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# FRACTIONAL EQUIAFFINE CURVATURES OF CURVES IN 3-DIMENSIONAL AFFINE SPACE

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Abstract. In this study, we investigate the equiaffine invariants of a parametrized curve in the 3-dimensional affine space $\mathbb{R}^3$ by using a simplification of Caputo fractional derivative. We introduce the so-called fractional equiaffine arclength function for a non-degenerate parametrized curve, providing the notions of fractional equiaffine frame and curvatures. Furthermore, we give the relations between the fractional and standard equiaffine curvatures.

**Keywords**: Affine differential geometry; Caputo fractional derivative; Equiaffine arclength; Equiaffine curvature.

**2010 Mathematics Subject Classification**: Primary 26A33, Secondary 53A04, 53A55.

## 1. Introduction

Fractional calculus extends to arbitrary orders the notions of classical derivative and integral of a function and has a remarkable historical background, which it can be found in [22]. This interesting field has applications ranging from physical phenomena ([20]), dynamical systems ([27]), viscoelasticity ([15], [24]) to medicine [8].

Recently, there have been ascending contributions to the differential geometric applications of fractional calculus. From the viewpoints of Riemannian and Finsler geometries, these

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contributions can be found in [4], [5]. Also, we refer to [1], [2], [9], [10], [11], [16], [17], [19], [25], [26], [28], [29] for the contributions to the differential geometries of curves and surfaces.

We will consider a simplification of Caputo fractional derivative as follows: let $f(t)$ and $g(x)$ be smooth functions and denote by $D^{\alpha}$ Caputo fractional derivative. Then the simplification, relating to the derivative of the composite function of $f(t)$ and $g(x)$, that we will use is given by

$$(D_x^{\alpha} f)(g(x)) = \frac{\alpha x^{1-\alpha}}{\Gamma(2-\alpha)} \frac{df}{dt} \frac{dg}{dx}. \tag{1.1}$$

The idea of using Equation (1.1) in the study of differential geometric curves was first proposed in [26] because of the reason that Caputo fractional derivative of composite functions is given by an infinite series. The derivative of composite functions, i.e. chain rule, is an essential tool for the parametrized objects in differential geometry. To overcome this difficulty in the case of Caputo fractional derivative, we will use Equation (1.1) in our calculations as did the authors in [26].

In this study, we perform Equation (1.1) in order to investigate the equiaffine invariants of the non-degenerate parametrized curves in the 3-dimensional affine space $\mathbb{R}^3$. Our motivation of investigating the equiaffine invariants is the following.

Let $\mathbf{r}(s)$ be a regular parametrized curve in a Euclidean space $\mathbb{E}^3$ by arclength and $\times$ denote the cross product. Let $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ be the Frenet frame along $\mathbf{r}(s)$ such that (see [21])

$$\mathbf{t} = \frac{d\mathbf{r}}{ds}, \quad \mathbf{n} = \frac{d^2\mathbf{r}/ds^2}{\|d^2\mathbf{r}/ds^2\|}, \quad \mathbf{b} = \mathbf{t} \times \mathbf{n},$$

where $\|.\|$ denotes the induced norm in $\mathbb{E}^3$ by the Euclidean scalar product.

If we use Equation (1.1) instead of the standard ordinary derivative, i.e. $d/ds$, then the set of Frenet vectors is again $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$. This situation changes for the equiaffine Frenet frame of a non-degenerate curve in $\mathbb{R}^3$. More explicitly, the equiaffine Frenet frame of a non-degenerate curve produced by Equation (1.1) is different than the standard equiaffine Frenet frame. This justifies why we consider the equiaffine invariants instead of Frenet invariants for the use of fractional derivative in the differential geometry of curves.

The main purpose of this study is to extend the results in [2] to 3-dimensional case where the authors ([2]) introduced the fractional equiaffine invariants of a non-degenerate curve in the affine plane $\mathbb{R}^2$. Since we use a different formula of derivative instead of the standard ordinary derivative, we will need a new equiaffine arclength function which differs by the standard one. The new equiaffine arclength function will depend on the dimension of affine space and the standard equaffine parameter of given non-degenerate curve. For this, we will provide a general formula for the fractional equiaffine arclength function of a non-degenerate curve in the $n-$dimensional affine space $\mathbb{R}^n$ $(n \geq 2)$ (see Definition 4.1). Then, in 3-dimensional context, we introduce the equiaffine Frenet curvatures of fractional type (see Definition 4.3) and obtain the properties between the fractional and standard equiaffine curvatures (Theorem 4.1 and Corollaries 4.1 and 4.2). Several examples are also provided by figures.

## 2. Fractional tools

Denote by $\Gamma(\alpha)$ the Euler gamma function depending on the parameter $\alpha \in \mathbb{R}$, which it is defined by ([14])

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt.$$

Throughout the paper we will assume $0 < \alpha \leq 1$. The Riemann–Liouville fractional integral of order $\alpha$ for a function $f(x)$ is defined by ([14], [22])

$$I^\alpha_{0+} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x \frac{f(\xi)}{(x-\xi)^{1-\alpha}} d\xi.$$

The Riemann–Liouville fractional derivative of order $\alpha$ is ([14], [22])

$$(\mathcal{D}^\alpha_{0+} f)(x) = \frac{d}{dx}(I^{1-\alpha}_{0+} f)(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^x \frac{f(\xi)}{(x-\xi)^\alpha} d\xi.$$

As can be seen, the Riemann–Liouville fractional derivative uses the ordinary integral of $f(x)$ and it is a nonlocal operator, i.e. the Riemann–Liouville derivative of $f(x)$ at a point $x_0$ is determined by nonlocal values of $f(x)$.

