**International Journal of Maps in Mathematics**

*Volume* ***5****, Issue* ***2****, 2022, Pages:****154-181***

ISSN: 2636-7467 (Online)

www.journalmim.com

# BIWARPED PRODUCT SUBMANIFOLDS WITH A SLANT BASE FACTOR

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Abstract. We study biwarped product submanifolds with a slant base factor in locally product Riemannian manifolds. We prove an existence theorem for such submanifolds. Then we give a necessary and sufficient condition for such a manifold to be a warped product. We establish a general inequality for such submanifolds. The equality case is also considered. Moreover, we give an application of this inequality.

**Keywords**: Biwarped product submanifold, Slant distribution, Invariant distribution, Anti-invariant distribution, Locally product Riemannian manifold

**2010 Mathematics Subject Classification**: Primary 53C15, Secondary 53B20.

## 1. Introduction

Let $(M_i, g_i)$ be Riemannian manifolds for $i \in \{0, 1, 2\}$ and let $f_{1,2} : M_0 \to (0, \infty)$ be smooth functions. Then the *biwarped product or twice warped product manifold* [5, 14] $M_0 \times_{f_1} M_1 \times_{f_2} M_2$ is the product manifold $\bar{M} = M_0 \times M_1 \times M_2$ endowed with the metric

$$g = \pi_0^*(g_0) \oplus (f_1 \circ \pi_0)^2 \pi_1^*(g_1) \oplus (f_2 \circ \pi_0)^2 \pi_2^*(g_2).$$

*Received:2021.12.02* *Revised:2022.03.08* *Accepted:2022.03.14*

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More precisely, for any vector fields $\bar{X}$ and $\bar{Y}$ of $\bar{M}$, we have

$$g(\bar{X},\bar{Y}) = g_0(\pi_{0_*}\bar{X},\pi_{0_*}\bar{Y}) + \sum_{i=1}^{2}(f_i\circ\pi_0)^2 g_i(\pi_{i_*}\bar{X},\pi_{i_*}\bar{Y}),$$

where $\pi_i : \bar{M} \to M_i$ is the canonical projection of $\bar{M}$ onto $M_i$, $\pi_i^*(g_i)$ is the pullback of $g_i$ by $\pi_i$ and the subscript $\pi_{i*}$ denotes the derivative map of $\pi_i$ for each $i$. The functions $f_1$ and $f_2$ are called *warping functions* and each manifold $(M_j, g_j)$, $j \in \{1,2\}$ is called a *fiber* of the biwarped product $\bar{M}$. The factor $(M_0, g_0)$ is called a *base manifold* of $\bar{M}$. As well known, the base manifold of $\bar{M}$ is totally geodesic and the fibers of $\bar{M}$ are totally umbilic in $\bar{M}$. We say that a biwarped product manifold is *trivial*, if the warping functions $f_1$ and $f_2$ are constants. Of course, biwarped product manifolds are natural generalizations of warped product manifolds [7] and special case of multiply warped product manifolds [14].

Let $M_0 \times_{f_1} M_1 \times_{f_2} M_2$ be a biwarped product manifold with the Levi-Civita connection $\bar{\nabla}$ and $\nabla^i$ denote the Levi-Civita connection of $M_i$ for $i \in \{0,1,2\}$. By usual convenience, we denote the set of lifts of vector fields on $M_i$ by $\mathcal{L}(M_i)$ and use the same notation for a vector field and for its lifts. On the other hand, since the map $\pi_0$ is an isometry and $\pi_1$ and $\pi_2$ are (positive) homotheties, they preserve the Levi-Civita connections. Thus there is no confusion using the same notation for a connection on $M_i$ and for its pullback via $\pi_i$. Then, the covariant derivative formulas [23] for a biwarped product manifold are given by

$$\bar{\nabla}_U V = \nabla^0_U V \tag{1.1}$$

$$\bar{\nabla}_V X = \bar{\nabla}_X V = V(\ln f_i)X \tag{1.2}$$

$$\bar{\nabla}_X Z = \begin{cases} 0 & if \quad i \neq j, \\ \nabla^i_X Z - g(X,Z)\nabla^0(\ln f_i) & if \quad i = j, \end{cases} \tag{1.3}$$

where $U, V \in \mathcal{L}(M_0)$, $X \in \mathcal{L}(M_i)$ and $Z \in \mathcal{L}(M_j)$.

The theory of warped product submanifolds has been become a popular research area since Chen [8] studied the warped product CR-submanifolds in Kaehler manifolds. Actually, several classes of warped product submanifolds appeared in the last eighteen years. Also, warped product submanifolds have been studied for different kinds of structures. Most of the studies related to the theory of warped product submanifolds can be found in Chen's book [10]. Recently, Taştan studied biwarped product submanifolds of a Kaehler manifold $(\bar{M}, J, g)$ of the form $M^T \times_f M^\perp \times_\sigma M^\theta$, where $M^T$ is a holomorphic, $M^\perp$ is a totally real and $M^\theta$ is a pointwise slant submanifold of $\bar{M}$ [20]. Afterwards, biwarped product submanifolds have been studying by many geometers for different kinds of structures (see, [2, 21, 22]).

In this paper, we study biwarped product submanifolds with a slant base factor in locally product Riemannian manifolds. More precisely, we consider biwarped product submanifolds of the form $M^\theta \times_f M^T \times_\sigma M^\perp$, where $M^\theta$ is a slant, $M^\perp$ is an anti-invariant and $M^T$ is an invariant submanifold of the locally product Riemannian manifold. After giving a non-trivial example and some auxiliary results, we prove an existence theorem for such submanifolds. Then, we investigate the behavior of the second fundamental form of such a submanifold and as a result, we get a condition for this kind of submanifold to be a warped product. Finally, we obtain an inequality for the squared norm of the second fundamental form in terms of the warping functions for such submanifolds. The equality case is also considered. Moreover, we give an application of this inequality for certain types of locally product Riemannian manifolds.

**Remark 1.1.** *Biwarped product submanifolds of the form $M^\theta \times_f M^T \times_\sigma M^\perp$ in locally product Riemannian manifolds were also studied in* [22]*. However, expect the first four equations of Lemma 5.1, our results are completely different from the results of* [22]*. Besides, biwarped product submanifolds of the form $M^\perp \times_f M^T \times_\sigma M^\theta$ in locally product Riemannian manifolds were studied in* [2]*, where $M^\theta$ is a proper pointwise slant submanifold of the locally product Riemannian manifold. But, the geometry of $M^\theta \times_f M^T \times_\sigma M^\perp$ and the geometry of $M^\perp \times_f M^T \times_\sigma M^\theta$ are quite different.*

## 2. Preliminaries

We first recall the fundamental definitions and notions needed for further study. In fact, we will give the notions for submanifolds of Riemannian manifolds in subsection 2.1. In subsection 2.2, we recall the definition of a locally product Riemannian manifold.

### 2.1. Riemannian submanifolds.

Let $M$ be a Riemannian manifold isometrically immersed in a Riemannian manifold $(\bar{M}, g)$ and $\bar{\nabla}$ be the Levi-Civita connection of $\bar{M}$ with respect to the metric g. Also, let $\nabla$ and $\nabla^\perp$ be the Levi-Civita connection and normal connection of $M$, respectively. Then the Gauss and Weingarten formulas [24] are given respectively by

$$\bar{\nabla}_V W = \nabla_V W + h(V, W) \qquad \textit{and} \qquad \bar{\nabla}_V Z = -A_Z V + \nabla^\perp_V Z. \tag{2.4}$$

Here $V, W$ are the tangent vector fields to $M$ and $Z$ is normal to $M$. In addition, $h$ is the second fundamental form and $A_Z$ is the Weingarten operator of $M$ associated with $Z$. Then, we have

$$g(h(V,W),Z) = g(A_Z V, W). \tag{2.5}$$

For a submanifold $M$ of a Riemannian manifold $\bar{M}$, the equation of Gauss is given by

$$\bar{R}(U,V,Z,W) = R(U,V,Z,W) + g(h(U,Z),h(V,W)) - g(h(U,W),h(V,Z)) \tag{2.6}$$

for any $U, V, Z, W \in \Gamma(TM)$, where $\bar{R}$ and $R$ are the curvature tensors on $\bar{M}$ and $M$ respectively. The mean curvature vector $H$ for an orthonormal frame $\{e_1, \ldots, e_m\}$ of tangent space $T_pM$, $p \in M$ on $M$ is defined by

$$H = \frac{1}{m} trace(h) = \frac{1}{m} \sum_{i=1}^{m} h(e_i, e_i), \tag{2.7}$$

where $m = dimM$. Also, we set

$$h_{ij}^r = g(h(e_i,e_j),e_r) \quad and \quad \| h \|^2 = \sum_{i,j=1}^{m} g(h(e_i,e_j),h(e_i,e_j)). \tag{2.8}$$

Moreover, the *sectional curvature* [24] of a plane section spanned by $e_i$ and $e_j$, denoted by $K_{ij}$, is

$$K_{ij} = R(e_i,e_j,e_j,e_i). \tag{2.9}$$

The *scalar curvature* [9] of $M$ of is given by

$$\tau(TM) = \sum_{1 \leq i \neq j \leq m} K_{ij}. \tag{2.10}$$

Let $G_r$ be a $r$-plane section on $TM$ and $\{e_1, \ldots, e_r\}$ any orthonormal basis of $G_r$. Then the scalar curvature $\tau(G_r)$ of $G_r$ is given by

$$\tau(G_r) = \sum_{1 \leq i \neq j \leq r} K_{ij}. \tag{2.11}$$

For a smooth function $f$ on $M$, the *Laplacian* of $f$ is defined by

$$\Delta f = \sum_{i=1}^{m} \{(\nabla_{e_i} e_i) f - e_i(e_i(f))\} = -\sum_{i=1}^{m} g(\nabla_{e_i} \nabla f, e_i), \tag{2.12}$$

where $\nabla f$ is the gradient of $f$ [9].

2.2. **Locally product Riemannian manifolds.** Let $\bar{M}$ be a Riemannian manifold. Suppose $\bar{M}$ is endowed with a tensor field

$$\mathcal{F}^2 = I, \qquad (\mathcal{F} \neq \mp I), \tag{2.13}$$

of type $(1,1)$. Here, $I$ is the identity endomorphism on $T\bar{M}$. Then, $(\bar{M}, g, \mathcal{F})$ called an *almost product manifold* and $\mathcal{F}$ is called an almost product structure. Also, we assume that $g$ and $\mathcal{F}$ satisfy

$$g(\mathcal{F}\bar{X}, \mathcal{F}\bar{Y}) = g(\bar{X}, \bar{Y}), \tag{2.14}$$

for all vector fields $\bar{X}, \bar{Y}$ tangent to $M$. Then, it is known that $(\bar{M}, g, \mathcal{F})$ is an *almost product Riemannian manifold*. Let $\bar{\nabla}$ be the Levi-Civita connection of $(\bar{M}, g, \mathcal{F})$. If we have

$$\bar{\nabla}\mathcal{F} \equiv 0, \tag{2.15}$$

then $(\bar{M}, g, \mathcal{F})$ is a *locally product Riemannian manifold*, (briefly, *l.p.R. manifold*).

