauss and Weingarten formulas with respect to $\widetilde{\nabla}$ and $\widetilde{\nabla}^*$ are, respectively, given by

$$\begin{aligned} \widetilde{\nabla}_X Y &= \nabla_X Y + g(A_N X, Y)N, && (3.13) \\ \widetilde{\nabla}_X N &= -A_N^* X + \kappa(X)N, && (3.14) \\ \widetilde{\nabla}^*_X Y &= \nabla^*_X Y + g(A_N^* X, Y)N, && (3.15) \\ \widetilde{\nabla}^*_X N &= -A_N X - \kappa(X)N && (3.16) \end{aligned}$$

for any $X, Y \in \Gamma(TM)$. It is easy to show that the induced connection ∇^* is the dual connection of ∇. Here, κ is a 1-form, A_N and A_N^* are the shape operators with respect to $\widetilde{\nabla}$ and its dual connection $\widetilde{\nabla}^*$, respectively.

Let R and $\widetilde{R}$ denote the Riemannian curvature tensor (M, g, ∇) and $(\widetilde{M}, \widetilde{g}, \widetilde{\nabla})$ respectively. Then the following relation holds

$$R(X,Y)Z = \widetilde{R}(X,Y)Z - g(A_N^* X, Z)A_N Y + g(A_N^* Y, Z)A_N X \tag{3.17}$$

for any $X, Y, Z \in \Gamma(TM)$ [15].

Let $\pi = Span\{X,Y\}$ be a plane section of $\Gamma(TM)$. Then the $K-$sectional curvature is defined by [20]

$$\widetilde{K}(\pi) = \frac{1}{2}\Big[\widetilde{g}(\widetilde{R}(X,Y)Y,X) + \widetilde{g}(\widetilde{R}^*(X,Y)Y,X)\Big] - \widetilde{g}(\widetilde{R}^0(X,Y)Y,X). \tag{3.18}$$

A hypersurface of $(\widetilde{M},\widetilde{g},\widetilde{\nabla})$ is called

i. totally geodesic with respect to $\widetilde{\nabla}$ (resp. $\widetilde{\nabla}^*$), if $A_N = 0$ (resp.$A_N^* = 0$).

ii. totally umbilical with respect to $\widetilde{\nabla}$ (resp. $\widetilde{\nabla}^*$), if there exists a smooth function ρ such that $A_N X = \rho X$ (resp. $A_N^* X = \rho X$).

iii. minimal with respect to $\widetilde{\nabla}$ (resp. $\widetilde{\nabla}^*$), if $trace A_N = 0$ (resp. $trace A_N^* = 0$).

For more details on statistical manifolds and their submanifolds, we refer to [15, 16].

4. Ricci Curvature

In this section, we shall give some relations involving Ricci curvatures of hypersurfaces immersed in $(\widetilde{M},\widetilde{g},\widetilde{\nabla})$.

Lemma 4.1. *Let (M,g) be a hypersurface of $(\widetilde{M},\widetilde{g},\widetilde{\nabla})$ and $\{e_1,\dots,e_n\}$ be an orthonormal basis of T_pM at a point $p \in M$. For any unit tangent vector X at a point p, we have the following equalities:*

$$Ric^0(X) = \widetilde{Ric}^0_{T_pM}(X) + trace A_N^0 g(A_N X, X) - g(A_N^2 X, X). \tag{4.19}$$

$$\begin{aligned}\sum_{j=2}^{n} g(R(X,e_j)e_j,X) &= \sum_{j=2}^{n} \widetilde{g}(\widetilde{R}(X,e_j)e_j,X) + g(A_N X,X) trace A_N^* \\ &\quad - g(A_N^* X, A_N X).\end{aligned} \tag{4.20}$$

$$\begin{aligned}\sum_{j=2}^{n} g(R(X,e_j)X,e_j) &= -\sum_{j=2}^{n} \widetilde{g}(\widetilde{R}^*(X,e_j)e_j,X) + g(A_N^* X,X) trace A_N \\ &\quad - g(A_N^* X, A_N X).\end{aligned} \tag{4.21}$$

Proof. In view of (2.8), the proof of (4.19) is straightforward.

