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THE SPECIAL CURVES OF FIBONACCI AND LUCAS CURVES

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Abstract. In this paper, we introduce the contrapedal, radial, inverse, conchoid and strophoid curves of Fibonacci and Lucas curves which are defined by Horadam and Shannon, [18]. Moreover, the graphs of these special curves are drawn by using Mathematica.

Keywords: Fibonacci curve, Lucas curve, Contrapedal curve, Radial curve, Inverse curve, Conchoid curve, Strophoid curve

2010 Mathematics Subject Classification: 53A04, 11B39.

1. Introduction

The plane curves in the Euclidean plane are one of the most essential subjects in differential geometry. Thanks to a growing interest in this subject, it is demonstrated that any plane curve brings about other plane curves through several constructions. Some of these are contrapedal, radial, inverse, conchoid and strophoid curves. Contrapedal curves are employed in many areas such as mathematics (see [16]) and physics (see [20]). Radial curve was studied by Robert Tucker in 1864, [25]. Geometrical inversion is originated from Jakob Steiner in 1824. In 1825, Adolphe Quetelet followed closely him by giving some examples. Apparently, it independently discovered by Giusto Bellavitis in 1836, by Stubbs and Ingram in 1842-3, and by Lord Kelvin who employed it in his electrical researches in 1845, [25]. Inverse curve has a important role in mathematics (see [6]). Conchoid is a plane curve invented

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by the Greek mathematician Nicomedes, who applied it to the problems of duplication the cube. The conchoid has been used by later mathematicians, notably Sir Isaac Newton, in the construction of various cubic curves, [23]. Conchoids make a significant contribution in many applications as optics (see [2]), astronomy (see [9]), engineering in medicine and biology (see [8], [12]), mechanical in fluid processing (see [21]), physics (see [22]), electromagnetic research (see [26]), etc. The *Conchoid of Nicomedes*, which is the conchoid of a line, and the *Limaçon of Pascal*, which is the conchoid of a circle, are the two most famous conchoids, [17]. Strophoid curve initially appears in work by the English mathematician Isaac Barrow, who was Isaac Newton's teacher, in 1670. However, the curve actually is described in his letters by Evangelista Torricelli before Barrow's work around 1645. In 1846, the strophoid, whose meaning is a "belt with a twist", was named by Montucci, [4]. J. Booth called it the logocyclic curve in his article in the 19th century, [3]. For further information about contrapedal, radial, inverse, conchoid, and strophoid curve, we recommend the reader to go through [7], [11], and [25].

The famous book called the Liber Abaci of Italian mathematician Leonardo de Pisa who is known as Fibonacci also posed a problem concerning the progeny of a single pair of rabbits which is the foundation of the Fibonacci sequence, [5]. During the time Fibonacci wrote Liber Abaci, Fibonacci numbers were not recognized as something special. The sequence was given the current name "Fibonacci numbers" by French mathematician Edouard Lucas who later created his own sequence based on the pattern set by Fibonacci. Lucas numbers are very similar to Fibonacci numbers in that they form a sequence of numbers and also closely related to Fibonacci numbers, [15].

In 1988, Horadam and Shannon defined Fibonacci and Lucas curves on Euclidean plane, (see [18]). Moreover, there are many articles about three dimensional Fibonacci curve, (see [13], [19]). In addition, Akyiğit, Erişir and Tosun studied on the evolute, parallel and pedal of Fibonacci and Lucas curves in 2015, (see [1]). In 2017, Özvatan and Pashaev had a study on generalized Fibonacci sequences and Binet-Fibonacci curves, (see [14]). They constructed Binet-Fibonacci curve in complex plane by extending Binet's formula to arbitrary real numbers. In this article, we are interested in investigation of the contrapedal, radial, inverse, conchoid and strophoid curves of Fibonacci and Lucas curves and obtaining the figures of these special curves.

1.1. **Fibonacci and Lucas Numbers.** This subsection gives a brief overview of Fibonacci and Lucas numbers. More detailed information about them can be found in [10] and [24].

1.1.1. *Fibonacci Numbers.*

Definition 1.1. *The nth Fibonacci number F_n is defined by*

$$F_n = F_{n-1} + F_{n-2}$$

with initial conditions

$$F_1 = F_2 = 1,$$

where $n \geq 3$. In this case, Fibonacci numbers are given by

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, \ldots, F_n, \ldots$$

The ratio of consecutive Fibonacci numbers gives us a new sequence:

$$\frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \ldots, \frac{F_{n+1}}{F_n}, \ldots$$

Lemma 1.1. *The ratio of two consecutive Fibonacci numbers approaches $\frac{1+\sqrt{5}}{2}$ as $n \to \infty$. More precisely,*

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \frac{1+\sqrt{5}}{2}.$$

Definition 1.2. *The positive root $\frac{1+\sqrt{5}}{2} = 1.618...$ of the equation $x^2 - x - 1 = 0$ is called golden ratio.*

Theorem 1.1. *Let α and β be the solutions of the quadratic equation $x^2 - x - 1 = 0$; so $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. Then, the relation that gives us the nth term of Fibonacci sequence is given by*

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \frac{\alpha^n - \beta^n}{\sqrt{5}},$$

where $n \geq 1$.

Corollary 1.1. *Let $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. Then,*

1. $\alpha\beta = -1$ ***2.*** $\alpha + \beta = 1$ ***3.*** $\alpha - \beta = \sqrt{5}$

4. $\alpha^2 + 1 = \sqrt{5}\alpha$ ***5.*** $\alpha = 2 - \beta^2$ ***6.*** $\alpha^2 + \beta^2 = 3$

1.1.2. *Lucas Numbers.*

Definition 1.3. *The nth Lucas number L_n is defined by*

$$L_n = L_{n-1} + L_{n-2}$$

with initial conditions

$$L_1 = 1, \quad L_2 = 3,$$

where $n \geq 3$. In this case, Lucas numbers are given by

$$1, 3, 4, 7, 11, 18, 29, 47, \dots, L_n, \dots$$

Lemma 1.2. *The ratio of two consecutive Lucas numbers approaches $\frac{1+\sqrt{5}}{2}$ as $n \to \infty$. That is,*

$$\lim_{n\to\infty} \frac{L_{n+1}}{L_n} = \frac{1+\sqrt{5}}{2}.$$

Theorem 1.2. *Let α and β be the solutions of the quadratic equation $x^2 - x - 1 = 0$; so $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. Then, the relation that gives us the nth term of Lucas sequence is given by*

$$L_n = \alpha^n + \beta^n,$$

where $n \geq 0$.

1.2. **Fibonacci and Lucas Curves.**

Definition 1.4. *Let $I \subseteq \mathbb{R}$ be an open interval of $\mathbb{R}$. Then, Fibonacci curve is defined by*

$$\begin{aligned} f &: I \to \mathbb{R}^2 \\ \theta &\mapsto f(\theta) = (x(\theta), y(\theta)), \end{aligned}$$

where

$$x(\theta) = \frac{\alpha^\theta - \alpha^{-\theta}\cos(\theta\pi)}{\sqrt{5}} \tag{1.1}$$

and

$$y(\theta) = \frac{-\alpha^{-\theta}\sin(\theta\pi)}{\sqrt{5}} \tag{1.2}$$

including $\alpha = \frac{1+\sqrt{5}}{2}$, [18].