Let $M_1(c_1)$ (resp. $M_2(c_2)$) be a real space form and have sectional curvature $c_1$ (resp. $c_2$). Then, the Riemannian curvature tensor $\bar{R}$ of l.p.R. manifold $\bar{M} = M_1 \times M_2$ has the form

$$\begin{aligned} \bar{R}(U,V)Z \;&= \frac{1}{4}(c_1 + c_2)\Big\{ g(V,Z)U - g(U,Z)V + g(\mathcal{F}V,Z)\mathcal{F}U - g(\mathcal{F}U,Z)\mathcal{F}V \Big\} \\ &= \frac{1}{4}(c_1 - c_2)\Big\{ g(V,Z)\mathcal{F}U - g(U,Z)\mathcal{F}V + g(\mathcal{F}V,Z)U - g(\mathcal{F}U,Z)V \Big\}, \end{aligned} \tag{2.16}$$

for all $U, V, Z \in \Gamma(T\bar{M})$ [24].

## 3. Skew semi-invariant submanifolds of order 1 in locally product Riemannian manifolds

We first recall the definition of the skew semi-invariant submanifolds of order 1 of a locally product Riemannian manifold and get some useful results for the further study.

Let $(\bar{M}, g, \mathcal{F})$ be a l.p.R. manifold and let $M$ be a submanifold of $\bar{M}$. If for $X \in \mathcal{D}_p$, the angle $\theta$ between $\mathcal{F}X$ and $\mathcal{D}_p$ is constant, i.e., it is independent of $p \in M$ and $X \in \mathcal{D}_p$, then $\mathcal{D}$ is called a *slant distribution* on $M$. $\theta$ is said the *slant angle* of the slant distribution $\mathcal{D}$. Thus, the invariant and anti-invariant distributions with respect to $\mathcal{F}$ are slant distributions with slant angle $\theta = 0$ and $\theta = \pi/2$, respectively. If the tangent bundle $TM$ of $M$ is slant [12, 15] then the submanifold $M$ of $\bar{M}$ is called a *slant submanifold*. A slant submanifold that is neither invariant nor anti-invariant is called a *proper* slant submanifold.

Let $M$ be a slant submanifold with slant angle $\theta$ of a locally product Riemannian manifold $(\bar{M}, g, \mathcal{F})$, for any $V \in \Gamma(TM)$, we write

$$\mathcal{F}V = PV + NV. \tag{3.17}$$

Here $PV$ is the tangential part of $\mathcal{F}V$ and $NV$ is the normal part of $\mathcal{F}V$. Then, for any $U, V \in \Gamma(TM)$ we have [15]

$$P^2V = \cos^2\theta V, \tag{3.18}$$

$$g(PU, PV) = \cos^2\theta g(U, V) \quad and \quad g(NU, NV) = \sin^2\theta g(U, V). \tag{3.19}$$

A submanifold $M$ of a locally product Riemannian manifold $(\bar{M}, g, \mathcal{F})$ is said a *skew semi-invariant submanifold of order 1* (briefly, s.s-i.) [18] if the tangent bundle $TM$ of $M$ has the form

$$TM = \mathcal{D}_\perp \oplus \mathcal{D}_T \oplus \mathcal{D}_\theta,$$

where $\mathcal{D}_\theta$ is slant distribution with slant angle $\theta$, $\mathcal{D}_T$ is an invariant distribution, i.e., $\mathcal{F}\mathcal{D}_T \subseteq \mathcal{D}_T$, $\mathcal{D}_\perp$ is an anti-invariant distribution, i.e. $\mathcal{F}\mathcal{D}_\perp \subseteq T^\perp M$. In that case, the normal bundle $T^\perp M$ of $M$ can be decomposed as

$$T^\perp M = N(\mathcal{D}_\theta) \oplus \mathcal{F}(\mathcal{D}_\perp) \oplus \bar{\mathcal{D}}_T, \tag{3.20}$$

where $\bar{\mathcal{D}}_T$ is the orthogonal complementary distribution of $N(\mathcal{D}_\theta) \oplus \mathcal{F}(\mathcal{D}_\perp)$ in $T^\perp M$ and it is an invariant subbundle of $T^\perp M$ with respect to $\mathcal{F}$.

**Remark 3.1.** *The class of s.s-i. submanifolds of order 1 of locally product Riemannian manifolds is a special subclass of skew semi-invariant submanifolds* [12] *and a natural generalization of invariant, anti-invariant* [1]*, semi-invariant* [6]*, slant* [15]*, semi-slant* [13] *and hemi-slant submanifolds* [19] *of locally product Riemannian manifolds.*

**Lemma 3.1.** [18] *Let $M$ be a proper s.s-i. submanifold of order 1 of a l.p.R. manifold $(\bar{M}, g, \mathcal{F})$. Then,*

$$g(\nabla_Z W, U) = -\csc^2\theta\Big\{g(A_{NPW}Z, U) + g(A_{NW}Z, \mathcal{F}U)\Big\}, \tag{3.21}$$

$$g(\nabla_Z W, X) = \sec^2\theta\Big\{g(A_{\mathcal{F}X}Z, PW) + g(A_{NPW}Z, X)\Big\}, \tag{3.22}$$

$$g(\nabla_U V, Z) = \csc^2\theta\Big\{g(A_{NPZ}U, V) + g(A_{NZ}U, \mathcal{F}V)\Big\}, \tag{3.23}$$

$$g(\nabla_U V, X) = g(A_{\mathcal{F}X}U, \mathcal{F}V), \tag{3.24}$$

$$g(\nabla_X Y, Z) = -\sec^2\theta\Big\{g(A_{\mathcal{F}Y}X, PZ) + g(A_{NPZ}X, Y)\Big\}, \tag{3.25}$$

$$g(\nabla_X Y, V) = -g(A_{\mathcal{F}Y}X, \mathcal{F}V), \tag{3.26}$$

$$g(\nabla_X Z, V) = -\csc^2\theta\Big\{g(A_{NPZ}X, V) + g(A_{NZ}X, \mathcal{F}V)\Big\}, \tag{3.27}$$

$$g(\nabla_Z X, V) = -g(A_{\mathcal{F}X}Z, \mathcal{F}V), \tag{3.28}$$

$$g(\nabla_U X, Z) = -\sec^2\theta\Big\{g(A_{\mathcal{F}X}U, PZ) + g(A_{NPZ}U, X)\Big\} \tag{3.29}$$

*for $Z, W \in \Gamma(\mathcal{D}_\theta)$, $U, V \in \Gamma(\mathcal{D}_T)$ and $X, Y \in \Gamma(\mathcal{D}_\perp)$.*

**Theorem 3.1.** *Let $M$ be a proper s.s-i. submanifold of order 1 of a locally product Riemannian manifold $(\bar{M}, g, \mathcal{F})$. Then the slant distribution $\mathcal{D}_\theta$ is totally geodesic iff the following equations hold*

$$g(A_{NPW}Z, V) = -g(A_{NW}Z, \mathcal{F}V), \tag{3.30}$$

$$g(A_{\mathcal{F}X}Z, PW) = -g(A_{NPW}Z, X), \tag{3.31}$$

*for $Z, W \in \Gamma(\mathcal{D}_\theta)$, $V \in \Gamma(\mathcal{D}_T)$ and $X \in \Gamma(\mathcal{D}_\perp)$.*

**Proof.** The distribution $\mathcal{D}_\theta$ is totally geodesic iff $g(\nabla_Z W, X) = 0$ and $g(\nabla_Z W, V) = 0$ for all $Z, W \in \Gamma(\mathcal{D}_\theta)$, $X \in \Gamma(\mathcal{D}_\perp)$ and $V \in \Gamma(\mathcal{D}_T)$. Thus, the assertions (3.30) and (3.31) follow from (3.21) and (3.22), respectively.

**Theorem 3.2.** *Let $M$ be a proper s.s-i. submanifold of order 1 of a locally product Riemannian manifold $(\bar{M}, g, \mathcal{F})$. Then the invariant distribution $\mathcal{D}_T$ is integrable iff the following equations hold*

$$g(A_{\mathcal{F}X}U, \mathcal{F}V) = g(A_{\mathcal{F}X}V, \mathcal{F}U), \tag{3.32}$$

$$g(A_{NPZ}U, V) + g(A_{NZ}U, \mathcal{F}V) = g(A_{NPZ}V, U) + g(A_{NZ}V, \mathcal{F}U), \tag{3.33}$$

*for $U, V \in \Gamma(\mathcal{D}_T)$, $X \in \Gamma(\mathcal{D}_\perp)$ and $Z \in \Gamma(\mathcal{D}_\theta)$.*

**Proof.** The distribution $\mathcal{D}_T$ is integrable iff $g([U,V],X)=0$ and $g([U,V],Z)=0$ for all $Z\in\Gamma(\mathcal{D}_\theta)$, $X\in\Gamma(\mathcal{D}_\perp)$ and $U,V\in\Gamma(\mathcal{D}_T)$. Thus, the assertions (3.32) and (3.33) follow from (3.23) and (3.24), respectively.

**Theorem 3.3.** *Let $M$ be a proper s.s-i. submanifold of order 1 of a locally product Riemannian manifold $(\bar{M},g,\mathcal{F})$. Then the anti-invariant distribution $\mathcal{D}_\perp$ is integrable iff the following equations hold*

$$g(A_{\mathcal{F}X}Y,\mathcal{F}V)=g(A_{\mathcal{F}Y}X,\mathcal{F}V), \tag{3.34}$$

$$g(A_{\mathcal{F}Y}X,PZ)=g(A_{\mathcal{F}X}Y,PZ), \tag{3.35}$$

*for $X,Y\in\Gamma(\mathcal{D}_\perp)$, $V\in\Gamma(\mathcal{D}_T)$ and $Z\in\Gamma(\mathcal{D}_\theta)$.*

**Proof.** The distribution $\mathcal{D}_\perp$ is integrable iff $g([X,Y],Z)=0$ and $g([X,Y],V)=0$ for all $Z\in\Gamma(\mathcal{D}_\theta)$, $X,Y\in\Gamma(\mathcal{D}_\perp)$ and $V\in\Gamma(\mathcal{D}_T)$. Thus, the assertions (3.34) and (3.35) follow from (3.25) and (3.26), respectively.

## 4. Biwarped Product Submanifolds in Locally Product Riemannian Manifolds

We first check that the existence of biwarped product submanifolds of the form, $M^T\times_f M^\perp\times_\sigma M^\theta$, $M^\perp\times_f M^\theta\times_\sigma M^T$ and $M_\theta\times_f M^T\times_\sigma M^\perp$, where $M^\perp$ is an anti-invariant, $M^\theta$ ia a proper slant and $M^T$ is an invariant submanifold of a l.p.R. manifold $(\bar{M},g,\mathcal{F})$.

M. Atçeken and B. Ṣahin independently proved that there do not exist (non-trivial) warped product semi-invariant submanifolds of the form $M^T\times_f M^\perp$ in a l.p.R. manifold $(\bar{M},g,\mathcal{F})$, such that $M^T$ is an invariant submanifold and $M^\perp$ is an anti-invariant submanifold of $(\bar{M},g,\mathcal{F})$ in [4, Theorem 3.1] and [16, Theorem 3.1], respectively. Again, M. Atçeken and B. Ṣahin independently proved that there do not exist (non-trivial) warped product semi-slant submanifolds of the form $M^T\times_f M^\theta$ in a l.p.R. manifold $\bar{M}$, such that $M^T$ is an invariant submanifold and $M^\theta$ is a proper slant submanifold of $\bar{M}$ in [3, Theorem 3.3] and [17, Theorem 3.1], respectively. Thus, we obtain the following result.

**Corollary 4.1.** *There do not exist (non-trivial) biwarped product submanifolds of the form $M^T\times_f M^\perp\times_\sigma M^\theta$ of a l.p.R. manifold $(\bar{M},g,\mathcal{F})$ such that $M^T$ is an invariant, $M^\perp$ is an anti-invariant and $M^\theta$ is a proper slant submanifold of $\bar{M}$.*

On the other hand, it was proved that there do not exist (non-trivial) warped product submanifolds of the form $M^\perp\times_f M^\theta$ in a l.p.R. manifold $\bar{M}$ such that $M^\perp$ is an anti-invariant submanifold and $M^\theta$ is a proper slant submanifold of $\bar{M}$ in [3, Theorem 3.4]. Thus, we deduce the following result.