Now we shall prove (4.20). From (3.17), we may write

$$\begin{aligned}g(R(e_1,e_2)e_2,e_1) &= \widetilde{g}(\widetilde{R}(e_1,e_2)e_2,e_1) - g(A_N^* e_1,e_2) g(A_N e_2,e_1) \\ &\quad + g(A_N^* e_2,e_2) g(A_N e_1,e_1).\end{aligned} \tag{4.22}$$

Taking trace in (4.22), we get

$$\begin{aligned}\sum_{j=2}^{n} g(R(e_1,e_j)e_j,e_1) &= \sum_{j=2}^{n} \widetilde{g}(\widetilde{R}(e_1,e_j)e_j,e_1)\\ &+g(A_Ne_1,e_1)\left(\sum_{j=2}^{n} g(A_Ne_j,e_j)-g(A_N^*e_1,e_1)\right)\\ &+\sum_{j=2}^{n}(g(A_N^*e_1,e_j)g(A_Ne_1,e_j))-g(A_N^*e_1,e_1)\\ &-g(A_N^*e_1,e_1)g(A_Ne_1,e_1),\end{aligned}$$

which is equivalent to

$$\begin{aligned}\sum_{j=2}^{n} g(R(e_1,e_j)e_j,e_1) &= \sum_{j=2}^{n} \widetilde{g}(\widetilde{R}(e_1,e_j)e_j,e_1)+g(A_Ne_1,e_1)\mathrm{trace}A_N^*\\ &-\sum_{j=2}^{n} g(A_N^*e_1,e_j)g(A_Ne_1,e_j). \qquad (4.23)\end{aligned}$$

Now, considering the fact that $\{e_1,\dots,e_n\}$ is an orthonormal basis of T_pM, we can write

$$\begin{aligned}A_N^*e_1 &= \lambda_1e_1+\cdots+\lambda_ne_n,\\ A_Ne_1 &= \mu_1e_1+\cdots+\mu_ne_n,\end{aligned}$$

where λ_i,μ_i are real numbers for each $i\in\{1,\dots,n\}$. Thus, we have

$$\begin{aligned}\sum_{j=2}^{n} g(A_N^*e_1,e_j)g(A_Ne_1,e_j) &= \lambda_1\mu_1+\cdots+\lambda_n\mu_n\\ &= g(A_N^*e_1,A_Ne_1). \qquad (4.24)\end{aligned}$$

Using (4.24) in (4.23), we get

$$\begin{aligned}\sum_{j=2}^{n} g(R(e_1,e_j)e_j,e_1) &= \sum_{j=2}^{n} \widetilde{g}(\widetilde{R}(e_1,e_j)e_j,e_1)\\ &+g(A_Ne_1,e_1)\mathrm{trace}A_N^*-g(A_N^*e_1,A_Ne_1). \qquad (4.25)\end{aligned}$$

Putting $X=e_1$ in (4.25) we obtain (4.21).

Now we shall prove (4.21). Using (3.11) and (3.17), we have

$$\begin{aligned}\widetilde{g}(\widetilde{R}^*(e_1,e_j)e_j,e_1) &= \widetilde{g}(\widetilde{R}(e_1,e_j)e_1,e_j)\\ &= -g(R(e_1,e_j)e_1,e_j)-g(A_N^*e_1,e_1)g(A_Ne_j,e_j)\\ &+g(A_N^*e_1,e_j)g(A_Ne_1,e_j). \qquad (4.26)\end{aligned}$$

Taking trace in (4.26), we get

$$\begin{aligned}\sum_{j=2}^{n} g(R(e_1,e_j)e_1,e_j) &= \sum_{j=2}^{n} \widetilde{g}(\widetilde{R}^*(e_1,e_j)e_j,e_1)+g(A_N^*e_1,e_j)\mathrm{trace}A_N \\ &\quad -g(A_N^*e_1,A_Ne_1). \end{aligned} \tag{4.27}$$

Putting $X=e_1$ in (4.27), we obtain (4.21).