The Caputo fractional derivative of order $\alpha$ for a function $f(x)$ is given by ([6])

$$(D^\alpha_{0+} f)(x) = I^{1-\alpha}_{0+}(\frac{df}{dx})(x) = \frac{1}{\Gamma(1-\alpha)} \int_0^x \frac{1}{(x-\xi)^\alpha} \frac{df(\xi)}{d\xi} d\xi.$$

Leibniz rule and the derivative of composite function for the Caputo fractional derivative are respectively defined by ([3])

$$(D^\alpha_{0+} fg)(x) = \sum_{i=0}^\infty \binom{\alpha}{i} \frac{d^i f}{dx^i} (D^{\alpha-i}_x g)(x) - \frac{f(0)g(0)}{\Gamma(1-\alpha)} x^{-\alpha}$$

and

$$(D^\alpha_{0+} f)(g(x)) = \sum_{i=1}^\infty \binom{\alpha}{i} \frac{x^{i-\alpha}}{\Gamma(i-\alpha+1)} \frac{d^i f(g(x))}{dx^i} + \frac{f(g(x)) - f(g(0))}{\Gamma(1-\alpha)} x^{-\alpha}. \tag{2.2}$$

Notice that the simplification (1.1) is obtained by extracting the term $i = 1$ in the infinite series in Equation (2.2).

For simplicity, we will use the following notation throughout the paper:

$$(D_{0+}^{\alpha}f)(x)=\frac{d^{\{\alpha\}}f}{dx^{\{\alpha\}}}.$$

## 3. Equiaffine Invariants

Let $\mathbb{R}^n$ denote the $n-$dimensional affine space ($n\geq 2$) and $\mathrm{Mat}(n,\mathbb{R})$ be the set of all square matrices of order $n$. We set

$$\mathrm{SL}(\mathbb{R}^n)=\{A\in \mathrm{Mat}(n,\mathbb{R}):\det(A)=1\}.$$

Then by an *equiaffine invariant* we mean an unchanged feature under the actions of $\mathrm{SL}(\mathbb{R}^n)$ and the translations of $\mathbb{R}^n$. For example, the volume is an equiaffine invariant (see e.g. [7]).

Denote by $[u_1,\dots,u_n]$ the determinant of the vectors $u_1,\dots,u_n\in\mathbb{R}^n$ where $u_k$ represents the $k$.-th column. Then the value of $[u_1,\dots,u_n]$ is an equiaffine invariant because it measures the volume of parallelopipedon determined by $u_1,\dots,u_n$.

Let $t\mapsto \mathbf{r}(t)$, $t\in I\subset\mathbb{R}$, a smooth parametrized curve in $\mathbb{R}^n$. We call the curve $\mathbf{r}(t)$ *non-degenerate* if, for every $t\in I$, (see [7] and also [12], [13], [18])

$$\left[\frac{d\mathbf{r}}{dt}(t),...,\frac{d^n\mathbf{r}}{dt^n}(t)\right]\neq 0.$$

For simplicity, by a curve we will mean a non-degenerate smooth parametrized curve throughout the paper. Then the *equiaffine arclength function* is defined by

$$\sigma(t)=\int^t\left[\frac{d\mathbf{r}}{du}(u),...,\frac{d^n\mathbf{r}}{du^n}(u)\right]^{2/(n^2+n)}du.$$

We call that the curve is parametrized by *equiaffine arclength* if, for every $\sigma\in J\subset\mathbb{R}$,

$$\left[\frac{d\mathbf{r}}{d\sigma}(\sigma),...,\frac{d^n\mathbf{r}}{d\sigma^n}(\sigma)\right]=1. \tag{3.3}$$

The set $\left\{\frac{d\mathbf{r}}{d\sigma}(\sigma),...,\frac{d^n\mathbf{r}}{d\sigma^n}(\sigma)\right\}$ is called the *equiaffine Frenet frame* of $\mathbf{r}(\sigma)$. When we differentiate Equation (3.3) with respect to the parameter $\sigma$, we may observe that

$$\left[\frac{d\mathbf{r}}{d\sigma}(\sigma),...,\frac{d^{n-1}\mathbf{r}}{d\sigma^{n-1}}(\sigma),\frac{d^{n+1}\mathbf{r}}{d\sigma^{n+1}}(\sigma)\right]=0,$$

where the following set are linearly dependent for every $\sigma\in J$:

$$\left\{\frac{d\mathbf{r}}{d\sigma}(\sigma),...,\frac{d^{n-1}\mathbf{r}}{d\sigma^{n-1}}(\sigma),\frac{d^{n+1}\mathbf{r}}{d\sigma^{n+1}}(\sigma)\right\}.$$

Hence, this gives the existence of smooth functions $\kappa_i(\sigma)$ on $J$ ($1\leq i\leq n-1$) such that

$$\frac{d^{n+1}\mathbf{r}}{d\sigma^{n+1}}(\sigma)+\sum_{i=1}^{n-1}\kappa_i(\sigma)\frac{d^i\mathbf{r}}{d\sigma^i}(\sigma)=0,$$

where

$$\kappa_i(\sigma) = (-1)^{n-i+1}\left[\frac{d\mathbf{r}}{d\sigma}(\sigma), ..., \frac{d^{i-1}\mathbf{r}}{d\sigma^{i-1}}(\sigma), \frac{d^{i+1}\mathbf{r}}{d\sigma^{i+1}}(\sigma), ..., \frac{d^{n+1}\mathbf{r}}{d\sigma^{n+1}}(\sigma)\right], \quad 1 \leq i \leq n-1.$$

The function $\kappa_i(\sigma)$ is called $i.-$th *equiaffine curvature* of the curve $\mathbf{r}(\sigma)$. The equiaffine curvatures are the equiaffine invariants in $\mathbb{R}^n$. In 3-dimensional case, that is, in the case $i \in \{1, 2\}$, we will use the notations $\kappa_1 = \kappa$ and $\kappa_2 = \tau$. In additon, the equiaffine Frenet vectors will be denoted by

$$\mathbf{T}(\sigma) = \frac{d\mathbf{r}}{d\sigma}(\sigma), \quad \mathbf{N}(\sigma) = \frac{d^2\mathbf{r}}{d\sigma^2}(\sigma), \quad \mathbf{B}(\sigma) = \frac{d^3\mathbf{r}}{d\sigma^3}(\sigma).$$

In consequence, the equiaffine equations of Frenet type are given in matrix form

$$\begin{bmatrix}\dot{\mathbf{T}}(\sigma)\\ \dot{\mathbf{N}}(\sigma)\\ \dot{\mathbf{B}}(\sigma)\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -\kappa(\sigma) & -\tau(\sigma) & 0\end{bmatrix}\begin{bmatrix}\mathbf{T}(\sigma)\\ \mathbf{N}(\sigma)\\ \mathbf{B}(\sigma)\end{bmatrix},$$

where $\dot{\mathbf{T}}(\sigma)$ is the derivative of $\mathbf{T}(\sigma)$ with respect to the arclength parameter $\sigma$.