**Corollary 4.2.** *There do not exist (non-trivial) biwarped product submanifolds of the form $M^\perp \times_f M^\theta \times_\sigma M^T$ of a l.p.R. manifold $(\bar{M}, g, \mathcal{F})$ such that $M^\perp$ is an anti-invariant, $M^\theta$ is a proper slant submanifold and $M^T$ is an invariant submanifold of $\bar{M}$.*

Now, we consider (non-trivial) biwarped product submanifolds in the form $M^\theta \times_f M^T \times_\sigma M^\perp$ in a l.p.R. manifold $(\bar{M}, g, \mathcal{F})$ such that $M^T$ is an invariant, $M^\perp$ is an anti-invariant and $M^\theta$ is a proper slant submanifold of $\bar{M}$. Firstly, we present an example of such a submanifold.

**Example 4.1.** *Consider the 8-dimensional Euclidean space $\mathbb{R}^8$ with standard metric $g$ and almost product structure $\mathcal{F}$ given by*

$$\mathcal{F}\partial_1 = \partial_1, \qquad \mathcal{F}\partial_2 = \partial_2, \qquad \mathcal{F}\partial_3 = -\partial_3, \qquad \mathcal{F}\partial_4 = -\partial_4,$$

$$\mathcal{F}\partial_5 = \partial_6, \qquad \mathcal{F}\partial_6 = \partial_5, \qquad \mathcal{F}\partial_7 = \partial_8, \qquad \mathcal{F}\partial_8 = \partial_7,$$

*where $\partial_k = \frac{\partial}{\partial x_k}$, $k \in \{1, \ldots, 8\}$ and $(x_1, x_2, \ldots, x_8)$ are natural coordinates of $\mathbb{R}^8$. Upon a straightforward calculation, we see that $(\mathbb{R}^8, \mathcal{F}, g)$ is a l.p.R. manifold. Let $M$ be a submanifold of $(\mathbb{R}^8, \mathcal{F}, g)$ given by*

$$x_1 = t\sin u, \qquad x_2 = t\cos u, \qquad x_3 = \frac{t}{\sqrt{2}}\cos v, \qquad x_4 = \frac{t}{\sqrt{2}}\sin v$$

$$x_5 = 2t\sin x, \qquad x_6 = 0, \qquad x_7 = 2t\cos x, \qquad x_8 = 0,$$

*where $u, v \in (0, \frac{\pi}{2})$ and $t > 0$. Then, the local frame of $TM$ is given by*

$$\begin{aligned}
Z &= \sin u\partial_1 + \cos u\partial_2 + \frac{1}{\sqrt{2}}\cos v\partial_3 + \frac{1}{\sqrt{2}}\sin v\partial_4 + 2\sin x\partial_5 + 2\cos x\partial_7,\\
U &= t\cos u\partial_1 - t\sin u\partial_2,\\
V &= -\frac{t}{\sqrt{2}}\sin v\partial_3 + \frac{t}{\sqrt{2}}\cos v\partial_4,\\
X &= 2t\cos x\partial_5 - 2t\sin x\partial_7.
\end{aligned}$$

*After some calculation, we see that $\mathcal{D}_\theta = span\{Z\}$ is a proper slant distribution with slant angle $\theta = \cos^{-1}(\frac{1}{11})$ and $\mathcal{D}_T = span\{U, V\}$ is an invariant distribution and $\mathcal{D}_\perp = span\{X\}$ is an anti-invariant distribution. Moreover, $\mathcal{D}_\theta$ is totally geodesic and both $\mathcal{D}_T$ and $\mathcal{D}_\perp$ are* *integrable distributions. If we denote the integral manifolds of $\mathcal{D}_\theta$, $\mathcal{D}_T$ and $\mathcal{D}_\perp$ by $M^\theta$, $M^T$ and $M^\perp$, respectively, then the induced metric tensor of $M$ is*

$$\begin{aligned} ds^2 &= \frac{11}{2}dt^2 + t^2(du^2 + \frac{1}{2}dv^2) + 4t^2dx^2 \\ &= g_{M^\theta} + t^2 g_{M^T} + (2t)^2 g_{M^\perp}. \end{aligned}$$

*Thus, $M = M^\theta \times_f M^T \times_\sigma M^\perp$ is a (non-trivial) biwarped product proper s.s-i. submanifold of order 1 of $(\mathbb{R}^8, \mathcal{F}, g)$ with warping functions $f = t$ and $\sigma = 2t$.*

## 5. Biwarped product proper skew semi-invariant submanifolds of order 1 of the form $M^\theta \times_f M^T \times_\sigma M^\perp$

First, we give a characterization for biwarped product proper s.s-i. submanifolds of order 1 of the form $M^\theta \times_f M^T \times_\sigma M^\perp$, where $M^\theta$ is a proper slant submanifold, $M^T$ is an invariant and $M^\perp$ is an anti invariant submanifold of a l.p.R. manifold $(\bar{M}, g, \mathcal{F})$. After that we investigate the behavior of the second fundamental form of such submanifolds and as a result, we give a condition for these submanifolds to be locally warped product. Firstly, we recall the following fact given in [11] to prove our theorem.

**Remark 5.1.** ([11, Remark 2.1]) *Suppose that the tangent bundle of a Riemannian manifold $M$ splits into an orthogonal sum $TM = \mathcal{D}_0 \oplus \mathcal{D}_1 \oplus \ldots \oplus \mathcal{D}_k$ of non-trivial distributions such that each $\mathcal{D}_j$ is spherical and its complement in $TM$ is autoparallel for $j \in \{1, 2, \ldots, k\}$. Then the manifold $M$ is locally isometric to a multiply warped product $M_0 \times_{f_1} M_2 \times_{f_2} \times \ldots \times_{f_k} M_k$.*

Now, we give one of the main theorems of this paper.

**Theorem 5.1.** *Let $M$ be a $(\mathcal{D}_\theta, \mathcal{D}_\perp)$-mixed geodesic proper s.s-i. submanifold of order 1 of a l.p.R. manifold $(\bar{M}, g, \mathcal{F})$. Then $M$ is a locally biwarped product submanifold of type $M^\theta \times_f M^T \times_\sigma M^\perp$ iff we have*

$$A_{NPZ}X = \cos^2\theta Z(\lambda)X, \tag{5.36}$$

$$A_{NZ}V + A_{NPZ}\mathcal{F}V = -\sin^2\theta Z(\mu)\mathcal{F}V \tag{5.37}$$

*for smooth functions $\lambda$ and $\mu$ satisfying $X(\lambda) = V(\lambda) = 0$ and $X(\mu) = V(\mu) = 0$ and*

$$g(A_{\mathcal{F}X}Z, PW) = -g(A_{NPW}Z, X), \tag{5.38}$$

$$g(A_{\mathcal{F}X}U, \mathcal{F}V) = 0, \tag{5.39}$$

$$g(A_{\mathcal{F}Y}X, \mathcal{F}U) = 0, \tag{5.40}$$

$$g(A_{\mathcal{F}X}Z, \mathcal{F}U) = 0, \tag{5.41}$$

$$g(A_{\mathcal{F}X}U, PZ) = -g(A_{NPZ}U, X), \tag{5.42}$$

*for* $Z, W \in \Gamma(\mathcal{D}_\theta)$, $U, V \in \Gamma(\mathcal{D}_T)$, $X, Y \in \Gamma(\mathcal{D}_\perp)$.

**Proof.** For any $Z \in \Gamma(\mathcal{D}_\theta)$, $U \in \Gamma(\mathcal{D}_T)$ and $X \in \Gamma(\mathcal{D}_\perp)$, using (2.4) and (3.17),

$$g(A_{NPZ}X, U) = -g(\bar{\nabla}_X NPZ, U) = -g(\bar{\nabla}_X \mathcal{F}PZ, U) + g(\bar{\nabla}_X P^2 Z, U).$$

By using (2.13) – (2.15) and (3.18), we find

$$g(A_{NPZ}X, U) = -g(\bar{\nabla}_X PZ, \mathcal{F}U) + \cos^2\theta g(\bar{\nabla}_X Z, U).$$

Here, using (2.4), we arrive to

$$g(A_{NPZ}X, U) = -g(\nabla_X PZ, \mathcal{F}U) + \cos^2\theta g(\nabla_X Z, U).$$

So, using (1.2), we conclude that

$$g(A_{NPZ}X, U) = -PZ(\ln\sigma)g(X, \mathcal{F}U) + \cos^2\theta Z(\ln\sigma)g(X, U) = 0. \tag{5.43}$$

Since $M$ is $(\mathcal{D}_\theta, \mathcal{D}_\perp)$-mixed geodesic, for $W \in \Gamma(\mathcal{D}_\theta)$ using (2.5), we find

$$g(A_{NPZ}X, W) = g(h(X, W), NPZ) = 0. \tag{5.44}$$

Next, by a similar argument, for $Y \in \Gamma(\mathcal{D}_\perp)$, using (2.4) and (3.17), we have

$$g(h(X, Y), NZ) = g(\bar{\nabla}_X Y, NZ) = g(\bar{\nabla}_X Y, \mathcal{F}Z) - g(\bar{\nabla}_X Y, PZ).$$

Then using (2.14),(2.15) and (1.2), we find

$$g(h(X, Y), NZ) = g(\bar{\nabla}_X \mathcal{F}Y, Z) + PZ(\ln\sigma)g(X, Y).$$

Hence using (2.4) and (2.5), we arrive to

$$g(h(X, Y), NZ) = -g(A_{\mathcal{F}Y}X, Z) + PZ(\ln\sigma)g(X, Y)$$

$$= -g(h(X, Z), \mathcal{F}Y) + PZ(\ln\sigma)g(X, Y).$$

In this equation, if we interchange $Z$ with $PZ$, then we have

$$g(h(X, Y), NPZ) = -g(h(X, PZ), \mathcal{F}Y) + \cos^2\theta Z(\ln\sigma)g(X, Y).$$

Since $M$ is $(\mathcal{D}_\theta, \mathcal{D}_\perp)$-mixed geodesic, we conclude that

$$g(A_{NPZ}X, Y) = \cos^2\theta Z(\ln\sigma)g(X, Y). \tag{5.45}$$

Moreover, we have $X(\ln\sigma) = V(\ln\sigma) = 0$, since $\sigma$ depends only on the points of $M^\theta$. So, we conclude that $\lambda = \ln\sigma$. Thus, from (5.43) – (5.45), it follows that (5.36). Now, we prove (5.37). For $Z \in \Gamma(\mathcal{D}_\theta)$, $V \in \Gamma(\mathcal{D}_T)$ and $X \in \Gamma(\mathcal{D}_\perp)$, using (2.4) and (3.17), we have

$$\begin{aligned} g(A_{NZ}V + A_{NPZ}\mathcal{F}V, X) &= g(A_{NZ}V, X) + g(A_{NPZ}\mathcal{F}V, X) \\ &= g(A_{NZ}X, V) + g(A_{NPZ}X, \mathcal{F}V) \\ &= -g(\bar{\nabla}_X NZ, V) - g(\bar{\nabla}_X NPZ, \mathcal{F}V) \\ &= -g(\bar{\nabla}_X NZ, V) - g(\bar{\nabla}_X \mathcal{F}PZ, \mathcal{F}V) \\ &\quad + g(\bar{\nabla}_X P^2 Z, \mathcal{F}V). \end{aligned}$$