Now, we shall give some relations involving $K-$Ricci curvature and $K-$scalar curvature which are defined by

$$Ric^k(X)=\sum_{j\neq i}^{n} K(e_i,e_j)$$

and

$$\tau^k(p)=\frac{1}{2}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\widetilde{g}(\widetilde{R}^0(e_i,e_j)e_j,e_i).$$

Theorem 4.1. *Let (M,g) be a minimal hypersurface with respect to $\widetilde{\widetilde{\nabla}}$ and $\widetilde{\nabla}^*$. Then we have*

$$Ric^k(X)+Ric^0(X)=\widetilde{Ric}^k_{T_pM}(X)+\widetilde{Ric}^0_{T_pM}(X) \tag{4.28}$$

for any unit vector $X\in T_pM$.

Proof. Under the assumption, we have from (4.19) and (4.20) that

$$\sum_{j=2}^{n} g(R(e_1,e_j)e_j,e_1) = \sum_{j=2}^{n} \widetilde{g}(\widetilde{R}(e_1,e_j)e_j,e_1)-g(A_N^*e_1,A_Ne_1) \tag{4.29}$$

and

$$\sum_{j=2}^{n} g(R(e_1,e_j)e_1,e_j) = -\sum_{j=2}^{n} \widetilde{g}(\widetilde{R}^*(e_1,e_j)e_j,e_1)-g(A_N^*e_1,A_Ne_1). \tag{4.30}$$

If the equations (4.29) and (4.30) are subtracted from side to side, we get

$$\begin{aligned}\sum_{j=2}^{n}\left[g(R(e_1,e_j)e_j,e_1)+g(R^*(e_1,e_j)e_j,e_1)\right] &= \sum_{j=2}^{n}\Big[\widetilde{g}(\widetilde{R}(e_1,e_j)e_j,e_1) \\ &\quad +\widetilde{g}(\widetilde{R}^*(e_1,e_j)e_j,e_1)\Big].\end{aligned}$$

In view of (3.18), we see that

$$\sum_{j=2}^{n}\left[K(e_1,e_j)+K^0(e_1,e_j)\right] = \sum_{j=2}^{n}\widetilde{K}(e_1,e_j)+\widetilde{K}^0(e_1,e_j). \tag{4.31}$$

Putting $X=e_1$ in (4.31), we obtain (4.28).

Remark 4.1. *Since*

$$A_N^0 X = A_N X + A_N^* X$$

for any $X \in \Gamma(TM)$, it is clear that if $A_N X = A_N^* X = 0$, then we have $A_N^0 = 0$. But the converse part of this claim is not correct in general. Considering this fact, the claim of Theorem 4.1 may not be correct when the hypersurface is minimal with respect to $\widetilde{\nabla}^0$.

Now, we recall the Chen-Ricci inequality for a Riemannian submanifold [17, 26]:

Theorem 4.2. *Let (M, g) be an $n-$dimensional submanifold of a Riemannian manifold $(\widetilde{M}, \widetilde{g})$. Then the following statements are true.*

i. *For any unit tangent vector X, we have*

$$Ric^0(X) \leqslant \frac{1}{4} n^2 \|H\|^2 + \widetilde{Ric}^0_{T_pM}(X). \tag{4.32}$$

ii. *The equality case of (4.32) holds for all unit tangent vectors of T_pM if and only if either p is a totally geodesic point or $n = 2$ and p is a totally umbilical point.*

Theorem 4.3. *Let (M, g) be a $n > 2$ minimal hypersurface with respect to $\widetilde{\nabla}$ and $\widetilde{\nabla}^*$. Then we have*

$$Ric^k(X) \geq \widetilde{Ric}^0_{T_pM}(X) \tag{4.33}$$

for any unit tangent vector $X \in T_PM$. The equality case of (4.33) holds for all $X \in T_pM$ if and only if $A_N X = -A_N^* X$.

Proof. Using the fact that $trace A_N = trace A_N^* = 0$, we see that $H = 0$ from Remark 4.1. In view of (4.32), we get

$$Ric^0(X) \leq \widetilde{Ric}^0_{T_pM}(X). \tag{4.34}$$

Using (4.34) in (4.28), we obtain (4.33). From Theorem 4.2, the equality case of (4.33) is satisfied if and only if $A_N^0 X = 0$ which shows that $A_N X = -A_N^* X$ for all $X \in T_pM$.

Now we recall the following theorem of T. Takahashi [25]:

Theorem 4.4. *The necessary condition for a submanifold of an Euclidean space to be a minimal immersion is that its Ricci curvature is negative semi-definite.*

In the following corollary, we obtain a similar claim of Theorem 4.4 for a minimal hypersurface with respect to $\widetilde{\nabla}$ and $\widetilde{\nabla}^*$ on statistical manifolds with constant curvatures.