## 4. Equiaffine invariants of fractional order

Let $\mathbf{r}(\sigma)$, $\sigma \in (a, b)$, $0 < a < b$, be a curve in $\mathbb{R}^n$, $n \geq 2$, parametrized by equiaffine arclength. Again, we consider the simplification (1.1) as

$$\frac{d^{\{\alpha\}}\mathbf{r}}{dt^{\{\alpha\}}}(\sigma(t)) = \frac{\alpha t^{1-\alpha}}{\Gamma(2-\alpha)}\frac{d\mathbf{r}}{d\sigma}(\sigma(t))\frac{d\sigma}{dt}(t). \tag{4.4}$$

Here $\alpha \in \mathbb{R}$ with $0 < \alpha \leq 1$ and Equation (4.4) becomes the classical chain rule provided $\alpha = 1$.

In the following, by using Equation (4.4) we introduce an equiaffine arclength function of fractional type.

**Definition 4.1.** *Let $\mathbf{r}(\sigma)$, $\sigma \in (a, b)$, $0 < a < b$, be a curve in $\mathbb{R}^n$ parametrized by equiaffine arclength. The following function $s(\sigma)$ is called equiaffine arclength function of the curve of order $0 < \alpha \leq 1$*

$$\sigma \mapsto s(\sigma) = \left(\frac{2\alpha + n - 1}{n+1}\left(\frac{\alpha}{\Gamma(2-\alpha)}\right)^{2/(n+1)}\sigma\right)^{(n+1)/(2\alpha+n-1)}. \tag{4.5}$$

It is obvious from Equation (4.5) that $s(\sigma)$ is a smooth function of $\sigma$ on $(a, b)$ and so is $\mathbf{r}(s(\sigma))$.

**Proposition 4.1.** *Let* $\mathbf{r}(s)$, $s \in (c,d)$, $0 < c < d$, *be a curve in* $\mathbb{R}^n$ *parametrized by equiaffine arclength of order* $0 < \alpha \le 1$. *Then, for every* $s \in (c,d)$,

$$\left[ \frac{d^{\{\alpha\}}\mathbf{r}}{ds^{\{\alpha\}}}(s), \frac{d}{ds}\left(\frac{d^{\{\alpha\}}\mathbf{r}}{ds^{\{\alpha\}}}(s)\right), ..., \frac{d^{n-1}}{ds^{n-1}}\left(\frac{d^{\{\alpha\}}\mathbf{r}}{ds^{\{\alpha\}}}(s)\right)\right] = 1.$$

**Proof.** Let $\sigma$ be the standard equiaffine arclength parameter of $\mathbf{r}(s)$. By Equation (4.5), we have $ds/d\sigma > 0$, yielding the existence of the inverse of the function $s(\sigma)$, namely,

$$s \mapsto \sigma(s) = \frac{n+1}{2\alpha+n-1}\left(\frac{\alpha}{\Gamma(2-\alpha)}\right)^{-2/(n+1)} s^{(2\alpha+n-1)/(n+1)}, \tag{4.6}$$

where $\sigma(s)$ is smooth on $s \in (c,d)$. Taking derivative in Equation (4.6) with respect to $s$,

$$\frac{d\sigma}{ds}(s) = \left(\frac{\alpha}{\Gamma(2-\alpha)}\right)^{-2/(n+1)} s^{2(\alpha-1)/(n+1)}. \tag{4.7}$$

From Equation (4.4) we have

$$\frac{d^{\{\alpha\}}\mathbf{r}}{ds^{\{\alpha\}}}(\sigma(s)) = \frac{\alpha s^{1-\alpha}}{\Gamma(2-\alpha)}\frac{d\mathbf{r}}{d\sigma}(\sigma(s))\frac{d\sigma}{ds}(s). \tag{4.8}$$

We successively differentiate Equation (4.8) with respect to $s$, obtaining

$$\frac{d}{ds}\left(\frac{d^{\{\alpha\}}\mathbf{r}}{ds^{\{\alpha\}}}(s)\right) = (...)\frac{d\mathbf{r}}{d\sigma}(\sigma(s)) + \frac{\alpha s^{1-\alpha}}{\Gamma(2-\alpha)}\left(\frac{d\sigma}{ds}(s)\right)^2\frac{d^2\mathbf{r}}{d\sigma^2}(\sigma(s)),$$
$$\vdots$$
$$\frac{d^{n-1}}{ds^{n-1}}\left(\frac{d^{\{\alpha\}}\mathbf{r}}{ds^{\{\alpha\}}}(s)\right) = (...)\frac{d\mathbf{r}}{d\sigma}(\sigma(s)) + (...)\frac{d^2\mathbf{r}}{d\sigma^2}(\sigma(s)) + ... + \frac{\alpha s^{1-\alpha}}{\Gamma(2-\alpha)}\left(\frac{d\sigma}{ds}(s)\right)^n\frac{d^n\mathbf{r}}{d\sigma^n}(\sigma(s)),$$

where since we want to find the value of the determinant determined by

$$\frac{d^{\{\alpha\}}\mathbf{r}}{ds^{\{\alpha\}}}(s), \frac{d}{ds}\left(\frac{d^{\{\alpha\}}\mathbf{r}}{ds^{\{\alpha\}}}(s)\right), ..., \frac{d^{n-1}}{ds^{n-1}}\left(\frac{d^{\{\alpha\}}\mathbf{r}}{ds^{\{\alpha\}}}(s)\right),$$

the coefficients denoted by $(\dots)$ will not effect our calculation. Noticing that $\mathbf{r}(\sigma)$ and $\sigma(s)$ are smooth, then the above derivatives exist. Hence,

$$\left[ \frac{d^{\{\alpha\}}\mathbf{r}}{ds^{\{\alpha\}}}(s), \frac{d}{ds}\left(\frac{d^{\{\alpha\}}\mathbf{r}}{ds^{\{\alpha\}}}(s)\right), ..., \frac{d^{n-1}}{ds^{n-1}}\left(\frac{d^{\{\alpha\}}\mathbf{r}}{ds^{\{\alpha\}}}(s)\right)\right] =$$
$$\left(\frac{\alpha s^{1-\alpha}}{\Gamma(2-\alpha)}\right)^n \left(\frac{d\sigma}{ds}(s)\right)^{(n^2+n)/2}\left[\frac{d\mathbf{r}}{d\sigma}(\sigma(s)), \frac{d^2\mathbf{r}}{d\sigma^2}(\sigma(s)), ..., \frac{d^n\mathbf{r}}{d\sigma^2}(\sigma(s))\right].$$