Using (2.14), (2.15), (3.17) and(3.18) and, we arrive to

$$\begin{aligned} g(A_{NZ}V + A_{NPZ}\mathcal{F}V, X) &= -g(\bar{\nabla}_X \mathcal{F}Z, V) + g(\bar{\nabla}_X PZ, V) - g(\bar{\nabla}_X PZ, V) \\ &\quad + \cos^2\theta g(\bar{\nabla}_X Z, \mathcal{F}V) + X(\cos^2\theta) g(Z, \mathcal{F}V) \\ &= -g(\bar{\nabla}_X \mathcal{F}Z, V) + \cos^2\theta g(\bar{\nabla}_X Z, \mathcal{F}V). \end{aligned}$$

Then, using (1.2), (2.4), (2.13) – (2.15), we find

$$\begin{aligned} g(A_{NZ}V + A_{NPZ}\mathcal{F}V, X) &= -g(\bar{\nabla}_X Z, \mathcal{F}V) + \cos^2\theta g(\nabla_X Z, \mathcal{F}V) \\ &= -g(\nabla_X Z, \mathcal{F}V) + \cos^2\theta g(\nabla_X Z, \mathcal{F}V) \\ &= -\sin^2\theta g(\nabla_X Z, \mathcal{F}V) \\ &= -\sin^2\theta Z(\ln\sigma) g(X, \mathcal{F}V). \end{aligned}$$

Since $g(X, \mathcal{F}V) = 0$, we conclude that

$$g(A_{NZ}V + A_{NPZ}\mathcal{F}V, X) = -\sin^2\theta Z(\ln\sigma) g(X, \mathcal{F}V) = 0. \tag{5.46}$$

Similarly, for $Z, W \in \Gamma(\mathcal{D}_\theta)$ and $V \in \Gamma(\mathcal{D}_T)$, using (2.4) and (3.17), we have

$$\begin{aligned} g(A_{NZ}V + A_{NPZ}\mathcal{F}V, W) &= g(A_{NZ}V, W) + g(A_{NPZ}\mathcal{F}V, W) \\ &= g(A_{NZ}W, V) + g(A_{NPZ}W, \mathcal{F}V) \\ &= -g(\bar{\nabla}_W NZ, V) - g(\bar{\nabla}_W NPZ, \mathcal{F}V) \\ &= -g(\bar{\nabla}_W NZ, V) - g(\bar{\nabla}_W \mathcal{F}PZ, \mathcal{F}V) \\ &\quad + g(\bar{\nabla}_W P^2 Z, \mathcal{F}V). \end{aligned}$$

Using (2.14), (2.15), (3.17) and (3.18), we arrive to

$$\begin{aligned} g(A_{NZ}V + A_{NPZ}\mathcal{F}V, W) &= -g(\bar{\nabla}_W \mathcal{F}Z, V) + g(\bar{\nabla}_W PZ, V) - g(\bar{\nabla}_W PZ, V) \\ &\quad + \cos^2\theta g(\bar{\nabla}_W Z, \mathcal{F}V) + W(\cos^2\theta) g(Z, \mathcal{F}V) \\ &= -g(\bar{\nabla}_W \mathcal{F}Z, V) + \cos^2\theta g(\bar{\nabla}_W Z, \mathcal{F}V) \\ &\quad + W(\cos^2\theta) g(Z, \mathcal{F}V). \end{aligned}$$

Then, using (1.2), (2.4), (2.13) – (2.15), we find

$$\begin{aligned} g(A_{NZ}V + A_{NPZ}\mathcal{F}V, W) &= -g(\bar{\nabla}_W Z, \mathcal{F}V) + \cos^2\theta g(\nabla_W Z, \mathcal{F}V) \\ &\quad + W(\cos^2\theta) g(Z, \mathcal{F}V) \\ &= -g(\nabla_W Z, \mathcal{F}V) + \cos^2\theta g(\nabla_W Z, \mathcal{F}V) + W(\cos^2\theta) g(Z, \mathcal{F}V) \\ &= -\sin^2\theta g(\nabla_W Z, \mathcal{F}V) + W(\cos^2\theta) g(Z, \mathcal{F}V) \\ &= -\sin^2\theta g(\nabla^\theta_W Z, \mathcal{F}V) + W(\cos^2\theta) g(Z, \mathcal{F}V). \end{aligned}$$

Since $g(\nabla^\theta_W Z, \mathcal{F}V) = 0$ and $g(Z, \mathcal{F}V) = 0$, we conclude that

$$g(A_{NZ}V + A_{NPZ}\mathcal{F}V, W) = -\sin^2\theta g(\nabla^\theta_W Z, \mathcal{F}V) + W(\cos^2\theta) g(Z, \mathcal{F}V) = 0. \tag{5.47}$$

On the other hand, for $Z \in \Gamma(\mathcal{D}_\theta)$ and $U, V \in \Gamma(\mathcal{D}_T)$, using (2.4) and (3.17), we get

$$\begin{aligned} g(A_{NZ}V + A_{NPZ}\mathcal{F}V, U) &= g(A_{NZ}V, U) + g(A_{NPZ}\mathcal{F}V, U) \\ &= g(A_{NZ}U, V) + g(A_{NPZ}U, \mathcal{F}V) \\ &= -g(\bar{\nabla}_U NZ, V) - g(\bar{\nabla}_U NPZ, \mathcal{F}V) \\ &= -g(\bar{\nabla}_U NZ, V) - g(\bar{\nabla}_U \mathcal{F}PZ, \mathcal{F}V) \\ &\quad + g(\bar{\nabla}_U P^2 Z, \mathcal{F}V). \end{aligned}$$

Using (2.14), (2.15), (3.17) and (3.18), we arrive to

$$\begin{aligned} g(A_{NZ}V + A_{NPZ}\mathcal{F}V, U) &= -g(\bar{\nabla}_U \mathcal{F}Z, V) + g(\bar{\nabla}_U PZ, V) - g(\bar{\nabla}_U PZ, V) \\ &\quad + \cos^2\theta g(\bar{\nabla}_U Z, \mathcal{F}V) + U(\cos^2\theta) g(Z, \mathcal{F}V) \\ &= -g(\bar{\nabla}_U \mathcal{F}Z, V) + \cos^2\theta g(\bar{\nabla}_U Z, \mathcal{F}V) \\ &\quad + U(\cos^2\theta) g(Z, \mathcal{F}V). \end{aligned}$$

Since $U[\cos^2\theta] = 0$, using (1.2), (2.4), (2.13) – (2.15), we find

$$\begin{aligned} g(A_{NZ}V + A_{NPZ}\mathcal{F}V, U) &= -g(\bar{\nabla}_U Z, \mathcal{F}V) + \cos^2\theta g(\nabla_U Z, \mathcal{F}V) \\ &= -g(\nabla_U Z, \mathcal{F}V) + \cos^2\theta g(\nabla_U Z, \mathcal{F}V) \\ &= -\sin^2\theta g(\nabla_U Z, \mathcal{F}V) \\ &= -\sin^2\theta Z(\ln f) g(U, \mathcal{F}V). \end{aligned}$$

So, we conclude that

$$g(A_{NZ}V + A_{NPZ}\mathcal{F}V, U) = -\sin^2\theta Z(\ln f) g(\mathcal{F}V, U). \tag{5.48}$$

Moreover, we have $X(\ln f) = V(\ln f) = 0$, since $f$ depends only on the points of $M^\theta$. So, we conclude that $\mu = \ln f$. Thus from (5.46) – (5.48), we get (5.37).

Next, we prove (5.38) – (5.42).We know $M$ is a biwarped product proper s.s-i. submanifold of order 1 of a locally product Riemannian manifold $(\bar{M}, g, \mathcal{F})$. Then, for $Z, W \in \Gamma(\mathcal{D}_\theta)$, using (1.1), we get $\nabla_Z W = \nabla^\theta_Z W$ and for $X \in \Gamma(\mathcal{D}_\perp)$,we have

$$g(\nabla_Z W, X) = \sec^2\theta\{g(A_{\mathcal{F}X} Z, PW) + g(A_{NPW} Z, X)\} = g(\nabla^\theta_Z W, X) = 0$$

from (3.22). Since $M^\theta$ is a proper slant submanifold, it follows that

$$g(A_{\mathcal{F}X}Z, PW) + g(A_{NPW}Z, X) = 0,$$

which gives (5.38). For $U, V \in \Gamma(\mathcal{D}_T)$ and $X, Y \in \Gamma(\mathcal{D}_\perp)$, using (1.3), we get $g(\nabla_U V, X) = g(\nabla^T_U V - g(U, V)\nabla(\ln f), X) = 0$. Then from (3.24) we find

$$g(\nabla_U V, X) = g(A_{\mathcal{F}X}U, \mathcal{F}V) = 0.$$

Therefore, we get (5.39). For $U \in \Gamma(\mathcal{D}_T)$ and $X, Y \in \Gamma(\mathcal{D}_\perp)$, using (1.3), we get $g(\nabla_X Y, U) = g(\nabla^\perp{}_X Y - g(X, Y)\nabla(\ln\sigma), U) = 0$. Then from (3.26) we find,

$$g(\nabla_X Y, U) = -g(A_{\mathcal{F}Y}X, \mathcal{F}U) = 0.$$

Hence, we conclude that (5.40). For $X \in \Gamma(\mathcal{D}_\perp)$, $Z \in \Gamma(\mathcal{D}_\theta)$ and $U \in \Gamma(\mathcal{D}_T)$, using (1.2), we write $g(\nabla_Z X, \mathcal{F}U) = g(Z(\ln\sigma)X, \mathcal{F}U) = Z(\ln\sigma)g(Z, \mathcal{F}U) = 0$. On the other hand, from (3.28) we find

$$g(\nabla_Z X, \mathcal{F}U) = -g(A_{\mathcal{F}X}Z, \mathcal{F}U) = 0.$$

Thus, we get (5.41). For $X \in \Gamma(\mathcal{D}_\perp)$, $Z \in \Gamma(\mathcal{D}_\theta)$ and $U \in \Gamma(\mathcal{D}_T)$, using (1.3), we have $g(\nabla_U X, Z) = 0$. Then, from (3.29) we find,

$$g(\nabla_U X, Z) = -\sec^2\theta\{g(A_{\mathcal{F}X}U, PZ) + g(A_{NPZ}U, X)\} = 0.$$

It follows (5.42).