Corollary 4.1. *Let $(\widetilde{M}, \widetilde{g}, \widetilde{\nabla})$ be of $K-$constant curvature with $c = 0$ and (M, g) be a minimal hypersurface with respect to $\widetilde{\nabla}$ and $\widetilde{\nabla}^*$. Then we have*

$$Ric^k(X) \geq 0$$

for any unit tangent vector $X \in T_pM$.

Now we shall give the following lemma for later uses:

Lemma 4.2. *Let (M, g) be a hypersurface of $(\widetilde{M}, \widetilde{g}, \widetilde{\nabla})$. Then the following relation is satisfied for any unit tangent vector $X \in T_pM$:*

$$\begin{aligned} Ric^k(X) &= \widetilde{Ric}_{T_pM}(X) - \frac{1}{2} g(A_N X, X) traceA_N^* \\ &\quad + \frac{1}{2} g(A_N^* X, X) traceA_N + g(A_N^0 X, X) traceA_N^0 + \left\| A_N^0 X \right\|^2 . \end{aligned} \tag{4.35}$$

Proof. From (3.18), we have

$$\begin{aligned} \widetilde{Ric}^k_{T_pM}(e_i) &= \frac{1}{2} \sum_{j=2}^{n} \widetilde{g}(\widetilde{R}(e_i, e_j)e_j, e_i) + \frac{1}{2} \sum_{j=2}^{n} \widetilde{g}(\widetilde{R}^*(e_i, e_j)e_j, e_i) \\ &\quad - \sum_{j=2}^{n} \widetilde{g}(\widetilde{R}^0(e_i, e_j)e_j, e_i). \end{aligned} \tag{4.36}$$

In view of (4.19), (4.20), and (4.21) in (4.36), we obtain

$$\begin{aligned} \widetilde{Ric}^k_{T_pM}(e_i) &= \frac{1}{2} \sum_{j=2}^{n} g(R(e_i, e_j)e_j, e_1) + \frac{1}{2} \sum_{j=2}^{n} g(R(e_1, e_j)e_j, e_1) \\ &\quad - \sum_{j=2}^{n} g(R^0 e_1, e_j)e_j, e_1) + \frac{1}{2} g(A_N e_1, e_1) traceA_N^* \\ &\quad - \frac{1}{2} g(A_N^* e_1, A_N e_1) - \frac{1}{2} g(A_N^* e_1, e_1) traceA_N \\ &\quad + \frac{1}{2} g(A_N e_1, A_N^* e_1) - g(A_N^0 e_1, e_1) traceA^0 + g(A_N^0 e_1, A_N^0 e_1). \end{aligned}$$

Putting $X = e_1$, the proof of (4.35) is straightforward.

From Lemma 4.2, we get the following corollary immediately:

Corollary 4.2. *Let (M, g) be a hypersurface of $(\widetilde{M}, \widetilde{g}, \widetilde{\nabla})$. Then the following inequality is satisfied for any unit tangent vector $X \in T_pM$:*

$$\begin{aligned} Ric^k(X) &\leqslant \widetilde{Ric}_{T_pM}(X) - \frac{1}{2} g(A_N X, X) traceA_N^* \\ &\quad + \frac{1}{2} g(A_N^* X, X) traceA_N + g(A_N^0 X, X) traceA_N^0. \end{aligned} \tag{4.37}$$

5. Scalar Curvature

In this section, we shall give some relations involving scalar curvatures of hypersurfaces immersed in $(\widetilde{M}, \widetilde{g}, \widetilde{\nabla})$.