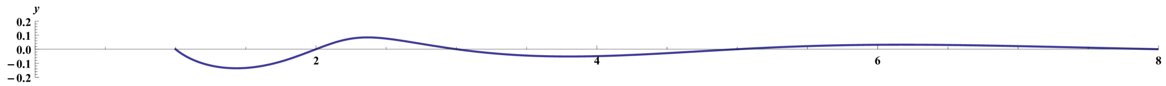


FIGURE 1. Fibonacci curve

In the interval $I = (2, 6)$, the graph of Fibonacci curve can be seen in Figure 1. By taking derivative of the equations (1.1) and (1.2) with respect to θ, we obtain that

$$\frac{dx}{d\theta} = x'(\theta) = \frac{\alpha^{-\theta}\left[\alpha^{2\theta} s + s\cos(\theta\pi) + \pi\sin(\theta\pi)\right]}{\sqrt{5}} \tag{1.3}$$

and

$$\frac{dy}{d\theta} = y'(\theta) = \frac{\alpha^{-\theta}\left[-\pi\cos(\theta\pi) + s\sin(\theta\pi)\right]}{\sqrt{5}}, \tag{1.4}$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ and $s = \log\left(\frac{1+\sqrt{5}}{2}\right)$. After taking derivative of the equations (1.3) and (1.4) with respect to θ, we obtain

$$\frac{d^2x}{d\theta^2} = x''(\theta) = \frac{\alpha^{-\theta}\left[(\pi^2 - s^2)\cos(\theta\pi) + \alpha^{2\theta}s^2 - 2\pi s\sin(\theta\pi)\right]}{\sqrt{5}} \tag{1.5}$$

and

$$\frac{d^2y}{d\theta^2} = y''(\theta) = \frac{\alpha^{-\theta}\left[2\pi s\cos(\theta\pi) + (\pi^2 - s^2)\sin(\theta\pi)\right]}{\sqrt{5}}, \tag{1.6}$$

[18], [1].

Definition 1.5. *Let $I \subseteq \mathbb{R}$ be an open interval of $\mathbb{R}$. Then, Lucas curve is defined by*

$$l : I \to \mathbb{R}^2$$

$$\theta \mapsto l(\theta) = (x(\theta), y(\theta)),$$

where

$$x(\theta) = \alpha^{\theta} + \alpha^{-\theta}\cos(\theta\pi) \tag{1.7}$$

and

$$y(\theta) = \alpha^{-\theta}\sin(\theta\pi) \tag{1.8}$$

including $\alpha = \frac{1+\sqrt{5}}{2}$, [18].

In the interval $I = (1, 5)$, the graph of Lucas curve can be seen in Figure 2.

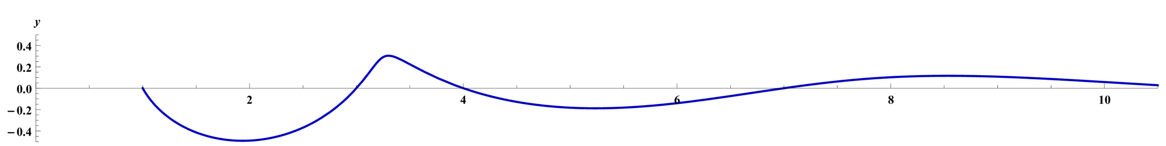


FIGURE 2. Lucas curve

By taking derivative of the equations (1.7) and (1.8) with respect to θ, we obtain that

$$\frac{dx}{d\theta} = x'(\theta) = \alpha^{-\theta}\big[s\alpha^{2\theta} - s\cos(\theta\pi) - \pi\sin(\theta\pi)\big] \tag{1.9}$$

and

$$\frac{dy}{d\theta} = y'(\theta) = \alpha^{-\theta}\big[\pi\cos(\theta\pi) - s\sin(\theta\pi)\big], \tag{1.10}$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ and $s = \log\left(\frac{1+\sqrt{5}}{2}\right)$. After taking derivative of the equations (1.9) and (1.10) with respect to θ, we obtain

$$\frac{d^2x}{d\theta^2} = x''(\theta) = \alpha^{-\theta}\left[\alpha^{2\theta}s^2 + (s^2-\pi^2)\cos(\theta\pi) + 2\pi s\sin(\theta\pi)\right] \tag{1.11}$$

and

$$\frac{d^2y}{d\theta^2} = y''(\theta) = \alpha^{-\theta}\big[-2\pi s\cos(\theta\pi) + (s^2-\pi^2)\sin(\theta\pi)\big], \tag{1.12}$$

[18], [1].

2. The Special Curves of Fibonacci Curve

In this section, we will present the special plane curves of Fibonacci curve by using equations (1.3), (1.4), (1.5) and (1.6).

2.1. The Contrapedal Curve of Fibonacci Curve. The parametric equation of contrapedal curve [1] of Fibonacci curve $f(\theta)$ with respect to point $P = (p_1, p_2)$ on the plane is that

$$Cp_f(\theta) = (A(\theta), B(\theta)), \tag{2.13}$$

where

$$A(\theta) = p_1 + \frac{\alpha^{-\theta}\left(s\alpha^{2\theta} + s\cos(\pi\theta) + \pi\sin(\pi\theta)\right)\left(\sqrt{5}s\left(\alpha^{4\theta}-1\right) - 5sp_1\alpha^{3\theta} + \alpha^{\theta}v_\theta\right)}{5\left(s^2\left(\alpha^{4\theta}+1\right) + 2s\alpha^{2\theta}(\pi\sin(\pi\theta) + s\cos(\pi\theta)) + \pi^2\right)}$$

and

$$B(\theta) = p_2 - \frac{\alpha^{-\theta}(\pi\cos(\pi\theta) - s\sin(\pi\theta))\left(\sqrt{5}s\left(\alpha^{4\theta}-1\right) - 5sp_1\alpha^{3\theta} + \alpha^{\theta}v_\theta\right)}{5\left(s^2\left(\alpha^{4\theta}+1\right) + 2s\alpha^{2\theta}(\pi\sin(\pi\theta) + s\cos(\pi\theta)) + \pi^2\right)}$$

including

$$v_\theta = \left((\sqrt{5}\pi\alpha^{\theta} - 5sp_2 - 5\pi p_1)\sin(\theta\pi) + 5(\pi p_2 - sp_1)\cos(\theta\pi)\right).$$

In Figure 3, Fibonacci curve which is represented by blue curve and the contrapedal curves $Cp_f(\theta)$ of Fibonacci curve $f(\theta)$ with respect to points $(0,6)$, $(3,4)$ $(2,2)$, and $(-1,-2)$ is

[1] Let $\alpha(t) = (x(t), y(t))$ be a regular plane curve and P be a fixed point on $\mathbb{R}^2$. The locus of bases of perpendicular lines from $P = (p_1, p_2)$ to a variable normal line to α is ***contrapedal curve*** and the equation of contrapedal curve of α is that $Cp_\alpha(t) = (f(t), g(t))$ where $f(t) = p_1 + \frac{(x(t)-p_1)x'(t)+(y(t)-p_2)y'(t)}{x'(t)^2+y'(t)^2}x'(t)$ and $g(t) = p_2 + \frac{(x(t)-p_1)x'(t)+(y(t)-p_2)y'(t)}{x'(t)^2+y'(t)^2}y'(t)$, [7].

plotted, from top to down respectively. As seen in the figure, in the interval where Fibonacci curve is injective, whether the contrapedal curve of Fibonacci curve is injective or not depends on given point P.

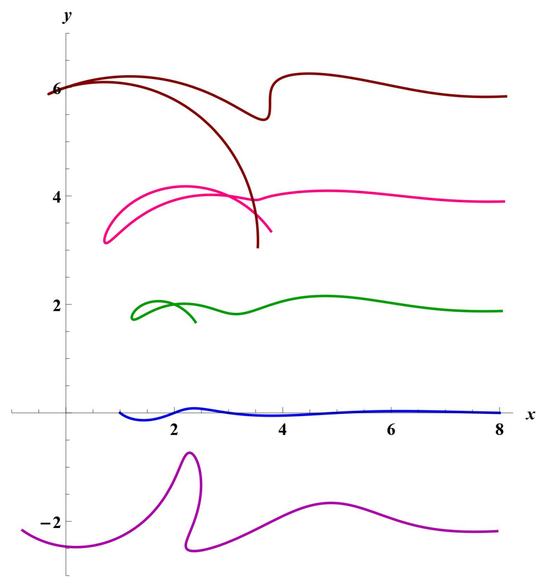


FIGURE 3. Fibonacci curve and its contrapedal curves

2.2. **The Radial Curve of Fibonacci Curve.** The parametric equation of radial curve[2] of Fibonacci curve $f(\theta)$ with respect to point $P = (p_1, p_2)$ on the plane is that