Conversely, assume that $M$ is a proper $(\mathcal{D}_\theta, \mathcal{D}_\perp)$-mixed geodesic s.s-i. submanifold of order 1 of a locally product Riemannian manifold $(\bar{M}, g, \mathcal{F})$ such that (5.36) – (5.42) hold. From (5.38), we get (3.31). On the other hand if we write $\mathcal{F}V$ instead of $V$ and $W$ instead of $Z$ in (5.37), we find $A_{NW}\mathcal{F}V + A_{NPW}V = -\sin^2\theta W(\mu)V$. If we take inner product of this equation with $Z \in \Gamma(\mathcal{D}_\theta)$, we get

$$\begin{aligned} g(A_{NW}\mathcal{F}V + A_{NPW}V, Z) &= g(A_{NW}Z, \mathcal{F}V) + g(A_{NPW}Z, V) \\ &= -\sin^2\theta W(\mu)g(V, Z) = 0. \end{aligned}$$

So, (3.30) holds. Thus from Theorem (3.1), the slant distribution $\mathcal{D}_\theta$ is totally geodesic and as a result, it is integrable. On the other hand, from (5.39), for all $U, V \in \Gamma(\mathcal{D}_T)$ and $X \in \Gamma(\mathcal{D}_\perp)$, we write $g(A_{\mathcal{F}X}V, \mathcal{F}U) = 0$. Thus, $g(A_{\mathcal{F}X}V, \mathcal{F}U) = g(A_{\mathcal{F}X}U, \mathcal{F}V)$, which is

(3.32). On the other hand, in (5.37), if we write $\mathcal{F}V$ instead of $V$, we find $A_{NZ}\mathcal{F}V+A_{NPZ}V = -\sin^2\theta Z(\mu)V$. If we take inner product of this equation with $U \in \Gamma(\mathcal{D}_T)$, we arrive at

$$\begin{aligned} g(A_{NZ}\mathcal{F}V + A_{NPZ}V, U) &= g(A_{NZ}\mathcal{F}V, U) + g(A_{NPZ}V, U) \\ &= -\sin^2\theta Z(\mu)g(V, U). \end{aligned} \tag{5.49}$$

Here, if we interchange $U$ and $V$ in (5.49), we find

$$\begin{aligned} g(A_{NZ}\mathcal{F}U + A_{NPZ}U, V) &= g(A_{NZ}\mathcal{F}U, V) + g(A_{NPZ}U, V) \\ &= -\sin^2\theta Z(\mu)g(U, V). \end{aligned} \tag{5.50}$$

From (5.49) and (5.50), we get $g(A_{NZ}U, \mathcal{F}V)+g(A_{NPZ}U, V) = g(A_{NZ}V, \mathcal{F}U)+g(A_{NPZ}V, U)$. This is (3.33). Thus, by Teorem 3.2, the invariant distribution $\mathcal{D}_T$ is integrable. On the other hand, for all $X, Y \in \Gamma(\mathcal{D}_\perp)$ and $U \in \Gamma(\mathcal{D}_T)$, we have $g(A_{\mathcal{F}Y}X, \mathcal{F}U) = 0$ from (5.40). It follows that $g(A_{\mathcal{F}Y}X, \mathcal{F}U) = g(A_{\mathcal{F}X}Y, \mathcal{F}U) = 0$. That is (3.34). Also, we get
$g(\nabla_X Y, Z) = -\sec^2\theta\{g(h(Y, PZ), \mathcal{F}X) + g(A_{NPZ}X, Y)\}$ from (3.25). Since $M$ is $(\mathcal{D}_\theta, \mathcal{D}_\perp)$-mixed geodesic, it follows that $g(h(Y, PZ), \mathcal{F}X) = 0$. Then, we find $g(\nabla_X Y, Z) = g(\nabla_Y X, Z)$. Thus (3.35) follows. Then by Theorem 3.3, the totally real distributions $\mathcal{D}_\perp$ is integrable. Let $M^\theta, M^T$ and $M^\perp$ be the integral manifolds of $\mathcal{D}_\theta, \mathcal{D}_T$ and $\mathcal{D}_\perp$, respectively. If we denote the second fundamental form of $M^T$ in $M$ by $h^T$, for $U, V \in \Gamma(\mathcal{D}_T)$ and $X \in \Gamma(\mathcal{D}_\perp)$, using (2.4), (3.24) and (5.39), we have

$$g(h^T(U, V), X) = g(\nabla_U V, X) = g(A_{\mathcal{F}X}U, \mathcal{F}V) = 0. \tag{5.51}$$

For any, $U, V \in \Gamma(\mathcal{D}_T)$ and $Z \in \Gamma(\mathcal{D}_\theta)$, using (2.4) and (3.23), we get

$$g(h^T(U, V), Z) = g(\nabla_U V, Z) = \csc^2\theta g(A_{NPZ}U, V) + g(A_{NZ}U, \mathcal{F}V).$$

At this equation, if we use (5.37), we have

$$g(h^T(U, V), Z) = \csc^2\theta g(A_{NPZ}V + A_{NZ}\mathcal{F}V, U) = -Z(\mu)g(V, U).$$

After some calculation, we obtain

$$g(h^T(U, V), Z) = g(-g(U, V)\nabla\mu, Z), \tag{5.52}$$

where $\nabla\mu$ is the gradient of $\mu$. Thus, from (5.51) and (5.52), we conclude that

$$h^T(U, V) = -g(U, V)\nabla\mu.$$

This equation says that $M^T$ is totally umbilic in $M$ with the mean curvature vector field $-\nabla\mu$. Now, we show that $-\nabla\mu$ is parallel. We have to satisfy $g(\nabla_U\nabla\mu, E) = 0$ for $U \in \Gamma(\mathcal{D}_T)$ and

$E \in (\mathcal{D}_T)^\perp = \mathcal{D}_\theta \oplus \mathcal{D}_\perp$. Here, we can put $E = Z + X$, where $Z \in \Gamma(\mathcal{D}_\theta)$ and $X \in \Gamma(\mathcal{D}_\perp)$. By direct computations, we obtain

$$\begin{aligned} g(\nabla_U \nabla\mu, E) &= \{Ug(\nabla\mu, E) - g(\nabla\mu, \nabla_U E)\} \\ &= U(E(\mu)) - [U, E](\mu) - g(\nabla\mu, \nabla_E U) \\ &= [U, E](\mu) + E(U(\mu)) - [U, E](\mu) - g(\nabla\mu, \nabla_E U) \\ &= -g(\nabla\mu, \nabla_E U) = -g(\nabla\mu, \nabla_Z U) - g(\nabla\mu, \nabla_X U), \end{aligned}$$

since $U(\mu) = 0$. Here, for any $W \in \Gamma(\mathcal{D}_\theta)$, we have $g(\nabla_Z U, W) = -g(U, \nabla_Z W) = 0$, since $M^\theta$ is totally geodesic in $M$. Thus, $\nabla_Z U \in \Gamma(\mathcal{D}_T)$ or $\nabla_Z U \in \Gamma(\mathcal{D}_\perp)$. In either case, we have

$$g(\nabla\mu, \nabla_Z U) = 0. \tag{5.53}$$

On the other hand, from (3.27), we have

$$g(\nabla_X U, W) = -g(U, \nabla_X W) = -\csc^2\theta\{g(A_{NPW}X, U) + g(A_{NW}X, \mathcal{F}U)\}.$$

Here, using (5.37), we obtain

$$g(\nabla_X U, W) = g(W(\mu)U, X) = 0.$$

That is, $\nabla_X U \in \Gamma(\mathcal{D}_T)$ or $\nabla_X U \in \Gamma(\mathcal{D}_\perp)$. In either case, we get

$$g(\nabla\mu, \nabla_X U) = 0. \tag{5.54}$$

From (5.53) and (5.54), we find

$$g(\nabla_U \nabla\mu, E) = 0.$$

Thus, $M^T$ is spherical, since it is also totally umbilic. Consequently, $\mathcal{D}_T$ is spherical.

Next, we show that $\mathcal{D}_\perp$ is spherical. Let $h^\perp$ denote the second fundamental form of $M^\perp$ in $M$. Then for $X, Y \in \Gamma(\mathcal{D}_\perp)$ and $U \in \Gamma(\mathcal{D}_T)$, using (2.4), (3.26) and (5.40), we have

$$g(h^\perp(X, Y), U) = g(\nabla_X Y, U) = -g(A_{\mathcal{F}Y}X, \mathcal{F}U) = 0. \tag{5.55}$$

On the other hand, for any $Z \in \Gamma(\mathcal{D}_\theta)$, using (3.25)

$$g(h^\perp(X, Y), Z) = -\sec^2\theta\{g(h(X, PZ), \mathcal{F}Y) + g(A_{NPZ}X, Y)\}.$$

Since $M$, $(\mathcal{D}_\theta, \mathcal{D}_\perp)$-mixed geodesic, $g(h(X, PZ), \mathcal{F}Y) = 0$. So, we have

$$g(h^\perp(X, Y), Z) = -g(A_{NPZ}X, Y).$$

Using (5.36), we obtain

$$g(h^\perp(X, Y), Z) = -Z(\lambda)g(X, Y).$$

By a direct calculation, we get

$$g(h^{\perp}(X,Y),Z) = -g(\nabla\lambda g(X,Y),Z), \tag{5.56}$$

where $\nabla\lambda$ is the gradient of $\lambda$. From (5.55) and (5.56), we obtain

$$h^{\perp}(X,Y) = -g(X,Y)\nabla\lambda.$$

So $M^{\perp}$ is totally umbilic in $M$ and the mean curvature vector field is $-\nabla\lambda$. What's left is to show that $-\nabla\lambda$ is parallel. We have to satisfy $g(\nabla_X\nabla\lambda, E) = 0$ for $X \in \Gamma(\mathcal{D}_{\perp})$ and $E \in (\mathcal{D}_{\perp})^{\perp} = \mathcal{D}_{\theta} \oplus \mathcal{D}_T$. The proof is similar to the parallelity of $-\nabla\mu$. So we omit it. $-\nabla\lambda$ is parallel. So, $M^{\perp}$ is spherical, since it is also totally umbilic. Consequently, $\mathcal{D}_{\perp}$ is spherical.

Lastly, we prove that $(\mathcal{D}_T)^{\perp} = \mathcal{D}_{\theta} \oplus \mathcal{D}_{\perp}$ and $(\mathcal{D}_{\perp})^{\perp} = \mathcal{D}_{\theta} \oplus \mathcal{D}_T$ are autoparallel. In fact, $\mathcal{D}_{\theta} \oplus \mathcal{D}_{\perp}$ is autoparallel iff all for four types of covariant derivatives $\nabla_Z W, \nabla_Z X, \nabla_X Z, \nabla_X Y$ are again in $\Gamma(\mathcal{D}_{\theta} \oplus \mathcal{D}_{\perp})$ for $Z, W \in \Gamma(\mathcal{D}_{\theta})$ and $X, Y \in \Gamma(\mathcal{D}_{\perp})$. This is equivalent to say that all four inner products $g(\nabla_Z W, U)$, $g(\nabla_Z X, U)$, $g(\nabla_X Z, U)$, $g(\nabla_X Y, U)$ vanish, where $U \in \Gamma(\mathcal{D}_T)$. Using (3.21) and (5.37), we find

$$\begin{aligned} g(\nabla_Z W, U) &= -\csc^2\theta\{g(A_{NPW}Z, U) + g(A_{NW}Z, \mathcal{F}U)\} \\ &= -\csc^2\theta g(A_{NPW}U + A_{NW}\mathcal{F}U, Z) \\ &= W(\mu)g(U,Z) = 0. \end{aligned}$$

Using (3.28) and (5.41), we find

$$g(\nabla_Z X, U) = -g(A_{\mathcal{F}X}Z, \mathcal{F}U) = 0.$$

By (3.27) and (5.37), we get

$$g(\nabla_X Z, U) = -\csc^2\theta\{g(A_{NPZ}X, U) + g(A_{NZ}X, \mathcal{F}U)\} = 0.$$

By (3.26) and (5.40), we find

$$g(\nabla_X Y, U) = -g(A_{\mathcal{F}Y}X, \mathcal{F}U) = 0.$$

Thus, $\mathcal{D}_{\theta} \oplus \mathcal{D}_{\perp}$ is autoparallel. On the other hand, $\mathcal{D}_{\theta} \oplus \mathcal{D}_T$ is autoparallel iff all four inner products $g(\nabla_Z W, X), g(\nabla_Z U, X), g(\nabla_U Z, X), g(\nabla_U V, X)$ vanish, where $Z, W \in \Gamma(\mathcal{D}_{\theta})$, $U, V \in \Gamma(\mathcal{D}_T)$ and $X \in \Gamma(\mathcal{D}_{\perp})$. Firstly, we have already $g(\nabla_Z U, X) = 0$ from above. Using (3.22) and (5.38), we get

$$g(\nabla_Z W, X) = \sec^2\theta\{g(A_{\mathcal{F}X}Z, PW) + g(A_{NPW}Z, X)\} = 0.$$

Using (3.24) and (5.39), we find

$$g(\nabla_U V, X) = g(A_{\mathcal{F}X}U, \mathcal{F}V) = 0.$$

And for last one, by (3.29) and (5.42), we get

$$g(\nabla_U Z, X) = -g(\nabla_U X, Z) = \sec^2\theta\{g(A_{\mathcal{F}X}U, PZ) + g(A_{NPZ}U, X)\} = 0.$$

So, $\mathcal{D}_\theta \oplus \mathcal{D}_T$ is autoparallel. Thus by Remark 5.1, $M$ is locally biwarped product submanifold of the form $M^\theta \times_f M^T \times_\sigma M^\perp$.