Because $\sigma$ is the standard equiaffine arclength parameter, the value of the determinant at the right hand side is 1, yielding

$$\left[ \frac{d^{\{\alpha\}}\mathbf{r}}{ds^{\{\alpha\}}}(s), \frac{d}{ds}\left(\frac{d^{\{\alpha\}}\mathbf{r}}{ds^{\{\alpha\}}}(s)\right), ..., \frac{d^{n-1}}{ds^{n-1}}\left(\frac{d^{\{\alpha\}}\mathbf{r}}{ds^{\{\alpha\}}}(s)\right)\right] = \left(\frac{\alpha s^{1-\alpha}}{\Gamma(2-\alpha)}\right)^n \left(\frac{d\sigma}{ds}(s)\right)^{(n^2+n)/2}.$$

Considering Equation (4.7) into the above last equation, we complete the proof.

Since we are interested in the 3−dimensional case, then Equation (4.5) is now

$$s(\sigma) = \left( \frac{\alpha+1}{2} \left( \frac{\alpha}{\Gamma(2-\alpha)} \right)^{1/2} \sigma \right)^{2/(\alpha+1)}. \tag{4.9}$$

Hence,

$$\sigma(s) = \left( \frac{\alpha}{\Gamma(2-\alpha)} \right)^{-1/2} \frac{2}{\alpha+1} s^{(\alpha+1)/2}$$

and

$$\frac{d\sigma}{ds}(s) = \left( \frac{\alpha s^{1-\alpha}}{\Gamma(2-\alpha)} \right)^{-1/2}. \tag{4.10}$$

**Definition 4.2.** *Let* $\mathbf{r}(s)$, $s \in (c,d)$, $0 < c < d$, *be a curve in* $\mathbb{R}^3$ *parametrized by equiaffine arclength of order* $0 < \alpha \leq 1$. *Then, the set* $\left\{\mathbf{T}^{\{\alpha\}}(s), \mathbf{N}^{\{\alpha\}}(s), \mathbf{B}^{\{\alpha\}}(s)\right\}$ *is called equiaffine Frenet frame of* $\mathbf{r}(s)$ *of order* $\alpha$, *where*

$$\mathbf{T}^{\{\alpha\}}(s) = \frac{d^{\{\alpha\}}\mathbf{r}}{ds^{\{\alpha\}}}(s), \quad \mathbf{N}^{\{\alpha\}}(s) = \frac{d}{ds}\left( \frac{d^{\{\alpha\}}\mathbf{r}}{ds^{\{\alpha\}}}(s) \right), \quad \mathbf{B}^{\{\alpha\}}(s) = \frac{d^2}{ds^2}\left( \frac{d^{\{\alpha\}}\mathbf{r}}{ds^{\{\alpha\}}}(s) \right).$$

Note that when $\alpha = 1$ the set $\left\{\mathbf{T}^{\{\alpha\}}(s), \mathbf{N}^{\{\alpha\}}(s), \mathbf{B}^{\{\alpha\}}(s)\right\}$ is equivalent to the standard equiaffine Frenet frame of $\mathbf{r}(s)$, that is, $\mathbf{T}^{\{1\}} = \mathbf{T}$, $\mathbf{N}^{\{1\}} = \mathbf{N}$, $\mathbf{B}^{\{1\}} = \mathbf{B}$.

By Proposition 4.1, we have

$$\left[\mathbf{T}^{\{\alpha\}}(s), \mathbf{N}^{\{\alpha\}}(s), \mathbf{B}^{\{\alpha\}}(s)\right] = 1. \tag{4.11}$$

Denote by a prime the ordinary derivative with respect to the parameter $s$, that is, $\mathbf{N}^{\{\alpha\}}(s) = \mathbf{T}^{\{\alpha\}\prime}(s)$ and $\mathbf{B}^{\{\alpha\}}(s) = \mathbf{N}^{\{\alpha\}\prime}(s)$. Then we differentiate Equation (4.11) with respect to $s$, obtaining

$$\left[\mathbf{T}^{\{\alpha\}}, \mathbf{N}^{\{\alpha\}}, \mathbf{B}^{\{\alpha\}\prime}\right] = 0,$$

where it can be seen that the set $\left\{\mathbf{T}^{\{\alpha\}}, \mathbf{N}^{\{\alpha\}}, \mathbf{B}^{\{\alpha\}\prime}\right\}$ is linearly dependent for every $s \in (c,d)$. Then there are some smooth functions on $(c,d)$ denoted by $\kappa^{\{\alpha\}}$ and $\tau^{\{\alpha\}}$ such that

$$\kappa^{\{\alpha\}}\mathbf{T}^{\{\alpha\}} + \tau^{\{\alpha\}}\mathbf{N}^{\{\alpha\}} + \mathbf{B}^{\{\alpha\}\prime} = 0.$$

Consequently, we can give the following.