$$R_f(\theta) = (R_1(\theta), R_2(\theta)), \tag{2.14}$$

where

$$R_1(\theta) = p_1 + \frac{\alpha^{-\theta}(\pi\cos(\theta\pi) - s\sin(\theta\pi))\left(s^2\left(\alpha^{4\theta}+1\right) + 2s\alpha^{2\theta}z_\theta + \pi^2\right)}{\sqrt{5}\left(s\alpha^{2\theta}\left((\pi^2 - 2s^2)\sin(\theta\pi) + 3\pi s\cos(\theta\pi)\right) + \pi\left(s^2+\pi^2\right)\right)}$$

and

$$R_2(\theta) = p_2 + \frac{\alpha^{-\theta}\left(s\alpha^{2\theta} + s\cos(\theta\pi) + \pi\sin(\theta\pi)\right)\left(s^2\left(\alpha^{4\theta}+1\right) + 2s\alpha^{2\theta}z_\theta + \pi^2\right)}{\sqrt{5}\left(s\alpha^{2\theta}\left((\pi^2 - 2s^2)\sin(\theta\pi) + 3\pi s\cos(\theta\pi)\right) + \pi\left(s^2+\pi^2\right)\right)}$$

including

$$z_\theta = \pi\sin(\theta\pi) + s\cos(\theta\pi).$$

From the equation (2.14), we can see that point P plays a role in just the translation of the created shape. In Figure 4, Fibonacci curve which is represented by blue curve and, from

[2]Let $\alpha(t) = (x(t), y(t))$ be a regular plane curve on $\mathbb{R}^2$. Suppose that lines are drawn from a fixed point $P = (p_1, p_2) \in \mathbb{R}^2$ such that these lines are equal and parallel to the radii of curvature of $\alpha(t)$. The locus of the end points is ***radial curve*** and the equation of radial curve is that $R_\alpha(t) = (f(t), g(t))$ where $f(t) = p_1 - \frac{y'(t)\left((x'(t))^2+(y'(t))^2\right)}{x'(t)y''(t)-x''(t)y'(t)}$ and $g(t) = p_2 + \frac{x'(t)\left((x'(t))^2+(y'(t))^2\right)}{x'(t)y''(t)-x''(t)y'(t)}$, [11].

left to right respectively, the $R_f(\theta)$ radial curves with respect to $(3,1)$ and $(6,1)$ points are plotted by restricting $x-$axis to $(-1,8)$ interval and $y-$axis to $(-1,3)$ interval. The figure indicates that the radial curve of Fibonacci curve is not injective.

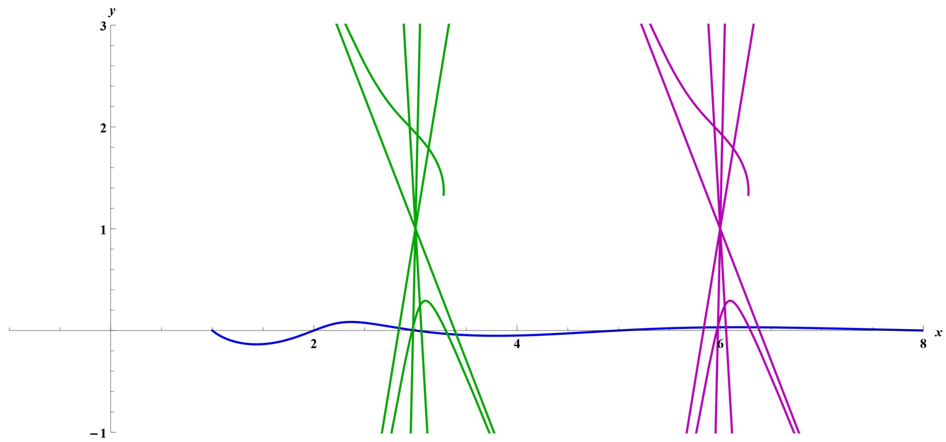


FIGURE 4. Fibonacci curve and its radial curves

2.3. **The Inverse Curve of Fibonacci Curve.** The parametric equation of inverse curve[3] of Fibonacci curve $f(\theta)$ with respect to point $R=(r_1,r_2)$ and value k is that

$$In_f(\theta)=\left(I_1(\theta),I_2(\theta)\right), \tag{2.15}$$

where

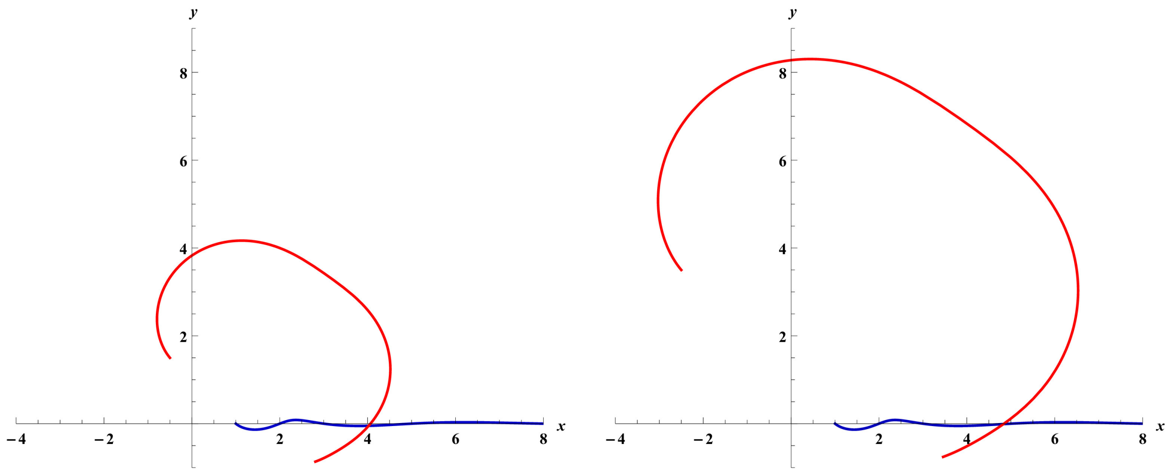
$$I_1(\theta)=r_1+k\frac{\sqrt{5}\left(\alpha^{\theta}-\alpha^{-\theta}\cos(\theta\pi)-\sqrt{5}r_1\right)}{\left(\alpha^{\theta}-\alpha^{-\theta}\cos(\theta\pi)-\sqrt{5}r_1\right)^2+\left(\alpha^{-\theta}\sin(\theta\pi)+\sqrt{5}r_2\right)^2}$$

and

$$I_2(\theta)=r_2-k\frac{\sqrt{5}\left(\alpha^{-\theta}\sin(\theta\pi)+\sqrt{5}r_2\right)}{\left(\alpha^{\theta}-\alpha^{-\theta}\cos(\theta\pi)-\sqrt{5}r_1\right)^2+\left(\alpha^{-\theta}\sin(\theta\pi)+\sqrt{5}r_2\right)^2}.$$

The equation (2.15) demonstrates that if the point R is kept constant, the value $k>0$ has a role in changing the size of the shape. The more we increase the value k, the more the figure enlarges by preserving its basic form. In contrast, the more we decrease the value k, the more the size of the shape is dwindled by preserving its basic form. That is, the value k is the radial ratio. In Figure 5, Fibonacci curve which is represented by blue curve and its inverse curves $In_f(\theta)$ with $k=5$ and $k=9$ with respect to the point $(2,-1)$ are plotted.

[3]Let $\alpha(t)=(x(t),y(t))$ be a regular plane curve and $R=(r_1,r_2)$ be a fixed point on $\mathbb{R}^2$. Suppose that a line L is drawn through R by intersecting α at P, and let Q be a point on L so that $|RP|.|RQ|=k$, a constant. Then, P and Q are inverse points, and the locus of Q is an ***inverse*** of α with respect to R. k may be negative, in which case P and Q lie on opposite sides of R. The parametric equation of inverse curve of α is that $In_\alpha(t)=(f(t),g(t))$ where $f(t)=r_1+k\frac{x(t)-r_1}{(x(t)-r_1)^2+(y(t)-r_2)^2}$ and $g(t)=r_2+k\frac{y(t)-r_2}{(x(t)-r_1)^2+(y(t)-r_2)^2}$, [11].