Next, we investigate the behavior of the second fundamental form $h$ of a non-trivial biwarped product s.s-i. submanifold of order 1 of a locally product Riemannian manifold $(\bar{M}, g, \mathcal{F})$ of the form $M^\theta \times_f M^T \times_\sigma M^\perp$.

**Lemma 5.1.** *Let $M$ be a biwarped product proper s.s-i. submanifold of order 1 of the form $M^\theta \times_f M^T \times_\sigma M^\perp$ of a l.p.R. manifold $(\bar{M}, g, \mathcal{F})$. Then for $h$ of $M$ in $(\bar{M}, g, \mathcal{F})$, we have*

$$g(h(U,V), NW) = -W(\ln f)g(U, \mathcal{F}V) + PW(\ln f)g(U,V), \tag{5.57}$$

$$g(h(Z,U), NW) = 0, \tag{5.58}$$

$$g(h(X,U), NW) = 0, \tag{5.59}$$

$$g(h(Z,U), \mathcal{F}X) = 0, \tag{5.60}$$

$$g(h(X,U), \mathcal{F}Y) = 0, \tag{5.61}$$

$$g(h(U,V), \mathcal{F}X) = 0, \tag{5.62}$$

*where $Z, W \in \Gamma(\mathcal{D}_\theta)$, $X, Y \in \Gamma(\mathcal{D}_\perp)$ and $U, V \in \Gamma(\mathcal{D}_T)$.*

**Proof.** For $U, V \in \Gamma(\mathcal{D}_T)$ and $W \in \Gamma(\mathcal{D}_\theta)$, using (2.4), (2.13) – (2.15) and (3.17), we have

$$\begin{aligned} g(h(U,V), NW) &= g(\bar{\nabla}_U V, NW) = -g(V, \bar{\nabla}_U NW) \\ &= -g(V, \bar{\nabla}_U \mathcal{F}W) + g(V, \bar{\nabla}_U PW) \\ &= -g(\mathcal{F}V, \bar{\nabla}_U W) + g(V, \nabla_U PW) \\ &= -g(\mathcal{F}V, \nabla_U W) + g(V, \nabla_U PW) \\ &= -W(\ln f)g(\mathcal{F}V, U) + PW(\ln f)g(U,V). \end{aligned}$$

Thus, we get (5.57). Now, using (2.4), (2.13) – (2.15) and (3.17), we get

$$\begin{aligned} g(h(Z,U),NW) &= g(\bar{\nabla}_Z U, NW) = -g(U, \bar{\nabla}_Z NW) \\ &= -g(U, \bar{\nabla}_Z \mathcal{F}W) + g(U, \bar{\nabla}_Z PW) \\ &= -g(\mathcal{F}U, \bar{\nabla}_Z W) + g(U, \nabla_Z PW) \\ &= g(W, \bar{\nabla}_Z(\mathcal{F}U)) - g(\nabla_Z U, PW) \\ &= g(W, \nabla_Z \mathcal{F}U) - g(\nabla_Z U, PW), \end{aligned}$$

for $Z, W \in \Gamma(\mathcal{D}_\theta)$ and $U \in \Gamma(\mathcal{D}_T)$. Here using (1.2), we get

$$g(h(Z,U),NW) = Z(\ln f)g(W,\mathcal{F}U) - Z(\ln f)g(U,PW) = 0$$

since $g(W,\mathcal{F}U) = g(U,PW) = 0$. So (5.58) follows. The proof of (5.59) is similar.
For $Z \in \Gamma(\mathcal{D}_\theta)$, $X \in \Gamma(\mathcal{D}_\perp)$ and $U \in \Gamma(\mathcal{D}_T)$, using (2.4), (2.13) – (2.15) and (3.17), we get

$$\begin{aligned} g(h(Z,U),\mathcal{F}X) &= g(\bar{\nabla}_Z U, \mathcal{F}X) = -g(U, \bar{\nabla}_Z \mathcal{F}X) \\ &= -g(\mathcal{F}U, \bar{\nabla}_Z X) = -g(\mathcal{F}U, \nabla_Z X) \\ &= -Z(\ln\sigma)g(\mathcal{F}U, X) = 0 \end{aligned}$$

since $g(\mathcal{F}U, X) = 0$. So (5.60) follows. Next, using (2.4), (2.13) – (2.15), (3.17) and (1.3) we get

$$\begin{aligned} g(h(X,U),\mathcal{F}Y) &= g(\bar{\nabla}_X U, \mathcal{F}Y) = -g(U, \bar{\nabla}_X \mathcal{F}Y) \\ &= -g(\mathcal{F}U, \bar{\nabla}_X Y) = -g(\mathcal{F}U, \nabla_X Y) \\ &= g(\nabla_X \mathcal{F}U, Y) = 0 \end{aligned}$$

for $U \in \Gamma(\mathcal{D}_T)$ and $X, Y \in \Gamma(\mathcal{D}_\perp)$. Thus, (5.61) follows. Lastly, using (2.4), (2.13) – (2.15), (3.17) and (1.3) we get

$$\begin{aligned} g(h(U,V),\mathcal{F}X) &= g(\bar{\nabla}_U V, \mathcal{F}X) = -g(V, \bar{\nabla}_U \mathcal{F}X) \\ &= -g(\mathcal{F}V, \bar{\nabla}_U X) = -g(\mathcal{F}V, \nabla_U X) = 0 \end{aligned}$$

for $U, V \in \Gamma(\mathcal{D}_T)$ and $X \in \Gamma(\mathcal{D}_\perp)$. So, we have (5.62). The other assertions can be obtained by a similar way.

The previous lemma shows partially us the behavior of the second fundamental form $h$ of the biwarped product proper s.s-i. submanifolds of order 1 of the form $M^\theta \times_f M^T \times_\sigma M^\perp$ in the normal subbundle $N(\mathcal{D}_\theta)$ and $\mathcal{F}(\mathcal{D}_\perp)$.

**Remark 5.2.** *The equations* (5.57), (5.58), (5.59) *and* (5.60) *also were obtained as Lemma 3.1-(ii), Lemma 3.1-(i), Lemma 3.3-(ii) and Lemma 3.3-(i), respectively in* [22].

By using (5.58) – (5.61), we immediately have the following result.

**Corollary 5.1.** *Let $M$ be a biwarped-product proper s.s-i. submanifold of order 1 of the form $M^\theta \times_f M^T \times_\sigma M^\perp$ of a locally product Riemannian manifold $(\bar{M}, g, \mathcal{F})$ such that the invariant normal subbundle $\bar{\mathcal{D}}_T = \{0\}$. Then $M$ is $(\mathcal{D}_T, \mathcal{D}_\perp)$ and $(\mathcal{D}_T, \mathcal{D}_\theta)$-mixed geodesic.*

Lastly, we give another main result of this section.

**Theorem 5.2.** *Let $M$ be a biwarped-product proper s.s-i. submanifold of order 1 in the form $M^\theta \times_f M^T \times_\sigma M^\perp$ of a l.p.R. manifold $(\bar{M}, g, \mathcal{F})$ such that its invariant normal subbundle $\bar{\mathcal{D}}_T = \{0\}$. Then $M$ is a locally warped product in the form $M^\theta \times M^T \times_\sigma M^\perp$ iff $M$ is $\mathcal{D}_T$-geodesic.*

**Proof.** If $M$ is a locally warped product of the form $M^\theta \times M^T \times_\sigma M^\perp$, then the warping function $f$ is constant. By (5.57), we have

$$g(h(U,V), NW) = -W(\ln f)g(U, \mathcal{F}V) + PW(\ln f)g(U,V) = 0$$

for $U, V \in \Gamma(\mathcal{D}_T)$ and $W \in \Gamma(\mathcal{D}_\theta)$, since $W(\ln f) = PW(\ln f) = 0$. Using this fact and (5.62), it follows that $h(U,V) = 0$. Which say us $M$ is $\mathcal{D}_T$-geodesic.
Conversely, let $M$ be $\mathcal{D}_T$-geodesic. Then for any $U, V \in \Gamma(\mathcal{D}_T)$ and $W \in \Gamma(\mathcal{D}_\theta)$, we have

$$W(\ln f)g(U, \mathcal{F}V) + PW(\ln f)g(U,V) = 0 \tag{5.63}$$

from (5.57). If we put $W = PW$ in (5.63) and using (3.18), we obtain

$$PW(\ln f)g(U, \mathcal{F}V) + \cos^2\theta W(\ln f)g(U,V) = 0. \tag{5.64}$$

If we replace $V$ by $\mathcal{F}V$ in (5.64), then (5.64) becomes

$$PW(\ln f)g(U,V) + \cos^2\theta W(\ln f)g(U, \mathcal{F}V) = 0. \tag{5.65}$$

From (5.63) and (5.65), we get

$$\sin^2\theta W(\ln f)g(U, \mathcal{F}V) = 0 \tag{5.66}$$

for any $U, V \in \Gamma(\mathcal{D}_T)$ and $W \in \Gamma(\mathcal{D}_\theta)$. Since (5.66) is true for any $U, V \in \Gamma(\mathcal{D}_T)$, it is also true for $\mathcal{F}V \in \Gamma(\mathcal{D}_T)$. So (5.66) becomes

$$\sin^2\theta W(\ln f)g(U,V) = 0. \tag{5.67}$$

Since $M$ is proper, $\sin\theta \neq 0$, we can deduce that $W(\ln f) = 0$ from (5.67). Namely, we find $f$ as a constant. Thus, $M$ must be a locally warped product in the form $M^\theta \times M^T \times_\sigma M^\perp$.

## 6. An inequality for non-trivial biwarped product s.s-i. submanifolds of order 1 of the form $M^\theta \times_f M^T \times_\sigma M^\perp$

In this section, we shall establish an inequality for the squared norm of the second fundamental form in terms of the warping functions for biwarped product skew semi-invariant submanifolds of order 1 of the form $M^\theta \times_f M^T \times_\sigma M^\perp$, where $M^\theta$ is a proper slant, $M^T$ is a invariant and $M^\perp$ is an anti-invariant submanifold in a l.p.R. manifold $(\bar{M}, g, \mathcal{F})$.

Let $M_0 \times_{f_1} M_1 \times_{f_2} M_2$ be a biwarped product submanifold in a Riemannian manifold $\bar{M}$. Then from [9], we write

$$\begin{aligned} K(X_0, X_i) = K_{0i} = &\ \frac{1}{f_i}((\nabla_{X_0} X_0)(f_i) - X_0(X_0(f_i))) \\ K(X_i, X_j) = K_{ij} = &\ -\frac{g(\nabla f_i, \nabla f_j)}{f_i f_j}, \quad i, j = 1, 2, \end{aligned} \tag{6.68}$$

for each unit vector $X_i$ tangent to $M_i$. If we consider the local orthonormal frame $\{e_1, e_2, \dots, e_m\}$ of $TM$, in view of Gauss equation (2.6), we derive

$$\tau(TM) = \bar{\tau}(TM) + \sum_{r=m+1}^{\bar{m}} \sum_{1 \leq i \neq j \leq m} \left( h_{ii}^r h_{jj}^r - (h_{ij}^r)^2 \right), \tag{6.69}$$

where $\bar{m} - m = dim T^\perp M$.