**Definition 4.3.** *Let* $\mathbf{r}(s)$, $s \in (c,d)$, $0 < c < d$, *be a curve in* $\mathbb{R}^3$ *parametrized by equiaffine arclength of order* $0 < \alpha \leq 1$. *Then the functions* $\kappa^{\{\alpha\}}(s)$ *and* $\tau^{\{\alpha\}}(s)$ *are called the equiaffine curvatures of* $\mathbf{r}(s)$ *of order* $\alpha$, *where*

$$\kappa^{\{\alpha\}}(s) = -\left[\mathbf{N}^{\{\alpha\}}(s), \mathbf{B}^{\{\alpha\}}(s), \mathbf{B}^{\{\alpha\}\prime}(s)\right] \tag{4.12}$$

*and*

$$\tau^{\{\alpha\}}(s) = \left[\mathbf{T}^{\{\alpha\}}(s), \mathbf{B}^{\{\alpha\}}(s), \mathbf{B}^{\{\alpha\}\prime}(s)\right]. \tag{4.13}$$

With this definition, we have the *equiaffine Frenet equations of order* $\alpha$ given in matrix form

$$\begin{bmatrix} \mathbf{T}^{\{\alpha\}\prime}(s) \\ \mathbf{N}^{\{\alpha\}\prime}(s) \\ \mathbf{B}^{\{\alpha\}\prime}(s) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\kappa^{\{\alpha\}}(s) & -\tau^{\{\alpha\}}(s) & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T}^{\{\alpha\}}(s) \\ \mathbf{N}^{\{\alpha\}}(s) \\ \mathbf{B}^{\{\alpha\}}(s) \end{bmatrix}.$$

We occasionally use the terms of fractional equiaffine arclength, Frenet vector and curvature instead of the equiaffine arclength, Frenet vector and curvature of order $\alpha$.

**Proposition 4.2.** *Let* $\mathbf{r}(s)$, $s \in (c, d)$, $0 < c < d$, *be a curve in* $\mathbb{R}^3$ *parametrized by equiaffine arclength of order* $0 < \alpha \leq 1$. *Denote by* $\left\{\mathbf{T}^{\{\alpha\}}(s), \mathbf{N}^{\{\alpha\}}(s), \mathbf{B}^{\{\alpha\}}(s)\right\}$ *and* $\{\mathbf{T}(\sigma), \mathbf{N}(\sigma), \mathbf{B}(\sigma)\}$ *the equiaffine Frenet frames of* $\mathbf{r}(s)$. *Then we have*

$$\begin{bmatrix} \mathbf{T}^{\{\alpha\}}(s) \\ \mathbf{N}^{\{\alpha\}}(s) \\ \mathbf{B}^{\{\alpha\}}(s) \end{bmatrix} = \begin{bmatrix} \left(\frac{\alpha s^{1-\alpha}}{\Gamma(2-\alpha)}\right)^{1/2} & 0 & 0 \\ \frac{1-\alpha}{2}\left(\frac{\alpha s^{-1-\alpha}}{\Gamma(2-\alpha)}\right)^{1/2} & 1 & 0 \\ \frac{\alpha^2-1}{4}\left(\frac{\alpha s^{-3-\alpha}}{\Gamma(2-\alpha)}\right)^{1/2} & \frac{1-\alpha}{2}s^{-1} & \left(\frac{\alpha s^{1-\alpha}}{\Gamma(2-\alpha)}\right)^{-1/2} \end{bmatrix} \begin{bmatrix} \mathbf{T}(\sigma(s)) \\ \mathbf{N}(\sigma(s)) \\ \mathbf{B}(\sigma(s)) \end{bmatrix},$$

*where* $\sigma$ *is the standard equiaffine arclength parameter.*

**Proof.** Denote by $\sigma$ the standard equiaffine parameter. By Equations (4.8) and (4.10), we write

$$\mathbf{T}^{\{\alpha\}}(s) = \left(\frac{\alpha s^{1-\alpha}}{\Gamma(2-\alpha)}\right)^{1/2} \mathbf{T}(\sigma(s)) \tag{4.14}$$

where $\mathbf{T}(\sigma(s)) = \frac{d\mathbf{r}}{d\sigma}(\sigma(s))$. Differentiating Equation (4.14) with respect to $s$,

$$\mathbf{N}^{\{\alpha\}}(s) = \frac{1-\alpha}{2}\left(\frac{\alpha s^{-1-\alpha}}{\Gamma(2-\alpha)}\right)^{1/2} \mathbf{T}(\sigma(s)) + \mathbf{N}(\sigma(s)) \tag{4.15}$$

and

$$\mathbf{B}^{\{\alpha\}}(s) = \frac{\alpha^2-1}{4}\left(\frac{\alpha s^{-3-\alpha}}{\Gamma(2-\alpha)}\right)^{1/2} \mathbf{T}(\sigma(s)) + \frac{1-\alpha}{2}s^{-1}\mathbf{N}(\sigma(s)) + \left(\frac{\alpha s^{1-\alpha}}{\Gamma(2-\alpha)}\right)^{-1/2} \mathbf{B}(\sigma(s)). \tag{4.16}$$

The proof is completed by expressing Equations (4.14), (4.15) and (4.16) in matrix form.

Proposition 4.2 indicates the difference between the fractional and standard equiaffine Frenet vectors. Now, we give the relations between the fractional and standard equiaffine curvatures.

**Theorem 4.1.** *Let* $\mathbf{r}(s)$*,* $s \in (c,d)$*,* $0 < c < d$*, be a curve in* $\mathbb{R}^3$ *parametrized by equiaffine arclength of order* $0 < \alpha \leq 1$*. The equiaffine curvatures* $\kappa^{\{\alpha\}}(s)$ *and* $\tau^{\{\alpha\}}(s)$ *of order* $\alpha$ *are invariants under the equiaffine transformations of* $\mathbb{R}^3$*. Furthermore, if the standard equiaffine curvatures of* $\mathbf{r}(s)$ *are denoted by* $\kappa(\sigma)$ *and* $\tau(\sigma)$*, then the following relations occur*

$$\kappa^{\{\alpha\}}(s) = \frac{(3+\alpha)(-1+\alpha)}{4}s^{-3} + \left(\frac{\alpha s^{1-\alpha}}{\Gamma(2-\alpha)}\right)^{-3/2}\kappa(\sigma(s)) - \frac{(1-\alpha)\Gamma(2-\alpha)}{2\alpha}s^{\alpha-2}\tau(\sigma(s)) \tag{4.17}$$