Let $\{e_1, e_2, \ldots, e_n\}$ be an orthonormal basis of T_pM at a point $p \in M$. We put

$$\sigma_{ij} = g(A_N e_i, e_j) \quad \text{and} \quad \sigma^*_{ij} = g(A^*_N e_i, e_j)$$

for any $i, j \in \{1, 2, \cdots, n\}$. From (3.17), we write

$$g(R(e_i, e_j)e_j, e_i) = \widetilde{g}(\widetilde{R}(e_i, e_j)e_j, e_i) - \sigma^*_{ij}\sigma_{ji} + \sigma^*_{jj}\sigma_{ii}. \tag{5.38}$$

Taking trace in (5.38), we get

$$\tau(p) = \widetilde{\tau}_{T_pM}(p) - \frac{1}{2}\sum_{i,j=1}^{n}\left(\sigma^*_{ij}\sigma_{ji} + \sigma^*_{jj}\sigma_{ii}\right). \tag{5.39}$$

Let us define

$$\left|A^0_N\right| = \left(\sum_{i,j=1}^{n} g(A^0_N e_i, e_j)\right)^2. \tag{5.40}$$

In light of the above facts, we shall state the following lemma:

Lemma 5.1. *For any hypersurface of $(\widetilde{M}, \widetilde{g}, \widetilde{\nabla})$, we have*

$$\begin{aligned} 2\sum_{i,j=1}^{n}\left(\sigma^*_{ij}\sigma_{ji} + \sigma^*_{jj}\sigma_{ii}\right) &= 4\left[traceA^0_N\right]^2 - \left[traceA_N\right]^2 - \left[traceA^*_N\right]^2 \\ &\quad +4\left|A^0_N\right| - \|A^*_N\|^2 - \|A_N\|. \end{aligned} \tag{5.41}$$

Proof. We can write

$$\begin{aligned} 2\sum_{i,j=1}^{n}\left(\sigma^*_{ij}\sigma_{ji} + \sigma^*_{jj}\sigma_{ii}\right) &= \left(\sum_{i,j=1}^{n}\sigma_{ii} + \sigma^*_{jj}\right)^2 - \left(\sum_{i=1}^{n}\sigma_{ii}\right)^2 - \left(\sum_{i=1}^{n}\sigma^*_{jj}\right)^2 \\ &\quad + \left(\sum_{i,j=1}^{n}\sigma^*_{ij} + \sigma^*_{ji}\right)^2 - \left(\sum_{i=1}^{n}\sigma^*_{ij}\right)^2 - \left(\sum_{i=1}^{n}\sigma_{ji}\right)^2. \end{aligned} \tag{5.42}$$

On the other hand, we also have

$$\begin{aligned}
\sum_{i,j=1}^{n} \sigma_{ii} + \sigma_{jj}^{*} &= \sum_{i,j=1}^{n} \left[g(A_N e_i, e_i) + g(A_N^{*} e_j, e_j) \right] \\
&= \sum_{i=1}^{n} g(A_N e_i, e_i) + \sum_{j=1}^{n} g(A_N^{*} e_j, e_j) \\
&= trace A_N + trace A_N^{*} \\
&= trace (A_N + A_N^{*}) \\
&= 2 trace A_N^{0}. \qquad (5.43)
\end{aligned}$$

In a similar way, we have

$$\begin{aligned}
\sum_{i,j=1}^{n} \sigma_{ij}^{*} + \sigma_{ji} &= \sum_{i,j=1}^{n} \left[g(A_N^{*} e_i, e_i) + g(A_N e_j, e_j) \right] \\
&= \sum_{i,j=1}^{n} g((A_N^{*} + A_N) e_i, e_j) \\
&= 2 \sum_{i,j=1}^{n} g(A_N^{0} e_i, e_j) \qquad (5.44)
\end{aligned}$$

The proof is straightforward from computing the other terms on the right-hand side of (5.42) in a similar way.

From the equation (5.39) and (5.41), we get the following lemma:

Proposition 5.1. *For any hypersurface of $(\widetilde{M}, \widetilde{g}, \widetilde{\nabla})$, we have*

$$\begin{aligned}
\tau(p) &= \widetilde{\tau}_{T_pM}(P) - 2\left[trace A_N^{0} \right]^2 + \frac{1}{2}\left[trace A_N \right]^2 + \frac{1}{2}\left[trace A_N^{*} \right]^2 \\
&\quad - 2\left| A_N^{0} \right| + \frac{1}{2}\left\| A_N^{*} \right\|^2 + \frac{1}{2}\left\| A_N \right\|^2. \qquad (5.45)
\end{aligned}$$

Theorem 5.1. *Let (M, g) be a hypersurface of $(\widetilde{M}, \widetilde{g}, \widetilde{\nabla})$. Then we have*

$$\tau(p) \geq \widetilde{\tau}_{T_pM}(p) - 2\left[trace A_N^{0} \right]^2 + \frac{1}{2}\left[trace A_N \right]^2 + \frac{1}{2}\left[trace A_N^{*} \right]^2 - 2\left| A_N^{0} \right| \qquad (5.46)$$

for any $p \in M$. The equality case of (5.46) holds for all $p \in M$ if and only if M is totally geodesic.