Now we are ready to prove the general inequality. Let $M$ be a $m = m_0 + m_1 + m_2$-dimensional biwarped product s.s-i. submanifolds of order 1 of type $M^\theta \times_f M^T \times_\sigma M^\perp$ in a locally product Riemannian manifold $(\bar{M}, g, \mathcal{F})$. A canonical orthonormal basis is given by $\{e_1, \dots, e_{m_0}, e_{m_0+1}, \dots, e_{m_0+m_1}, e_{m_0+m_1+1}, \dots, e_{m_0+m_1+m_2}, e_{m+1}, \dots, e_{\bar{m}}\}$ of $T\bar{M}$ such that $\{e_1, \dots, e_{m_0}\}$ is an orthonormal basis of $TM^\theta$, $\{e_{m_0+1}, \dots, e_{m_0+m_1}\}$ is an orthonormal basis of $TM^T$, $\{e_{m_0+m_1+1}, \dots, e_{m_0+m_1+m_2}\}$ is an orthonormal basis of $TM^\perp$, $\{e_{m+1}, \dots, e_{\bar{m}}\}$ is an orthonormal basis of $T^\perp M$.

**Theorem 6.1.** *Let $M = M^\theta \times_f M^T \times_\sigma M^\perp$ be an $m$-dimensional non-trivial biwarped product s.s-i. submanifold $M$ of order 1 of an $\bar{m}$-dimensional locally product Riemannian manifold $(\bar{M}, g, \mathcal{F})$. Then*

***(i)*** *the second fundamental form of $M$ satisfies*

$$\begin{aligned} \tfrac{1}{2} \parallel h \parallel^2 \geq &\ \bar{\tau}(TM) - \bar{\tau}(TM^\theta) - \bar{\tau}(TM^T) - \bar{\tau}(TM^\perp) \\ &- m_1 \frac{\Delta f}{f} - m_2 \frac{\Delta \sigma}{\sigma} + m_1 m_2 \frac{g(\nabla f, \nabla \sigma)}{f \sigma}, \end{aligned} \tag{6.70}$$

*where* $m_1 = dimM^T$ *and* $m_2 = dimM^{\perp}$.

***(ii)*** *The equality case of the inequality* (6.70) *holds identically iff* $M^\theta$ *is also totally geodesic in* $\bar{M}$*, and both* $M^T$ *and* $M^\perp$ *are totally umbilic in* $\bar{M}$.

**Proof.** Putting $U = W = e_i$ and $V = Z = e_j$ in Gauss equation (2.6), we obtain

$$\bar{R}(e_i, e_j, e_j, e_i) = R(e_i, e_j, e_j, e_i) + g(h(e_i, e_j), h(e_i, e_j)) - g(h(e_i, e_i), h(e_j, e_j)).$$

Taking summation, over $1 \leq i, j \leq m (i \neq j)$ in above equation, we obtain

$$2\bar{\tau}(TM) = 2\tau(TM) - m^2 \parallel H \parallel^2 + \parallel h \parallel^2 .$$

Then from (2.11), we derive

$$\begin{aligned}
\frac{1}{2} \parallel h \parallel^2 \quad = \quad & \frac{m^2}{2} \parallel H \parallel^2 + \bar{\tau}(TM) - \sum_{1 \leq i < j \leq m_0} K_{ij} \\
& - \sum_{m_0+1 \leq i < j \leq m_0+m_1} K_{ij} - \sum_{m_0+m_1+1 \leq i < j \leq m_0+m_1+m_2} K_{ij} - \sum_{i=1}^{m_0} \sum_{j=m_0+1}^{m_0+m_1} K_{ij} \\
& - \sum_{i=1}^{m_0} \sum_{j=m_0+m_1+1}^{m_0+m_1+m_2} K_{ij} - \sum_{i=m_0+1}^{m_0+m_1} \sum_{j=m_0+m_1+1}^{m_0+m_1+m_2} K_{ij}.
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}
\frac{1}{2} \parallel h \parallel^2 \quad = \quad & \frac{m^2}{2} \parallel H \parallel^2 + \bar{\tau}(TM) - \tau(TM^\theta) - \tau(TM^T) - \tau(TM^\perp) \\
& - \sum_{i=1}^{m_0} \sum_{j=m_0+1}^{m_0+m_1} K_{ij} - \sum_{i=1}^{m_0} \sum_{j=m_0+m_1+1}^{m_0+m_1+m_2} K_{ij} - \sum_{i=m_0+1}^{m_0+m_1} \sum_{j=m_0+m_1+1}^{m_0+m_1+m_2} K_{ij}.
\end{aligned}$$

Last three terms of first line of above equation can be obtained by using (6.69), then we get

$$\begin{aligned}
\frac{1}{2} \parallel h \parallel^2 \quad = \quad & \frac{m^2}{2} \parallel H \parallel^2 + \bar{\tau}(TM) \\
& - \bar{\tau}(TM^\theta) - \sum_{r=m+1}^{\bar{m}} \sum_{1 \leq i \neq t \leq m_0} \left( h_{ii}^r h_{tt}^r - (h_{it}^r)^2 \right) \\
& - \bar{\tau}(TM^T) - \sum_{r=m+1}^{\bar{m}} \sum_{m_0+1 \leq j \neq l \leq m_0+m_1} \left( h_{jj}^r h_{ll}^r - (h_{jl}^r)^2 \right) \\
& - \bar{\tau}(TM^\perp) - \sum_{r=m+1}^{\bar{m}} \sum_{m_0+m_1+1 \leq a \neq b \leq m_0+m_1+m_2} \left( h_{aa}^r h_{bb}^r - (h_{ab}^r)^2 \right) \\
& - \sum_{i=1}^{m_0} \sum_{j=m_0+1}^{m_0+m_1} K_{ij} - \sum_{i=1}^{m_0} \sum_{j=m_0+m_1+1}^{m_0+m_1+m_2} K_{ij} - \sum_{i=m_0+1}^{m_0+m_1} \sum_{j=m_0+m_1+1}^{m_0+m_1+m_2} K_{ij}.
\end{aligned} \tag{6.71}$$

Now, using (6.68), for a biwarped product submanifold, we find

$$\sum_{i=1}^{m_0} \sum_{j=m_0+1}^{m_0+m_1} K_{ij} = m_1 \frac{\Delta f}{f}, \qquad \sum_{i=1}^{m_0} \sum_{j=m_0+m_1+1}^{m_0+m_1+m_2} K_{ij} = m_2 \frac{\Delta \sigma}{\sigma}$$

and

$$\sum_{i=m_0+1}^{m_0+m_1}\sum_{j=m_0+m_1+1}^{m_0+m_1+m_2} K_{ij} = -m_1m_2\frac{g(\nabla f,\nabla\sigma)}{f\sigma}.$$

If we use these equations in (6.71), we obtain

$$\begin{aligned}
\frac{1}{2}\parallel h\parallel^2 \;=\;& \frac{m^2}{2}\parallel H\parallel^2 + \bar{\tau}(TM) - m_1\frac{\Delta f}{f} - m_2\frac{\Delta\sigma}{\sigma} + m_1m_2\frac{g(\nabla f,\nabla\sigma)}{f\sigma}\\
& -\bar{\tau}(TM^\theta) - \sum_{r=m+1}^{\bar{m}}\sum_{1\leq i\neq t\leq m_0}\Big(h^r_{ii}h^r_{tt} - (h^r_{it})^2\Big)\\
& -\bar{\tau}(TM^T) - \sum_{r=m+1}^{\bar{m}}\sum_{m_0+1\leq j\neq l\leq m_0+m_1}\Big(h^r_{jj}h^r_{ll} - (h^r_{jl})^2\Big)\\
& -\bar{\tau}(TM^\perp) - \sum_{r=m+1}^{\bar{m}}\sum_{m_0+m_1+1\leq a\neq b\leq m_0+m_1+m_2}\Big(h^r_{aa}h^r_{bb} - (h^r_{ab})^2\Big).
\end{aligned}$$

If we arrange this equation, we arrive to

$$\begin{aligned}
\frac{1}{2}\parallel h\parallel^2 \;=\;& \frac{m^2}{2}\parallel H\parallel^2 + \bar{\tau}(TM) - m_1\frac{\Delta f}{f} - m_2\frac{\Delta\sigma}{\sigma} + m_1m_2\frac{g(\nabla f,\nabla\sigma)}{f\sigma}\\
& -\bar{\tau}(TM^\theta) - \bar{\tau}(TM^T) - 2\bar{\tau}(TM^\perp)\\
& \sum_{r=m+1}^{\bar{m}}\sum_{1\leq i\neq t\leq m_0}(h^r_{it})^2 + \sum_{r=m+1}^{\bar{m}}\sum_{m_0+1\leq j\neq l\leq m_0+m_1}(h^r_{jl})^2\\
& + \sum_{r=m+1}^{\bar{m}}\sum_{m_0+m_1+1\leq a\neq b\leq m_0+m_1+m_2}(h^r_{ab})^2 - \sum_{r=m+1}^{\bar{m}}\sum_{1\leq i\neq t\leq m_0}(h^r_{ii}h^r_{tt})\\
& - \sum_{r=m+1}^{\bar{m}}\sum_{m_0+1\leq j\neq l\leq m_0+m_1}(h^r_{jj}h^r_{ll})\\
& - \sum_{r=m+1}^{\bar{m}}\sum_{m_0+m_1+1\leq a\neq b\leq m_0+m_1+m_2}(h^r_{aa}h^r_{bb}).
\end{aligned}$$

Adding and substracting the term $\frac{1}{2}\sum_{r=m+1}^{\bar{m}}\Big((h_{11}^r)^2+\ldots+(h_{mm}^r)^2\Big)$ in the above equation, we find that

$$\begin{aligned}\frac{1}{2}\parallel h\parallel^2 &= \frac{m^2}{2}\parallel H\parallel^2+\bar{\tau}(TM)-m_1\frac{\Delta f}{f}-m_2\frac{\Delta\sigma}{\sigma}+m_1m_2\frac{g(\nabla f,\nabla\sigma)}{f\sigma}\\ &-\bar{\tau}(TM^\theta)-\bar{\tau}(TM^T)-\bar{\tau}(TM^\perp)\\ &+\sum_{r=m+1}^{\bar{m}}\sum_{1\le i\neq t\le m_0}(h_{it}^r)^2+\sum_{r=m+1}^{\bar{m}}\sum_{m_0+1\le j\neq l\le m_0+m_1}(h_{jl}^r)^2\\ &+\sum_{r=m+1}^{\bar{m}}\sum_{m_0+m_1+1\le a\neq b\le m_0+m_1+m_2}(h_{ab}^r)^2-\sum_{r=m+1}^{\bar{m}}\sum_{1\le i\neq t\le m_0}(h_{ii}^rh_{tt}^r)\\ &-\sum_{r=m+1}^{\bar{m}}\sum_{m_0+1\le j\neq l\le m_0+m_1}(h_{jj}^rh_{ll}^r)\\ &-\sum_{r=m+1}^{\bar{m}}\sum_{m_0+m_1+1\le a\neq b\le m_0+m_1+m_2}(h_{aa}^rh_{bb}^r)\\ &+\frac{1}{2}\sum_{r=m+1}^{\bar{m}}\Big((h_{11}^r)^2+\ldots+(h_{mm}^r)^2\Big)\\ &-\frac{1}{2}\sum_{r=m+1}^{\bar{m}}\Big((h_{11}^r)^2)+\ldots+(h_{mm}^r)^2\Big).\end{aligned}\tag{6.72}$$