(A) when $R=(2,-1)$ and $k=5$ (B) when $R=(2,-1)$ and $k=9$

FIGURE 5. Fibonacci curve and its inverse curves

Moreover, if one keeps the point R constant and gets the negative of the value k, then the shape is rotated around the point R at a rotation of $180°$.

Firstly, we start to make R become the origin. So, $(I_1', I_2') = (I_1, I_2) - (r_1, r_2) = (I_1 - r_1, I_2 - r_2)$ then we get that

$$I_1' = k\frac{\sqrt{5}\left(\alpha^{\theta} - \alpha^{-\theta}\cos(\theta\pi) - \sqrt{5}r_1\right)}{\left(\alpha^{\theta} - \alpha^{-\theta}\cos(\theta\pi) - \sqrt{5}r_1\right)^2 + \left(\alpha^{-\theta}\sin(\theta\pi) + \sqrt{5}r_2\right)^2},$$
$$I_2' = -k\frac{\sqrt{5}\left(\alpha^{-\theta}\sin(\theta\pi) + \sqrt{5}r_2\right)}{\left(\alpha^{\theta} - \alpha^{-\theta}\cos(\theta\pi) - \sqrt{5}r_1\right)^2 + \left(\alpha^{-\theta}\sin(\theta\pi) + \sqrt{5}r_2\right)^2}.$$

We know that to rotate a point $180°$ counterclockwise about the origin, we need to multiply the $x-$ and $y-$coordinates by -1 i.e. $(x, y) \to (-x, -y)$. Therefore, we get that

$$I_1'' = -k\frac{\sqrt{5}\left(\alpha^{\theta} - \alpha^{-\theta}\cos(\theta\pi) - \sqrt{5}r_1\right)}{\left(\alpha^{\theta} - \alpha^{-\theta}\cos(\theta\pi) - \sqrt{5}r_1\right)^2 + \left(\alpha^{-\theta}\sin(\theta\pi) + \sqrt{5}r_2\right)^2},$$
$$I_2'' = k\frac{\sqrt{5}\left(\alpha^{-\theta}\sin(\theta\pi) + \sqrt{5}r_2\right)}{\left(\alpha^{\theta} - \alpha^{-\theta}\cos(\theta\pi) - \sqrt{5}r_1\right)^2 + \left(\alpha^{-\theta}\sin(\theta\pi) + \sqrt{5}r_2\right)^2}.$$

Finally, we make the point R center again. So, $(I_1''', I_2''') = (I_1'', I_2'') + (r_1, r_2) = (I_1' + r_1, I_2' + r_2)$ then we get that

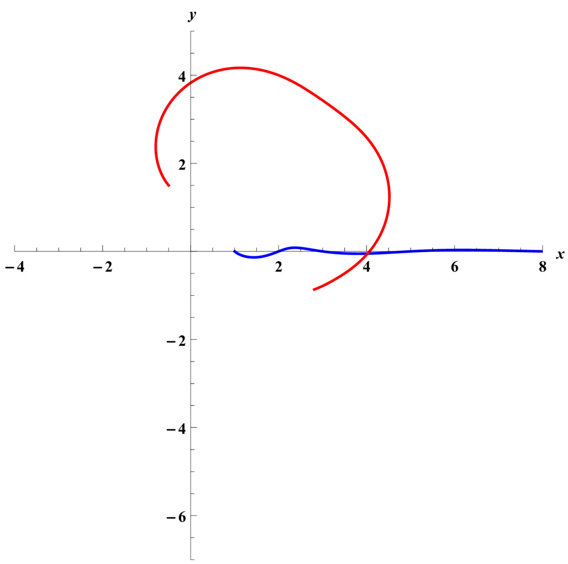
$$\begin{aligned} I_1''' &= r_1 - k\frac{\sqrt{5}\left(\alpha^{\theta} - \alpha^{-\theta}\cos(\theta\pi) - \sqrt{5}r_1\right)}{\left(\alpha^{\theta} - \alpha^{-\theta}\cos(\theta\pi) - \sqrt{5}r_1\right)^2 + \left(\alpha^{-\theta}\sin(\theta\pi) + \sqrt{5}r_2\right)^2}, \\ I_2''' &= r_2 + k\frac{\sqrt{5}\left(\alpha^{-\theta}\sin(\theta\pi) + \sqrt{5}r_2\right)}{\left(\alpha^{\theta} - \alpha^{-\theta}\cos(\theta\pi) - \sqrt{5}r_1\right)^2 + \left(\alpha^{-\theta}\sin(\theta\pi) + \sqrt{5}r_2\right)^2}. \end{aligned} \tag{2.16}$$

In addition, if we write $-k$ instead of k in the equation (2.15), then we obtain that

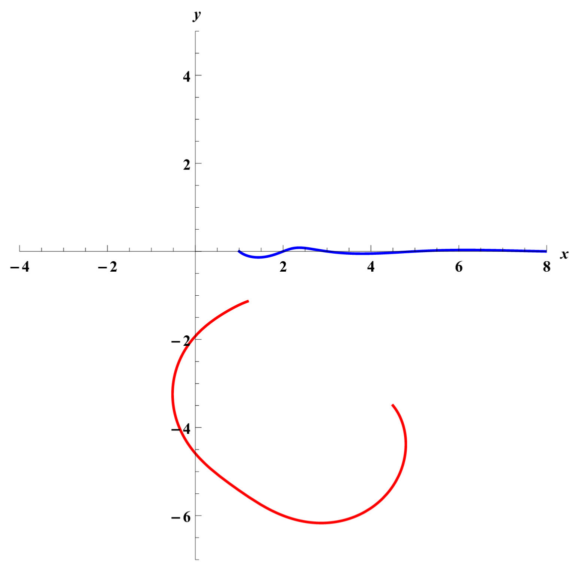
$$\begin{aligned} I_1(\theta) &= r_1 - k\frac{\sqrt{5}\left(\alpha^{\theta}-\alpha^{-\theta}\cos(\theta\pi)-\sqrt{5}r_1\right)}{\left(\alpha^{\theta}-\alpha^{-\theta}\cos(\theta\pi)-\sqrt{5}r_1\right)^2+\left(\alpha^{-\theta}\sin(\theta\pi)+\sqrt{5}r_2\right)^2}, \\ I_2(\theta) &= r_2 + k\frac{\sqrt{5}\left(\alpha^{-\theta}\sin(\theta\pi)+\sqrt{5}r_2\right)}{\left(\alpha^{\theta}-\alpha^{-\theta}\cos(\theta\pi)-\sqrt{5}r_1\right)^2+\left(\alpha^{-\theta}\sin(\theta\pi)+\sqrt{5}r_2\right)^2}. \end{aligned} \tag{2.17}$$

Consequently, from the equations (2.16) and (2.17), we see that the statement is true.

In Figure 6, Fibonacci curve which is represented by blue curve and its inverse curves $In_f(\theta)$ with $k = 5$ and $k = -5$ with respect to the point $(2, -1)$ are plotted.