*and*

$$\tau^{\{\alpha\}}(s) = \frac{(3+\alpha)(1-\alpha)}{4}s^{-2} + \left(\frac{\alpha s^{1-\alpha}}{\Gamma(2-\alpha)}\right)^{-1}\tau(\sigma(s)). \tag{4.18}$$

**Proof.** Since $\kappa^{\{\alpha\}}(s)$ and $\tau^{\{\alpha\}}(s)$ are defined by determinants (see Definition 4.3), those are invariant under the equiaffine transformations of $\mathbb{R}^3$. This is the proof of first part. Differentiating (4.16) with respect to $s$,

$$\frac{d(\mathbf{B}^{\{\alpha\}})}{ds}(s) = p(s)\mathbf{T}(\sigma(s)) + q(s)\mathbf{N}(\sigma(s)), \tag{4.19}$$

where

$$p(s) = \frac{(3+\alpha)(1-\alpha^2)}{8}\left(\frac{\alpha s^{-5-\alpha}}{\Gamma(2-\alpha)}\right)^{1/2} - \left(\frac{\alpha s^{1-\alpha}}{\Gamma(2-\alpha)}\right)^{-1}\kappa(\sigma(s))$$

and

$$q(s) = \frac{(3+\alpha)(-1+\alpha)}{4}s^{-2} - \left(\frac{\alpha s^{1-\alpha}}{\Gamma(2-\alpha)}\right)^{-1}\tau(\sigma(s)).$$

If we consider Equations (4.15), (4.16) and (4.19) in Equation (4.12), after some manipulations, we derive Equation (4.17). Analogously, Equation (4.18) is obtained by substituting equations (4.14), (4.16) and (4.19) into (4.13). This completes the proof.

As consequences, we can state the following results.

**Corollary 4.1.** *Let* $\mathbf{r}(s)$*,* $s \in (c,d)$*,* $0 < c < d$*, be a curve in* $\mathbb{R}^3$ *parametrized by equiaffine arclength of order* $0 < \alpha \leq 1$*. If the equiaffine curvatures of* $\mathbf{r}(s)$ *vanish identically, then*

$$\kappa^{\{\alpha\}}(s) = \frac{(3+\alpha)(-1+\alpha)}{4}s^{-3}$$

*and*

$$\tau^{\{\alpha\}}(s) = \frac{(3+\alpha)(1-\alpha)}{4}s^{-2}.$$

**Proof.** It follows by Equations (4.17) and (4.18).

**Corollary 4.2.** *Let $\mathbf{r}(\sigma)$, $\sigma \in (a,b)$, $0<a<b$, be a curve in $\mathbb{R}^3$ parametrized by equiaffine arclength. If the equiaffine curvatures of $\mathbf{r}(\sigma)$ of order $0<\alpha\leq 1$ vanish identically, then*

$$\kappa(\sigma) = \frac{(3+\alpha)(1-\alpha)}{(1+\alpha)^2}\sigma^{-3} \tag{4.20}$$

*and*

$$\tau(\sigma) = -\frac{(3+\alpha)(1-\alpha)}{(1+\alpha)^2}\sigma^{-2}. \tag{4.21}$$

**Proof.** If $\tau^{\{\alpha\}}(s) = 0$ for every $s$ then from Equation (4.18) we have

$$\tau(\sigma(s)) = \frac{(3+\alpha)(-1+\alpha)}{4}\left(\frac{\alpha s^{-1-\alpha}}{\Gamma(2-\alpha)}\right). \tag{4.22}$$

Equation (4.21) is obtained by considering Equation (4.9) in Equation (4.22). Analogously, if $\kappa^{\{\alpha\}} = 0$ for every $s$ then Equation (4.17) is now

$$\kappa(\sigma(s)) = \frac{(3+\alpha)(1-\alpha)}{4}\left(\frac{\alpha s^{-1-\alpha}}{\Gamma(2-\alpha)}\right)^{3/2} + \frac{(1-\alpha)\Gamma(2-\alpha)}{2\alpha}\left(\frac{\alpha s^{-(1+\alpha)/3}}{\Gamma(2-\alpha)}\right)^{3/2}\tau(\sigma(s)). \tag{4.23}$$

Substituting Equations (4.9) and (4.21) into Equation (4.23), we derive Equation (4.20).

## 5. Examples

**Example 5.1.** *Consider in $\mathbb{R}^3$ the following curve (see Figure 1)*

$$\mathbf{r}(\sigma) = \left(\sigma, \frac{\sigma^2}{2}, \frac{\sigma^3}{3}\right), \quad \sigma \in (a,b), \quad 0<a<b,$$

*where $\sigma$ is the equiaffine arclength parameter of $\mathbf{r}(\sigma)$, that is, $[\mathbf{T}(\sigma), \mathbf{N}(\sigma), \mathbf{B}(\sigma)] = 1$, for every $\sigma \in (a,b)$. Because $\mathbf{B}(\sigma) = (0,0,1)$, the equiaffine curvatures $\kappa(\sigma)$ and $\tau(\sigma)$ are identically 0. By Corollary 4.1, the equiaffine curvatures of the curve of order $0<\alpha\leq 1$ are $\kappa^{\{\alpha\}}(s) = (3+\alpha)(-1+\alpha)/(4s^3)$ and $\tau^{\{\alpha\}}(s) = (3+\alpha)(1-\alpha)/(4s^2)$, where $s$ is the equiaffine arclength parameter of order $\alpha$. The graphs of the curvature functions $\kappa^{\{\alpha\}}(s)$ and $\tau^{\{\alpha\}}(s)$ can be drawn in Figures 2 and 3 up to different values of $\alpha$.*

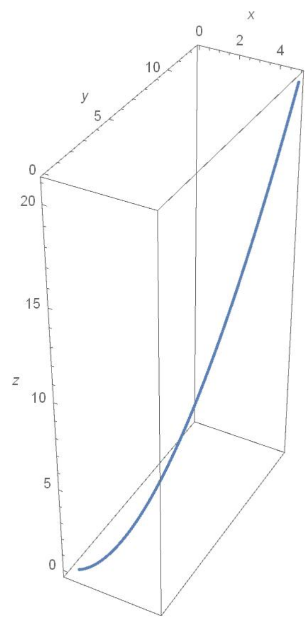


FIGURE 1. $\mathbf{r}(\sigma) = \left(\sigma, \frac{\sigma^2}{2}, \frac{\sigma^3}{3}\right)$, $\sigma \in [1/2, 5]$, with vanishing equiaffine curvatures.