Proof. The proof of (5.46) is straightforward from (5.45). The equality case of (5.46) holds for all $p \in M$ if and only if we have $A_N^{*} = A_N = 0$. Using the fact that $A_N^{0} = \frac{1}{2}(A_N X + A_N^{*} X)$, we obtain $A_N^{0} X = 0$ for any $X \in T_pM$. This shows that M is totally geodesic. The converse part of the proof is straightforward.

Now we shall give the following lemma for later uses:

Lemma 5.2. *For any hypersurface of $(\widetilde{M}, \widetilde{g}, \widetilde{\nabla})$, we have*

$$\tau^k(p) = \widetilde{\tau}^k(p) + (traceA_N^0)^2 + trace(A_N^0)^2. \tag{5.47}$$

Proof. Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of T_pM at a point $p \in M$. From Lemma 4.2, we write

$$\begin{aligned} Ric^k(e_i) &= \widetilde{Ric}_{T_pM}(e_i) - \frac{1}{2}g(A_N e_i, e_i) traceA_N^* + \frac{1}{2}g(A_N^* e_i, e_i) traceA_N \\ &\quad + traceA_N^0 g(A_N^0 e_i, e_i) + \left\|A_N^0 e_i\right\|^2 \end{aligned} \tag{5.48}$$

for any $i \in \{1, \ldots, n\}$. Taking trace in (5.48), we get

$$\begin{aligned} \tau^k(p) &= \widetilde{\tau}^k_{T_pM}(p) + \frac{1}{2} traceA_N traceA_N^* - \frac{1}{2} traceA_N traceA_N^* \\ &\quad + (traceA_N^0)^2 + \sum_{i=1}^{n} g(A_N^0 e_i, A_N^0 e_i) \end{aligned} \tag{5.49}$$

which is equivalent to (5.47).

As a result of Lemma 5.2, we obtain the following corollaries:

Corollary 5.1. *Let (M, g) be a hypersurface of $(\widetilde{M}, \widetilde{g}, \widetilde{\nabla})$. Then we have*

$$\tau(p) \leq \widetilde{\tau}(p) + trace(A_N^0)^2 \tag{5.50}$$

for any $p \in M$. The equality case of (5.50) holds for all $p \in M$ if and only if M is minimal.

Corollary 5.2. *Let (M, g) be a totally umbilical hypersurface of $(\widetilde{M}, \widetilde{g}, \widetilde{\nabla})$. Then we have*

$$\widetilde{\tau}_{T_pM}(p) < \tau_{T_pM}(p) \tag{5.51}$$

Proof. If (M, g) is a totally umbilical hypersurface, then there exists a smooth function ρ^0 on M such that we can write $A_N^0 X = \rho^0 X$ for any $X \in \Gamma(TM)$. Thus, we obtain from (5.47) that

$$\widetilde{\tau}_{T_pM}(p) = \tau_{T_pM}(p) + (n^2 - n)\lambda^2. \tag{5.52}$$

In view (5.52), we have (5.51).

6. Examples

Now we shall give an example satisfying some results obtained in this paper:

Example 6.1. *Let us consider a hypersurface M given by*

$$M = \{(cosx_1, sinx_1, x_2, x_3) : x_1 \in (0, 2\pi], x_2, x_3 \in \mathbb{R}\}$$

in $\mathbb{E}^4$. The natural tangent vector fields of M are given by

$$e_1 = -sinx_1\partial_1 + cosx_1\partial_2, \quad e_2 = \partial_3, \quad e_3 = \partial_4$$

and the normal vector field of M is given by

$$N = cosx_1\partial_1 + sinx_1\partial_2,$$

where $\{\partial_1, \partial_2, \partial_3, \partial_4\}$ is the natural basis of $\mathbb{E}^4$. By a straightforward computation, we easily have