(A) when $R = (2, -1)$ and $k = 5$



(B) when $R = (2, -1)$ and $k = -5$

FIGURE 6. Fibonacci curve and its inverse curve with negative value k

2.4. **The Conchoid Curve of Fibonacci Curve.** The parametric equation of conchoid curve[4] of Fibonacci curve $f(\theta)$ with respect to point $R = (r_1, r_2)$ and value k is that

$$C_f(\theta) = (c_1(\theta), c_2(\theta)), \tag{2.18}$$

where

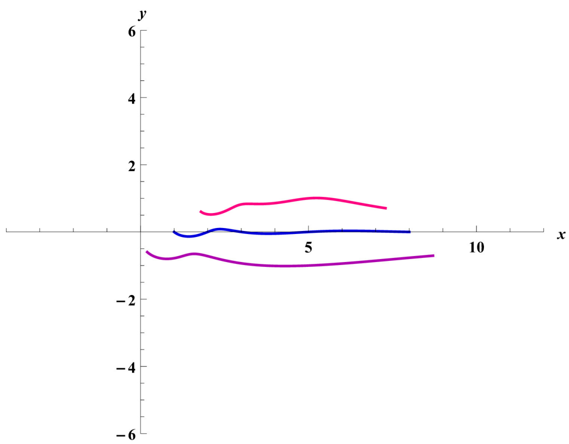
$$c_1(\theta) = \frac{\alpha^{\theta}-\alpha^{-\theta}\cos(\theta\pi)}{\sqrt{5}} \pm k\frac{\left(\alpha^{\theta}-\alpha^{-\theta}\cos(\theta\pi)-\sqrt{5}r_1\right)}{\sqrt{\left(\alpha^{\theta}-\alpha^{-\theta}\cos(\theta\pi)-\sqrt{5}r_1\right)^2+\left(\alpha^{-\theta}\sin(\theta\pi)+\sqrt{5}r_2\right)^2}}$$

[4]Let $\alpha(t) = (x(t), y(t))$ be a regular plane curve and $R = (r_1, r_2)$ be fixed point on $\mathbb{R}^2$. Suppose that a line L is drawn through R by intersecting α at Q. The locus of points P_1 and P_2 on L such that $|P_1Q| = |QP_2| = k$, a constant is the ***conchoid curve*** of α with respect to $R = (r_1, r_2)$. The parametric equation of conchoid curve of α is $C_\alpha(t) = (f(t), g(t))$ where $f(t) = x(t) \pm k\frac{x(t)-r_1}{\sqrt{(x(t)-r_1)^2+(y(t)-r_2)^2}}$ and $g(t) = y(t) \pm k\frac{y(t)-r_2}{\sqrt{(x(t)-r_1)^2+(y(t)-r_2)^2}}$, [11].

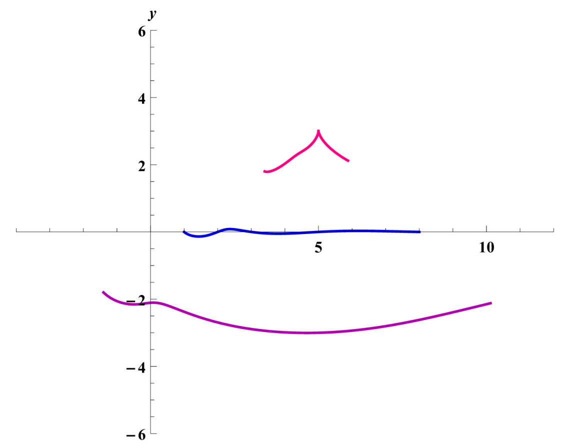
and

$$c_2(\theta) = \frac{-\alpha^{-\theta}\sin(\theta\pi)}{\sqrt{5}} \mp k \frac{\left(\alpha^{-\theta}\sin(\theta\pi) + \sqrt{5}r_2\right)}{\sqrt{\left(\alpha^{\theta} - \alpha^{-\theta}\cos(\theta\pi) - \sqrt{5}r_1\right)^2 + \left(\alpha^{-\theta}\sin(\theta\pi) + \sqrt{5}r_2\right)^2}}.$$

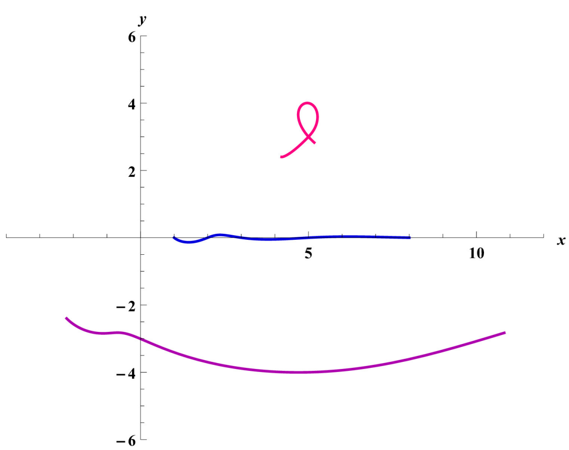
In Figure 7, Fibonacci curve and its conchoid curves $C_f(\theta)$ with respect to different values k and the point $(5,3)$ are plotted. The blue, purple and pink curves in the figure, respectively, represent Fibonacci curve, the locus of P_1 and the locus of P_2. As it is seen in this figure, if we fix the point R, whether its conchoid curve is injective or not depends on the value k in the interval which Fibonacci curve is injective.



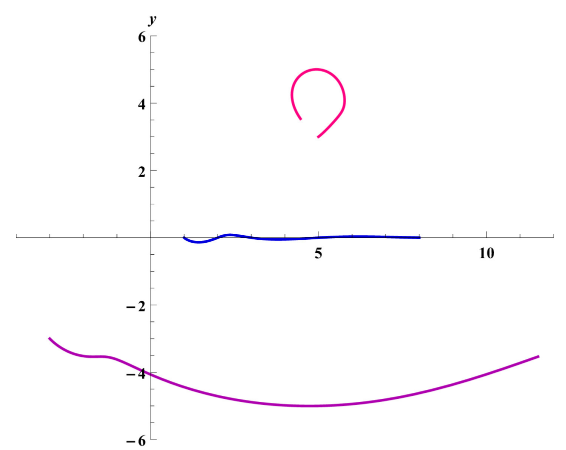
(A) when $R = (5,3)$ and $k = 1$



(B) when $R = (5,3)$ and $k = 3$



(C) when $R = (5,3)$ and $k = 4$



(D) when $R = (5,3)$ and $k = 5$

FIGURE 7. Fibonacci curve and its conchoid curves

2.5. **The Strophoid Curve of Fibonacci Curve.** The parametric equation of strophoid curve[5] of Fibonacci curve $f(\theta)$ with respect to points $R=(r_1,r_2)$ and $A=(a_1,a_2)$ is that

$$S_f(\theta)=\left(s_1(\theta),s_2(\theta)\right), \tag{2.19}$$

where

$$s_1(\theta)=\frac{\alpha^{\theta}-\alpha^{-\theta}\cos(\theta\pi)}{\sqrt{5}}\pm\frac{\left(\alpha^{\theta}-\alpha^{-\theta}\cos(\theta\pi)-\sqrt{5}r_1\right)\omega_\theta}{\sqrt{\left(\alpha^{\theta}-\alpha^{-\theta}\cos(\theta\pi)-\sqrt{5}r_1\right)^2+\left(-\alpha^{-\theta}\sin(\theta\pi)-\sqrt{5}r_2\right)^2}}$$

and

$$s_2(\theta)=-\frac{\alpha^{-\theta}\sin(\theta\pi)}{\sqrt{5}}\pm\frac{\left(-\alpha^{-\theta}\sin(\theta\pi)-\sqrt{5}r_2\right)\omega_\theta}{\sqrt{\left(\alpha^{\theta}-\alpha^{-\theta}\cos(\theta\pi)-\sqrt{5}r_1\right)^2+\left(-\alpha^{-\theta}\sin(\theta\pi)-\sqrt{5}r_2\right)^2}}$$

including

$$\omega_\theta=\frac{1}{\sqrt{5}}\sqrt{\left(\sqrt{5}a_1-\alpha^{\theta}+\alpha^{-\theta}\cos(\theta\pi)\right)^2+\left(\sqrt{5}a_2+\alpha^{-\theta}\sin(\theta\pi)\right)^2}.$$

In Figure 8, Fibonacci curve and its strophoid curves $S_f(\theta)$ with respect to $R=(4,1)$ and $A=(-1,-1)$ are plotted. The blue, purple and pink curves, respectively, in the figure represent Fibonacci curve, the locus of P_1 and the locus of P_2.