Here, by (2.7), we have

$$\parallel H\parallel^2=\frac{1}{m^2}\sum_{r=m+1}^{\bar{m}}\Big((h_{11}^r)^2+\ldots+(h_{mm}^r)^2\Big)+2\sum_{r=m+1}^{\bar{m}}\sum_{1\le i\neq j\le m}(h_{ii}^rh_{jj}^r).$$

Using this equation in (6.72), we obtain

$$\begin{aligned}\frac{1}{2}\|h\|^2 &= \frac{m^2}{2}\parallel H\parallel^2+\bar{\tau}(TM)-m_1\frac{\Delta f}{f}-m_2\frac{\Delta\sigma}{\sigma}+m_1m_2\frac{g(\nabla f,\nabla\sigma)}{f\sigma}\\ &-\bar{\tau}(TM^\theta)-\bar{\tau}(TM^T)-\bar{\tau}(TM^\perp)\\ &+\sum_{r=m+1}^{\bar{m}}\sum_{1\le i\neq t\le m_0}(h_{it}^r)^2+\sum_{r=m+1}^{\bar{m}}\sum_{m_0+1\le j\neq l\le m_0+m_1}(h_{jl}^r)^2\\ &+\sum_{r=m+1}^{\bar{m}}\sum_{m_0+m_1+1\le a\neq b\le m_0+m_1+m_2}(h_{ab}^r)^2-\frac{m^2}{2}\parallel H\parallel^2\\ &+\frac{1}{2}\sum_{r=m+1}^{\bar{m}}\Big((h_{11}^r)^2+\ldots+(h_{mm}^r)^2\Big).\end{aligned}\tag{6.73}$$

Now, the inequality (6.70) comes from (6.73). The equality sign in (6.70) holds iff

$$\begin{gathered}\sum_{r=m+1}^{\bar{m}}\sum_{1\le i\neq t\le m}(h_{it}^r)^2=0\qquad and\\ \sum_{r=m+1}^{\bar{m}}\Big((h_{11}^r)^2+\ldots+(h_{mm}^r)^2\Big)=0.\end{gathered}\tag{6.74}$$

It follows that, $h_{ij}^r = g(h(e_i, e_j), e_r) = 0$ for $i, j \in 1, \ldots, m$ and $r \in m+1, \ldots, \bar{m}$. Which says us $h \equiv 0$. For a biwarped product submanifold of the form $M = M^\theta \times_f M^T \times_\sigma M^\perp$, we know already that $M^\theta$ is totally geodesic in $M$ and both $M^T$ and $M^\perp$ are totally umbilic in $M$. Since, the second fundamental form $h$ of $M$ vanishes, identically, it follows that $M^\theta$ is also totally geodesic in $\bar{M}$ and both $M^T$ and $M^\perp$ are also totally umbilic in $\bar{M}$.

Now we give an application of the inequality (6.70).

**Theorem 6.2.** *Let $M = M^\theta \times_f M^T \times_\sigma M^\perp$ be an $m$-dimensional non-trivial biwarped product s.s-i. submanifold $M$ of order 1 of an $\bar{m}$-dimensional locally product Riemannian manifold $(\bar{M} = M_1(c_1) \times M_2(c_2), \mathcal{F}, g)$. Then the squared norm of the second fundemental form $h$ of $M$ satisfies*

$$\begin{aligned} \| h \|^2 \geq \; & \frac{1}{2}(c_1 + c_2)\Big(m_0 m_1 + m_0 m_2 + m_1 m_2\Big) - 2m_1 \frac{\Delta f}{f} - 2m_2 \frac{\Delta \sigma}{\sigma} \\ & + 2m_1 m_2 \frac{g(\nabla f, \nabla \sigma)}{f\sigma}, \end{aligned} \tag{6.75}$$

*where $m_0 = dim M^\theta$, $m_1 = dim M^T$, $m_2 = dim M^\perp$ and $m_0 + m_1 + m_2 = m$.*

**Proof.** In (2.16), substituting $X = e_i$, $Y = Z = e_j$ and take inner product with $e_i$ in the above equation, we obtain

$$\begin{aligned} \bar{R}(e_i, e_j, e_j, e_i) = \; & \frac{1}{4}(c_1 + c_2)\Big\{ g(e_j, e_j) g(e_i, e_i) - g(e_i, e_j) g(e_j, e_i) \\ & + g(\mathcal{F}e_j, e_j) g(\mathcal{F}e_i, e_i) - g(\mathcal{F}e_i, e_j) g(\mathcal{F}e_j, e_i)\Big\} \\ & + \frac{1}{4}(c_1 - c_2)\Big\{ g(e_j, e_j) g(\mathcal{F}e_i, e_i) - g(e_i, e_j) g(\mathcal{F}e_j, e_i) \\ & + g(\mathcal{F}e_j, e_j) g(e_i, e_i) - g(\mathcal{F}e_i, e_j) g(e_j, e_i)\Big\}. \end{aligned}$$

Taking summation over basis vectors of $TM$ for $1 \leq i \neq j \leq m$, we get

$$\begin{aligned} 2\bar{\tau}(TM) = \; & \frac{1}{4}(c_1 + c_2)\Big\{ \sum_{1 \leq i \neq j \leq m} g(e_j, e_j) g(e_i, e_i) - \sum_{1 \leq i \neq j \leq m} g(e_i, e_j)^2 \\ & + \sum_{1 \leq i \neq j \leq m} g(\mathcal{F}e_j, e_j) g(\mathcal{F}e_i, e_i) - \sum_{1 \leq i \neq j \leq m} g(\mathcal{F}e_i, e_j) g(\mathcal{F}e_j, e_i)\Big\} \\ & + \frac{1}{4}(c_1 - c_2)\Big\{ \sum_{1 \leq i \neq j \leq m} g(e_j, e_j) g(\mathcal{F}e_i, e_i) - \sum_{1 \leq i \neq j \leq m} g(e_i, e_j) g(\mathcal{F}e_j, e_i) \\ & + \sum_{1 \leq i \neq j \leq m} g(\mathcal{F}e_j, e_j) g(e_i, e_i) - \sum_{1 \leq i \neq j \leq m} g(\mathcal{F}e_i, e_j) g(e_j, e_i)\}\Big\}. \end{aligned}$$

Let $M$ be an $m$-dimensional non-trivial biwarped product s.s-i. submanifold $M$ of order 1 of an $\bar{m}$-dimensional locally product Riemannian manifold $\bar{M} = M_1(c_1) \times M_2(c_2)$ in the form $M^\theta \times_f M^T \times_\sigma M^\perp$. We choose the orthonormal frame fields of $TM^\theta$ and $TM^T$

as $\{e_1 = \sec\theta Pe_1, \ldots, e_{m_0} = \sec\theta Pe_{m_0}\}$ and $\{\mathcal{F}e_{m_0+1} = e_{m_0+1}, \ldots, \mathcal{F}e_t = e_t, \mathcal{F}e_{t+1} = -e_{t+1}, \ldots, \mathcal{F}e_{m_0+m_1} = -e_{m_0+m_1}\}$, respectively. Also, we choose the orthonormal frame fields of $TM^\perp$ as $\{e_{m_0+m_1+1}, \ldots, e_{m_0+m_1+m_2}\}$. Here, for $1 \leq i \leq m_0$, we have $g(\mathcal{F}e_i, e_i) = \cos\theta$ and for $1 \leq i \neq j \leq m_0$, we have $g(\mathcal{F}e_i, e_j) = 0$, since $M^\theta$ is a slant submanifold with slant angle $\theta$. Also, for $m_0 + 1 \leq i \leq t$, we have $g(\mathcal{F}e_i, e_i) = 1$ and for $t + 1 \leq i \leq m_0 + m_1$, we have $g(\mathcal{F}e_i, e_i) = -1$. Moreover, for $m_0 + m_1 + 1 \leq i \leq m_0 + m_1 + m_2 = m$, we have $g(\mathcal{F}e_i, e_i) = 0$ and for $m_0 + m_1 + 1 \leq i \neq j \leq m_0 + m_1 + m_2 = m$, we have $g(\mathcal{F}e_i, e_j) = 0$, since $M^\perp$ is an anti-invariant submanifold. Thus, using these facts, we obtain the following

$$\sum_{m_0+1\leq i\neq j\leq m_0+m_1} g(\mathcal{F}e_j, e_j)g(\mathcal{F}e_i, e_i) = m_1 - 3,$$

$$\sum_{1\leq i\neq j\leq m_0} g(\mathcal{F}e_j, e_j)g(\mathcal{F}e_i, e_i) = (m_0 - 1)\cos^2\theta,$$

$$\sum_{m_0+1\leq i\neq j\leq m_0+m_1} g(e_j, e_j)g(\mathcal{F}e_i, e_i) = 2t - m_1 - 1,$$

$$\sum_{1\leq i\neq j\leq m_0} g(e_j, e_j)g(\mathcal{F}e_i, e_i) = (m_0 - 1)\cos\theta,$$

$$\sum_{m_0+m_1+1\leq i\neq j\leq m} g(\mathcal{F}e_j, e_j)g(\mathcal{F}e_i, e_i) = \sum_{m_0+m_1+1\leq i\neq j\leq m} g(e_j, e_j)g(\mathcal{F}e_i, e_i) = 0,$$

and

$$\sum_{1\leq i\neq j\leq m} g(\mathcal{F}e_i, e_j)g(e_j, e_i) = \sum_{1\leq i\neq j\leq m} g(\mathcal{F}e_i, e_j)g(\mathcal{F}e_j, e_i) = 0.$$

Thus, we find

$$\begin{aligned} 2\bar{\tau}(TM) \;=\; & \frac{1}{4}(c_1 + c_2)\Big\{m(m-1) + m_1 - 3 + (m_0 - 1)\cos^2\theta\Big\} \\ & +\frac{1}{4}(c_1 - c_2)\Big\{2(2t - m_1 - 1) + 2(m_0 - 1)\cos\theta\Big\}. \end{aligned} \tag{6.76}$$

Similarly for $TM^\theta$, $TM^T$ and $TM^\perp$, we derive

$$\begin{aligned} 2\bar{\tau}(TM^\theta) \;=\; & \frac{1}{4}(c_1 + c_2)\Big\{m_0(m_0 - 1) + (m_0 - 1)\cos^2\theta\Big\} \\ & +\frac{1}{4}(c_1 - c_2)\Big\{2(m_0 - 1)\cos\theta\Big\} \end{aligned} \tag{6.77}$$

$$\begin{aligned} 2\bar{\tau}(TM^T) \;=\; & \frac{1}{4}(c_1 + c_2)\Big\{m_1(m_1 - 1) + m_1 - 3\Big\} \\ & +\frac{1}{4}(c_1 - c_2)\Big\{2(2t - m_1 - 1)\Big\} \end{aligned} \tag{6.78}$$

$$2\bar{\tau}(TM^\perp) = \frac{1}{4}(c_1 + c_2)\Big\{m_2(m_2 - 1)\Big\}. \tag{6.79}$$

Thus, using (6.76) – (6.79) in (6.70), we get the inequality (6.75).

**Acknowledgments.** All authors except the first one are supported by 1002- Scientific and Technological Research Projects Funding Program of TUBITAK project number 119F207. Moreover, the second author is supported by the Scientific Research Projection Coordination Unit of İstanbul University with project numbered 25839.

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