$$\widetilde{\nabla}^0_{e_1} e_1 = -cosx_1\partial_1 - sinx_1\partial_2, \quad \widetilde{\nabla}^0_{e_2} e_2 = 0, \quad \widetilde{\nabla}^0_{e_3} e_3 = 0$$

and $\widetilde{\nabla}^0_{e_i} e_j = 0$ for $i \neq j \in \{1, 2, 3\}$. From (2.6), we get

$$A^0_N = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{6.53}$$

Now, suppose that the connections $\widetilde{\nabla}$ and $\widetilde{\nabla}^*$ are satisfied the following relations:

$$\widetilde{\nabla}_{e_1} e_1 = -2\cos x_1\partial_1 \quad \widetilde{\nabla}_{e_2} e_2 = e_2, \quad \widetilde{\nabla}_{e_3} e_3 = e_3, \tag{6.54}$$

$$\widetilde{\nabla}^*_{e_1} e_1 = -2\sin x_1\partial_2, \quad \widetilde{\nabla}^*_{e_2} e_2 = -e_2, \quad \widetilde{\nabla}^*_{e_3} e_3 = -e_3,$$

and $\widetilde{\nabla}_{e_i} e_j = \widetilde{\nabla}^*_{e_i} e_j = 0$ for $i \neq j \in \{1, 2, 3\}$. Then we get

$$\widetilde{K}(e_1, e_2) = \widetilde{K}(e_1, e_3) = \widetilde{K}(e_2, e_3) = 0$$

and

$$\widetilde{Ric}_{T_pM}(e_1) = \widetilde{Ric}_{T_pM}(e_2) = \widetilde{Ric}_{T_pM}(e_1) = \widetilde{\tau}_{T_pM}(p) = 0. \tag{6.55}$$

In view of (3.13), (3.15) and (6.54), we have

$$A_N = \begin{bmatrix} -2\cos^2 x_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad A^*_N = \begin{bmatrix} -2\sin^2 x_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{6.56}$$

From these facts, it is clear that

$$K(e_1, e_2) = K(e_1, e_3) = K(e_2, e_3) = 0, \tag{6.57}$$

$$Ric(e_1) = Ric(e_2) = Ric(e_3) = \tau(p) = 0. \tag{6.58}$$

Considering (6.53), (6.55), (6.56) and (6.58), we see that the hypersurface M is satisfied the claims of Theorem 4.1, Theorem 4.3, Corollary 4.2, Theorem 5.1, and Corollary 5.1.

Example 6.2. *Let us consider the following hypersurface*

$$M = \{(x_1, x_2, x_3, 0) : \forall x_1, x_2, x_3 \in \mathbb{R}\}$$

in $\mathbb{E}^4$. Then it is clear that $T_pM = Span\{e_1, e_2, e_3\}$ and $N = \partial_4$ such that $e_i = \partial_i$ for $i \in \{1, 2, 3\}$. Suppose that the connection ∇ and $\widetilde{\nabla}^*$ are satisfied

$$\widetilde{\nabla}_{e_1} e_1 = \partial_1 + \partial_4, \qquad \widetilde{\nabla}^*_{e_1} e_1 = -\partial_1 - \partial_4,$$

$$\widetilde{\nabla}_{e_1} e_1 = \partial_2 + \partial_4, \qquad \widetilde{\nabla}^*_{e_2} e_2 = -\partial_2 - \partial_4,$$

and the other component of $\widetilde{\nabla}_{e_i} e_j$ are equal to zero for $i, j \in \{1, 2, 3\}$. Then we have

$$\nabla_{e_1} e_1 = \partial_1, \qquad \nabla^*_{e_1} e_1 = -\partial_1,$$

$$\nabla_{e_2} e_2 = \partial_2, \qquad \nabla^*_{e_2} e_2 = -\partial_2,$$

and other component of $\nabla_{e_i} e_j$ are equal to zero for $i, j \in \{1, 2, 3\}$. By a straightforward computation, we obtain M is minimal with respect to $\widetilde{\nabla}$ and $\widetilde{\nabla}^*$, and $c = 0$. Also, we see that the hyperplane M satisfies of Corollary 4.1 by $Ric^k(X) = 0$ for any $X \in T_pM$ at any point $p \in M$.

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