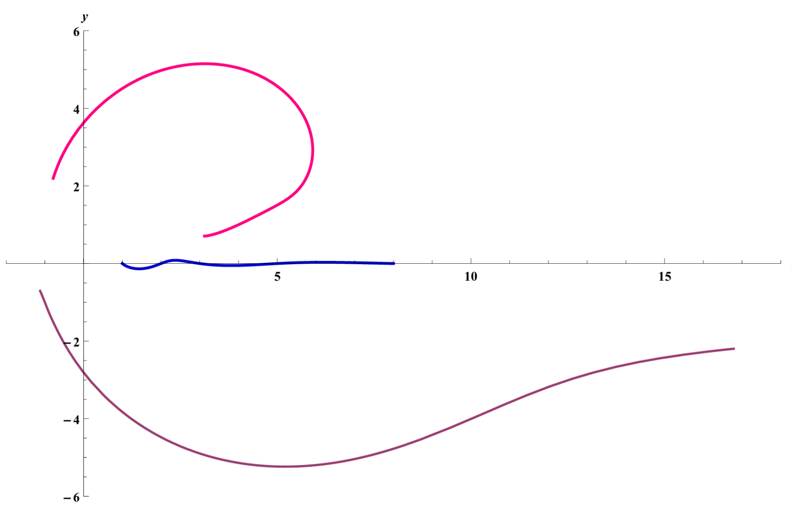


FIGURE 8. Fibonacci curve and its strophoid curve when $R=(4,1)$ and $A=(-1,-1)$

[5]Let $\alpha(t)=(x(t),y(t))$ be a regular plane curve and $R=(r_1,r_2)$ and $A=(a_1,a_2)$ be two fixed points on $\mathbb{R}^2$. Here, the point R is called the pole point. The locus of points P_1 and P_2 on a line L through R and intersecting α at a point Q such that $|P_2Q|=|QP_1|=|QA|$ is the ***strophoid curve*** of α with respect to R and A. The parametric equation of strophoid curve of α is $S_\alpha(t)=(f(t),g(t))$ where $f(t)=x(t)\pm\frac{1}{\sqrt{1+m^2}}\left[(a_1-x(t))^2+(a_2-y(t))^2\right]^{1/2}$ and $g(t)=y(t)\pm\frac{m}{\sqrt{1+m^2}}\left[(a_1-x(t))^2+(a_2-y(t))^2\right]^{1/2}$ included $m=\frac{y(t)-r_2}{x(t)-r_1}$, [11].

3. The Special Curves of Lucas Curves

In this section, we will find the equations of special plane curves of Lucas curve by using equations (1.9), (1.10), (1.11) and (1.12) and give their graphs.

3.1. The Contrapedal Curve of Lucas Curve. The parametric equation of contrapedal curve of Lucas curve $l(\theta)$ with respect to point $P=(p_1,p_2)$ on the plane is that

$$Cp_l(\theta)=(A(\theta),B(\theta)), \tag{3.20}$$

where

$$A(\theta)=p_1-\frac{\alpha^{-\theta}\left(s\alpha^{2\theta}-s\cos(\theta\pi)-\pi\sin(\theta\pi)\right)\left(s\left(1-\alpha^{4\theta}+p_1\alpha^{3\theta}\right)+\alpha^{\theta}v_\theta\right)}{s^2\left(\alpha^{4\theta}+1\right)+\pi^2-2s\alpha^{2\theta}(\pi\sin(\theta\pi)+s\cos(\theta\pi))}$$

and

$$B(\theta)=p_2-\frac{\alpha^{-\theta}(\pi\cos(\theta\pi)-s\sin(\theta\pi))\left(s\left(1-\alpha^{4\theta}+p_1\alpha^{3\theta}\right)+\alpha^{\theta}v_\theta\right)}{s^2\left(\alpha^{4\theta}+1\right)+\pi^2-2s\alpha^{2\theta}(\pi\sin(\theta\pi)+s\cos(\theta\pi))}$$

including

$$v_\theta=\pi\alpha^{\theta}\sin(\theta\pi)+(\pi p_2-sp_1)\cos(\theta\pi)-(\pi p_1+sp_2)\sin(\theta\pi).$$

In Figure 9, Lucas curve which is represented by blue curve and its contrapedal curves $Cp_l(\theta)$ with respect to $(4,3)$ and $(1,-3)$ are plotted, from top to down respectively. As it can be seen in the figure, whether the contrapedal curve of Lucas curve is injective depends on point P in the interval where Lucas curve is injective.

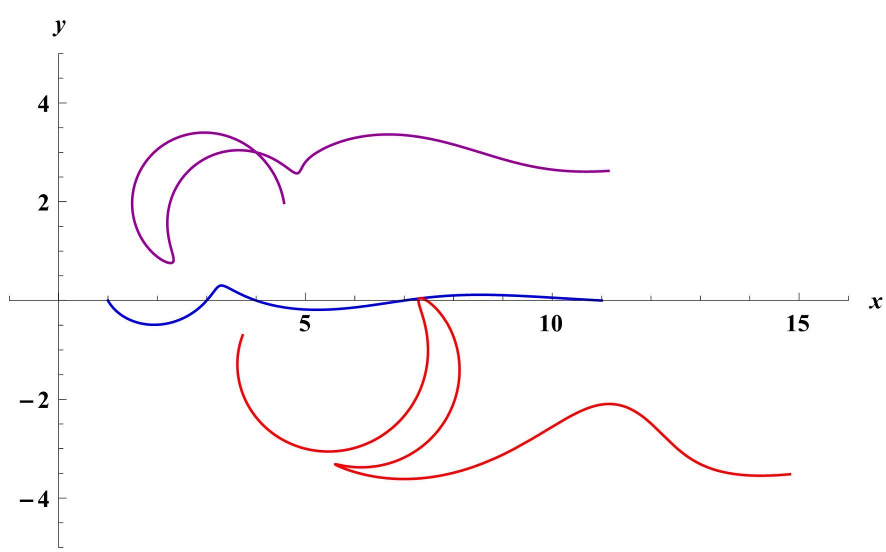


Figure 9. Lucas curve and its contrapedal curves

3.2. **The Radial Curve of Lucas Curve.** The parametric equation of radial curve of Lucas curve $l(\theta)$ with respect to point $P = (p_1, p_2)$ is that

$$R_l(\theta) = (R_1(\theta), R_2(\theta))\,, \tag{3.21}$$

where

$$R_1(\theta) = p_1 - \frac{\alpha^{-\theta}(\pi\cos(\theta\pi) - s\sin(\theta\pi))\left(s^2\left(\alpha^{4\theta}+1\right) - 2s\alpha^{2\theta}z_\theta + \pi^2\right)}{\pi\left(s^2+\pi^2\right) - s\alpha^{2\theta}\left((\pi^2-2s^2)\sin(\theta\pi) + 3\pi s\cos(\theta\pi)\right)}$$

and

$$R_2(\theta) = p_2 + \frac{\alpha^{-\theta}\left(s\alpha^{2\theta} - s\cos(\theta\pi) - \pi\sin(\theta\pi)\right)\left(s^2\left(\alpha^{4\theta}+1\right) - 2s\alpha^{2\theta}z_\theta + \pi^2\right)}{\pi(s^2+\pi^2) - s\alpha^{2\theta}\left(3\pi s\cos(\theta\pi) + (\pi^2-2s^2)\sin(\theta\pi)\right)}$$

including

$$z_\theta = \pi\sin(\theta\pi) + s\cos(\theta\pi).$$

It can be understood from the equation (3.21) that point P plays a role in the translation of the shape created by radial curve. In Figure 10, Lucas curve which is represented by blue curve and its radial curves $R_l(\theta)$, from left to right respectively, at $(-1, 2)$ and $(6, 2)$ points have been plotted by restricting $x-$axis to $(-5, 11)$ interval and $y-$axis to $(-10, 10)$ interval. The figure indicates that the radial curve of Lucas curve is not injective.