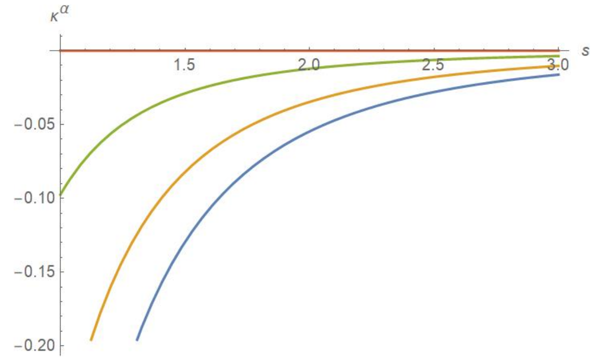


FIGURE 2. The graphs of $\kappa^{\{\alpha\}}(s) = (3+\alpha)(-1+\alpha)/(4s^3)$, $s \in [1,3]$, in blue for $\alpha = 0.5$, in yellow for $\alpha = 0.7$, in green for $\alpha = 0.9$ and in red for $\alpha = 1$.

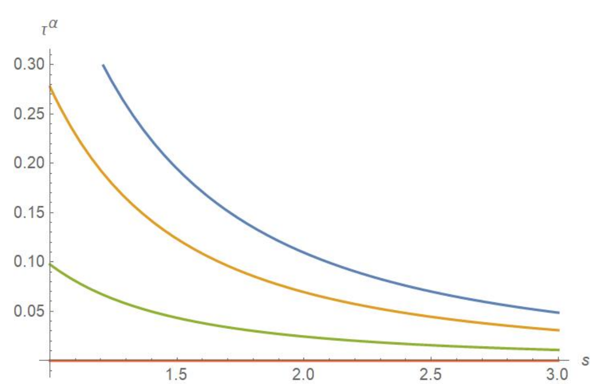


FIGURE 3. The graphs of $\tau^{\{\alpha\}}(s) = (3+\alpha)(1-\alpha)/(4s^2)$, $s \in [1,3]$, in blue for $\alpha = 0.5$, in yellow for $\alpha = 0.7$, in green for $\alpha = 0.9$ and in red for $\alpha = 1$.

**Example 5.2.** *Let $0 < \alpha \leq 1$. We take in $\mathbb{R}^3$ the following curve (see Figure 4)*

$$\mathbf{r}(s) = \frac{\Gamma(2-\alpha)}{\alpha}\left(\frac{s^{\alpha}}{\alpha}, \frac{s^{\alpha+1}}{\alpha+1}, \frac{s^{\alpha+2}}{2(\alpha+2)}\right), \quad s \in (c,d), \quad 0 < c < d,$$

*where $s$ is the equiaffine arclength parameter of $\mathbf{r}(s)$ of order $\alpha$, that is,*

$$[\mathbf{T}^{\{\alpha\}}(s), \mathbf{N}^{\{\alpha\}}(s), \mathbf{B}^{\{\alpha\}}(s)] = 1,$$

*for every $s \in (c,d)$. Because $\mathbf{B}^{\{\alpha\}}(s) = (0,0,1)$, the equiaffine curvatures $\kappa^{\{\alpha\}}(s)$ and $\tau^{\{\alpha\}}(s)$ of order $\alpha$ are identically $0$. By Corollary 4.2, the standard equiaffine curvatures of $\mathbf{r}(s)$ are $\kappa(\sigma) = (3+\alpha)(1-\alpha)(1+\alpha)^{-2}\sigma^{-3}$ and $\tau(\sigma) = -(3+\alpha)(1-\alpha)(1+\alpha)^{-2}\sigma^{-2}$, where $\sigma$ is the equiaffine arclength parameter (see Figures 5 and 6).*

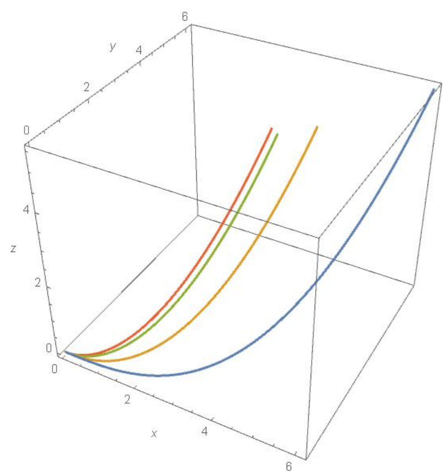


FIGURE 4. $\mathbf{r}(s) = \frac{\Gamma(2-\alpha)}{\alpha}\left(\frac{s^{\alpha}}{\alpha}, \frac{s^{\alpha+1}}{\alpha+1}, \frac{s^{\alpha+2}}{2(\alpha+2)}\right)$, $s \in [1/2, 5]$, with vanishing equiaffine curvatures of order $0 < \alpha \leq 1$. In blue for $\alpha = 0.5$, in yellow for $\alpha = 0.7$, in green for $\alpha = 0.9$ and in red for $\alpha = 1$.

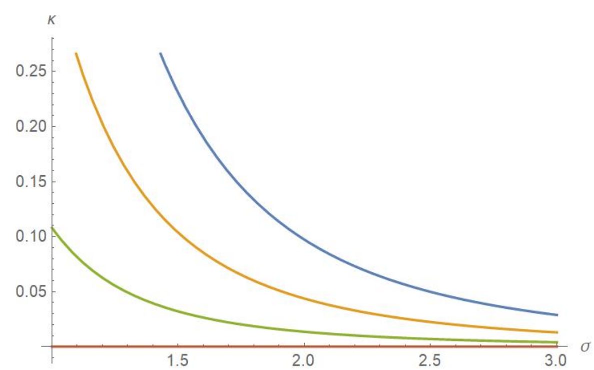


FIGURE 5. The graphs of $\kappa(\sigma) = (3+\alpha)(1-\alpha)(1+\alpha)^{-2}\sigma^{-3}$, $\sigma \in [1,3]$, in blue for $\alpha = 0.5$, in yellow for $\alpha = 0.7$, in green for $\alpha = 0.9$ and in red for $\alpha = 1$.