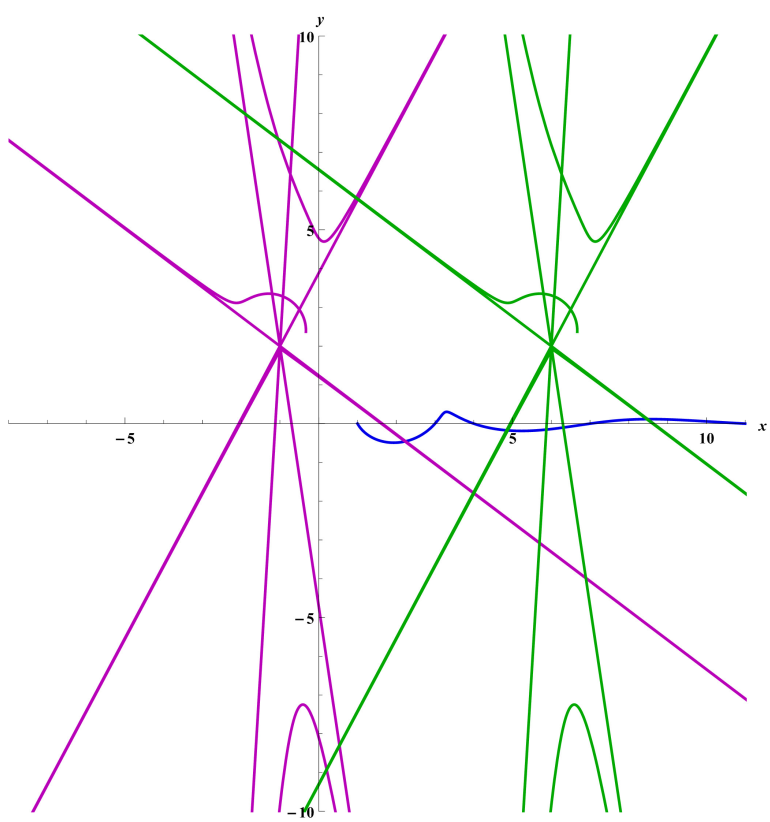


FIGURE 10. Lucas curve and its radial curves

3.3. The Inverse Curve of Lucas Curve. The parametric equation of inverse curve of Lucas curve $l(\theta)$ with respect to point $R=(r_1,r_2)$ and value k is that

$$In_l(\theta)=\left(I_1(\theta),I_2(\theta)\right), \tag{3.22}$$

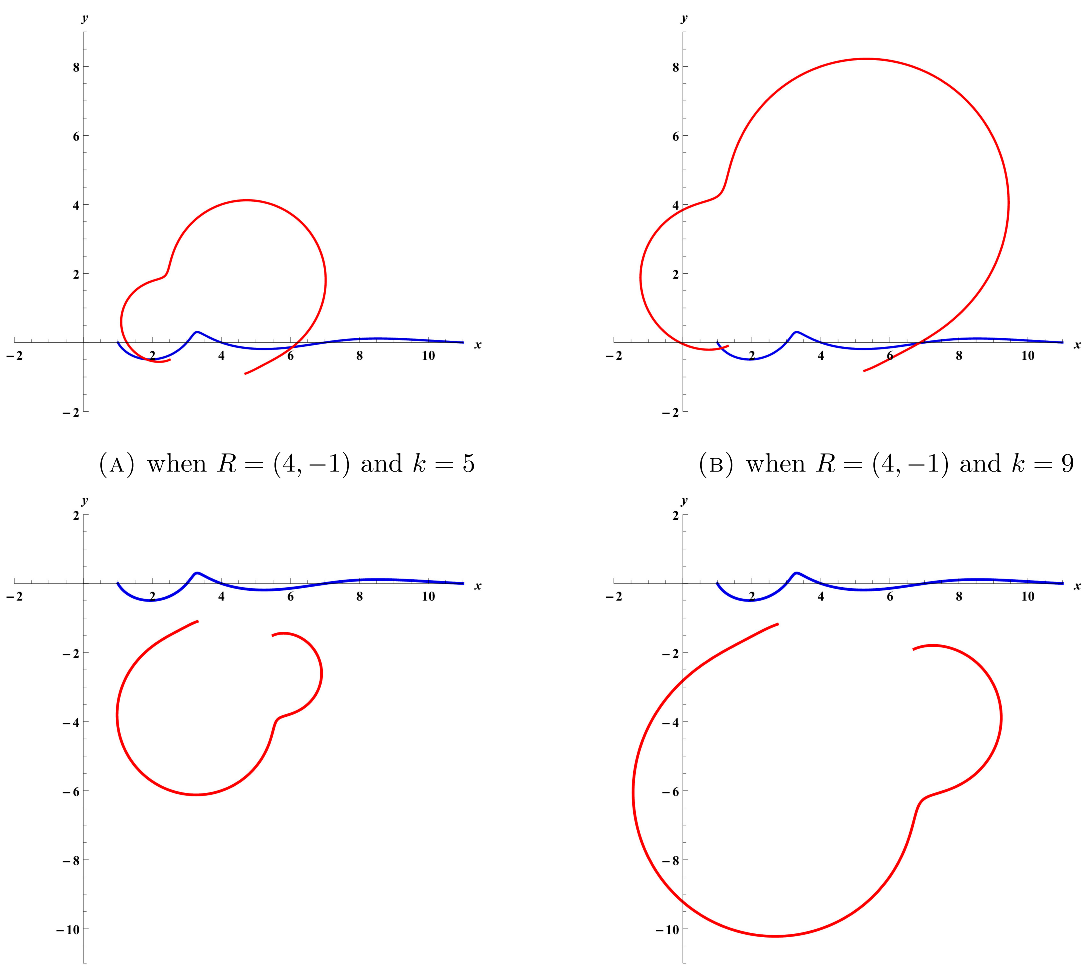
where

$$I_1(\theta)=r_1+k\frac{\alpha^{\theta}+\alpha^{-\theta}\cos(\theta\pi)-r_1}{\left(\alpha^{\theta}+\alpha^{-\theta}\cos(\theta\pi)-r_1\right)^2+\left(\alpha^{-\theta}\sin(\theta\pi)-r_2\right)^2}$$

and

$$I_2(\theta)=r_2+k\frac{\alpha^{-\theta}\sin(\theta\pi)-r_2}{\left(\alpha^{\theta}+\alpha^{-\theta}\cos(\theta\pi)-r_1\right)^2+\left(\alpha^{-\theta}\sin(\theta\pi)-r_2\right)^2}.$$

Results obtained by investigating the special cases of value k for the inverse curve of Fibonacci curve are also valid for the inverse curve of Lucas curve. In Figure 11, Lucas curve which is represented by blue curve and its inverse curves $In_l(\theta)$ for $k=-5$, $k=5$, $k=-9$, and $k=9$ with respect to the point $(4,-1)$ are plotted.



(A) when $R=(4,-1)$ and $k=5$

(B) when $R=(4,-1)$ and $k=9$

(C) when $R=(4,-1)$ and $k=-5$

(D) when $R=(4,-1)$ and $k=-9$

FIGURE 11. Lucas curve and its inverse curves

3.4. **The Conchoid Curve of Lucas Curve.** The parametric equation of conchoid curve of Lucas curve $l(\theta)$ with respect to point $R = (r_1, r_2)$ and value k is that

$$C_l(\theta) = (c_1(\theta), c_2(\theta)), \tag{3.23}$$

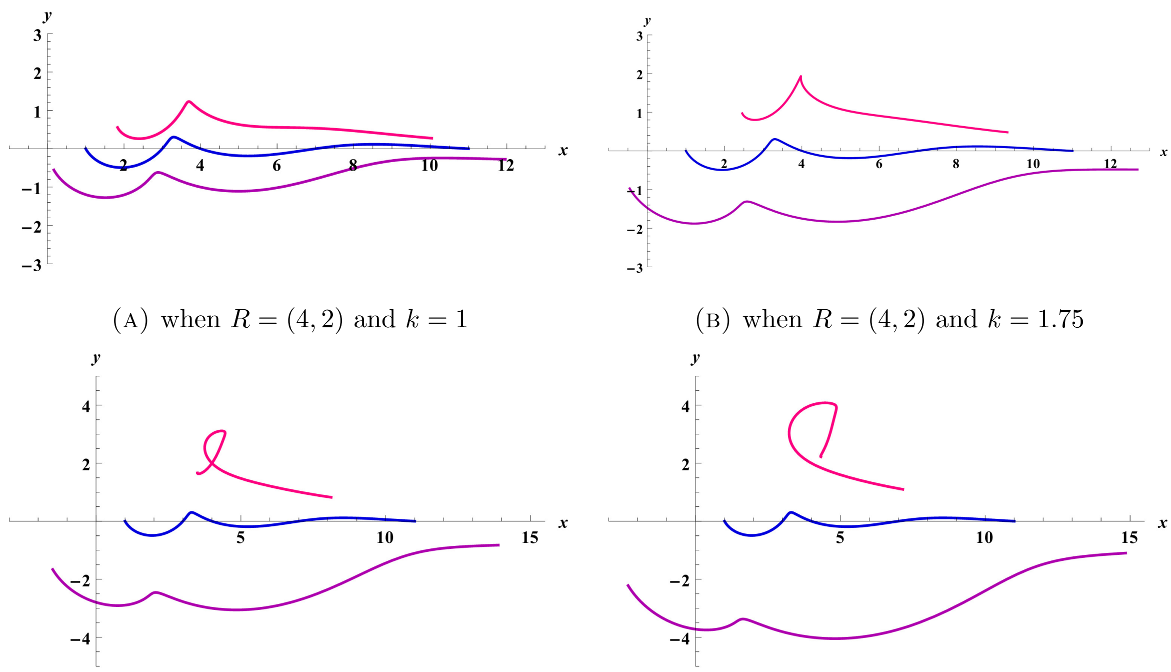
where

$$c_1(\theta) = \alpha^\theta + \alpha^{-\theta}\cos(\theta\pi) \pm k \frac{\alpha^\theta + \alpha^{-\theta}\cos(\theta\pi) - r_1}{\sqrt{\left(\alpha^\theta + \alpha^{-\theta}\cos(\theta\pi) - r_1\right)^2 + \left(\alpha^{-\theta}\sin(\theta\pi) - r_2\right)^2}}$$

and

$$c_2(\theta) = \alpha^{-\theta}\sin(\theta\pi) \pm k \frac{\alpha^{-\theta}\sin(\theta\pi) - r_2}{\sqrt{\left(\alpha^\theta + \alpha^{-\theta}\cos(\theta\pi) - r_1\right)^2 + \left(\alpha^{-\theta}\sin(\theta\pi) - r_2\right)^2}}.$$

In Figure 12, Lucas curve and its conchoid curves $C_l(\theta)$ with respect to different values k and the point $(4, 2)$ are plotted. The blue, purple and pink curves represent Lucas curve, the locus of P_1 and the locus of P_2, respectively, in the figure. As it is seen in this figure, whether its conchoid curve is injective depends on value k in the interval where Lucas curve is injective.



(A) when $R = (4, 2)$ and $k = 1$

(B) when $R = (4, 2)$ and $k = 1.75$

(C) when $R = (4, 2)$ and $k = 3$

(D) when $R = (4, 2)$ and $k = 4$

FIGURE 12. Lucas curve and its conchoid curves

3.5. **The Strophoid Curve of Lucas Curve.** The parametric equation of strophoid curve of Lucas curve $l(\theta)$ with respect to points $R=(r_1,r_2)$ and $A=(a_1,a_2)$ is that

$$S_l(\theta)=(s_1(\theta),s_2(\theta))\,, \tag{3.24}$$

where

$$s_1(\theta)=\alpha^\theta+\alpha^{-\theta}\cos(\theta\pi)\pm\frac{\left(\alpha^\theta+\alpha^{-\theta}\cos(\theta\pi)-r_1\right)\omega_\theta}{\sqrt{\left(\alpha^\theta+\alpha^{-\theta}\cos(\theta\pi)-r_1\right)^2+\left(\alpha^{-\theta}\sin(\theta\pi)-r_2\right)^2}}$$

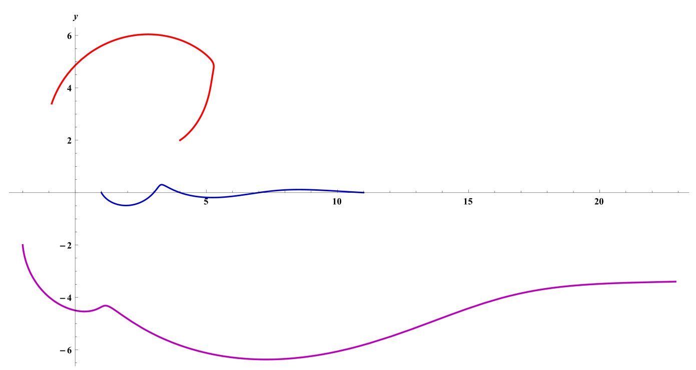
and

$$s_2(\theta)=\alpha^{-\theta}\sin(\theta\pi)\pm\frac{\left(\alpha^{-\theta}\sin(\theta\pi)-r_2\right)\omega_\theta}{\sqrt{\left(\alpha^\theta+\alpha^{-\theta}\cos(\theta\pi)-r_1\right)^2+\left(\alpha^{-\theta}\sin(\theta\pi)-r_2\right)^2}}$$

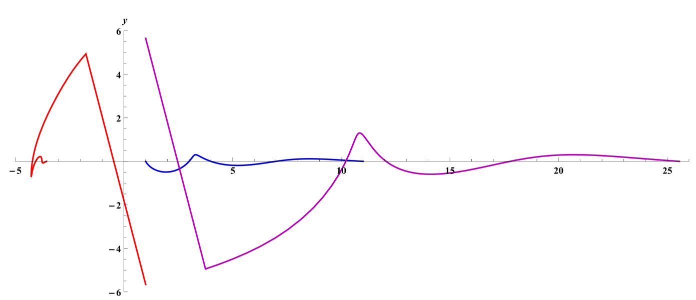
including

$$\omega_\theta=\sqrt{\left(a_1-\alpha^\theta-\alpha^{-\theta}\cos(\theta\pi)\right)^2+\left(a_2-\alpha^{-\theta}\sin(\theta\pi)\right)^2}.$$

In Figure 13, Lucas curve and its strophoid curves $S_l(\theta)$ with respect to different points R and A are plotted. The blue, purple and red curves represent Lucas curve, the locus of P_1 and the locus of P_2, respectively, in the figure. As it is seen in this figure, whether its strophoid curve has a critical point depends on A and R in the interval where Lucas curve has not any critical point.



(A) when $R=(4,2)$ and $A=(-1,3)$



(B) when $R=(1,0)$ and $A=(-3,-4)$

FIGURE 13. Lucas curve and its strophoid curves

4. Conclusion

In this study, firstly the notions of contrapedal, radial, inverse, conchoid and strophoid curves of the Fibonacci and Lucas curves have been investigated. Afterwards, their graphs which have been plotted by using Mathematica are examined in the interval $I = (2, 6)$ for Fibonacci curve and in the interval $I = (1, 5)$ for Lucas curve.

We have obtained some results from the notions and figures which is acquired.

- As illustrated in Figure 3 and Figure 9, if their contrapedal curves are injective or not depends on given point P in the intervals where Fibonacci and Lucas curves are injective.
- From equations (2.14) and (3.21), it is clear that the point P has a role in the translation of the figure which is created. Figure 4 and Figure 10 illustrate that their radial curves are not injective.
- The equations (2.15) and (3.22) reveals that if one fixes the point R, the value $k > 0$ has a role in changing the size of inverse curves which belongs to Fibonacci and Lucas curves. As the value k increases, the size of the shape enlarges by preserving the main form. Conversely, as the value k decreases, the size becomes smaller by preserving the main form. Moreover, if one keeps the point R constant and gets the negative of the value k, then the shape is rotated around the point R at a rotation of $180°$.
- From Figure 7 and Figure 12, it can be seen that in the interval where Fibonacci and Lucas curves are injective, if one fixes the point R, the value k is an important factor in the injectivity of their conchoid curves.
- It can be observed from Figure 13 that in the interval where Lucas curve has not any critical point whether its strophoid curve has at least one critical point or not depends on the given points R and A.

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