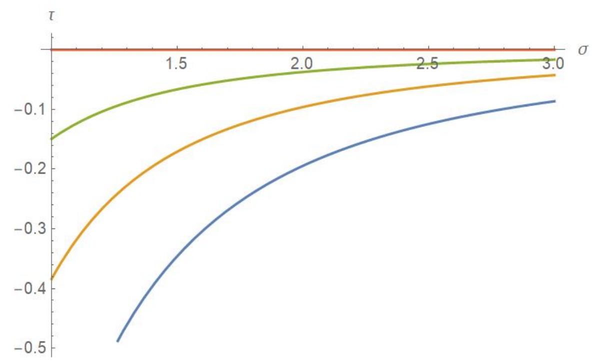


FIGURE 6. The graphs of $\tau(\sigma) = -(3+\alpha)(1-\alpha)(1+\alpha)^{-2}\sigma^{-2}$, $\sigma \in [1,3]$, in blue for $\alpha = 0.5$, in yellow for $\alpha = 0.7$, in green for $\alpha = 0.9$ and in red for $\alpha = 1$.

## 6. Discussions

The results of the present study may give new ideas relating to using of fractional derivative in the differential geometry of curves. For example, when emposing some natural conditions on curvatures, the classification of curves is a central problem. Or, the extension of results in 3-dimenisonal case to higher dimensions is an important problem again. Hence, the following two problems can be posed:

(1) The first one is the problem of finding parametric equations of curves when their fractional curvatures $\kappa^{\{\alpha\}}$ and $\tau^{\{\alpha\}}$ are constant. Indeed, solving this problem is equivalent to solve the following vector differential equation

$$\kappa_0^{\{\alpha\}}\mathbf{T}^{\{\alpha\}} + \tau_0^{\{\alpha\}}\mathbf{N}^{\{\alpha\}} + \mathbf{B}^{\{\alpha\}\prime} = 0, \tag{6.24}$$

where $\kappa_0^{\{\alpha\}}$ and $\tau_0^{\{\alpha\}}$ are some constants. As an example, we will find the equation of a curve that satisfies $\kappa_0^{\{\alpha\}}(s) = 0 = \tau_0^{\{\alpha\}}(s)$, for every $s$. Then Equation (6.24) is now $\mathbf{B}^{\{\alpha\}\prime} = 0$, or equivalently,

$$\frac{d^3}{ds^3}\left(\frac{d^{\{\alpha\}}\mathbf{r}}{ds^{\{\alpha\}}}\right) = 0.$$

Integrating,

$$\frac{d^{\{\alpha\}}\mathbf{r}}{ds^{\{\alpha\}}} = \mathbf{a} + \mathbf{b}s + \mathbf{c}\frac{s^2}{2}, \tag{6.25}$$

where $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$. Since $\left[\mathbf{T}^{\{\alpha\}}, \mathbf{N}^{\{\alpha\}}, \mathbf{B}^{\{\alpha\}}\right] = 1$, we may choose that $\mathbf{a} = (1,0,0)$, $\mathbf{b} = (0,1,0)$ and $\mathbf{c} = (0,0,1)$. Now if we consider Equation (4.4) into Equation (6.25) then we have

$$\frac{d\mathbf{r}}{ds} = \frac{\Gamma(2-\alpha)}{\alpha}\left(s^{-1+\alpha}, s^{\alpha}, s^{1+\alpha}\right).$$

After integrating the above last equation, up to a translation of $\mathbb{R}^3$, we find the parametrization of the curve that we are looking for. Consequently, the general solution of the posed problem can be obtained by following the similar steps.

(2) The second idea is to find the relations in higher dimensions between the fractional and standard equiaffine curvatures, that is, the analogous ones of equations (4.17) and (4.18). In particular, the main purpose of this problem is to express the relations into one equation between the fractional and standard equiaffine curvatures. For this, given a curve $\mathbf{r}(s)$ in $\mathbb{R}^n$ parametrized by equiaffine arclength of order $\alpha$ then the $i.-$th equiaffine curvature of order $\alpha$ can be defined by

$$\kappa_i^{\{\alpha\}} = (-1)^{n-i+1} \left[ \frac{d^{\{\alpha\}}\mathbf{r}}{ds^{\{\alpha\}}}, ..., \frac{d^{i-2}}{ds^{i-2}} \left( \frac{d^{\{\alpha\}}\mathbf{r}}{ds^{\{\alpha\}}} \right), \frac{d^{i}}{ds^{i}} \left( \frac{d^{\{\alpha\}}\mathbf{r}}{ds^{\{\alpha\}}} \right), ..., \frac{d^{n}}{ds^{n}} \left( \frac{d^{\{\alpha\}}\mathbf{r}}{ds^{\{\alpha\}}} \right) \right],$$

where $i \in \{1, ..., n-1\}$. The problem proposes to establish a unique relation between $\kappa_i^{\{\alpha\}}$ and $\kappa_i$ that holds for some $i \in \{1, ..., n-1\}$.

## 7. Conclusions

The simplification of Caputo fractional derivative given by Equation (1.1) effects the study of curves in terms of their equiaffine invariants in two ways. Given a curve $\mathbf{r}(s)$, then the first effect is obtaining a different equiaffine Frenet frame of $\mathbf{r}(s)$ from the standard one (Proposition 4.2). This situation is not valid for the Euclidean setting. The second effect can be seen on the fractional equiaffine curvatures (see Equations (4.17) and (4.18)) where the value of the terms containing the arclength $s$ take a large value around an initial time and converges to zero for $s \to \infty$. See also Figures 2 and 3. This intention of the fractional equiaffine curvatures refers to the memory effect of fractional derivative which is decreasing for a long period of time ([23]).

As can be observed in the figures of Section 4, as $\alpha$ goes to 1 the geometric notions defined by using the derivative formula (1) approach to the standard ones. This implies that the idea proposed in the present study is consistent with the classical theory.

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