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ON THE MANNHEIM PARTNER OF A CUBIC BEZIER CURVE IN E^3

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Abstract. In this study we have examined, Mannheim partner of a cubic Bezier curve based on the control points with matrix form in E^3. Frenet vector fields and also curvatures of Mannheim partner of the cubic Bezier curve are examined based on the Frenet apparatus of the first cubic Bezier curve in E^3.

Keywords: Bézier curves, Mannheim partner, Cubic Bezier curve

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1. Introduction and Preliminaries

French engineer Pierre Bézier, who used Bézier curves to design automobile bodies studied with them in 1962. But the study of these curves was first developed in 1959 by mathematician Paul de Casteljau using de Casteljau's algorithm, a numerically stable method to evaluate Bézier curves. A Bézier curve is frequently used in computer graphics and related fields, in vector graphics, used in animation as a tool to control motion. To guarantee smoothness, the control point at which two curves meet must be on the line between the two control points on either side. In animation applications, such as Adobe Flash and Synfig, Bézier curves are used to outline, for example, movement. Users outline the wanted path in Bézier curves, and the application creates the needed frames for the object to move along the path. For 3D animation Bézier curves are often used to define 3D paths as well as 2D

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curves for key-frame interpolation. We have been motivated by the following studies. First Bezier-curves with curvature and torsion continuity has been examined in [6]. Also in [2], [3] and [7] Bezier curves and surfaces have been given. In [4] Bézier curves are designed for Computer-Aided Geometric. Recently equivalence conditions of control points and application to planar Bezier curves have been examined. In [8] Frenet apparatus of the cubic Bézier curves has been examined in E^3. Before, the 5^{th} order Bézier curve and its, first, second, and third based on the control points of the 5^{th} order Bézier Curve in E^3 are examined too in [12]. We have already examine in cubic Bézier curves and involutes in [8] and [9], respectively. Also Bertrand mate of a cubic Bezier curve based on the control points with matrix form has been examined with Frenet apparatus in [11]. Here we will examine the Mannheim partner of a cubic Bezier curve, based on the control points with matrix representation.

The set, whose elements are Frenet vector fields and the curvatures of a curve $\alpha(t) \subset \mathbf{E}^3$, is called Frenet apparatus of the curves. Let $\alpha(t)$ be the curve, with $\eta = \|\alpha'(t)\| \neq 1$ and Frenet apparatus be $\{T(t), N(t), B(t), \kappa(t), \tau(t)\}$. Frenet vector fields are given for a non arc-length curve

$$T(t) = \frac{\alpha'(t)}{\|\alpha'(t)\|}, \qquad N(t) = B(t)\Lambda T(t), \qquad B(t) = \frac{\alpha'(t)\Lambda\alpha''(t)}{\|\alpha'(t)\Lambda\alpha''(t)\|},$$

$$\kappa(t) = \frac{\left\|\alpha'(t)\Lambda\alpha''(t)\right\|}{\|\alpha'(t)\|^3} \quad \text{and} \quad \tau(t) = \frac{\left\langle \alpha'(t)\Lambda\alpha''(t), \alpha'''(t)\right\rangle}{\|\alpha'(t)\Lambda\alpha''(t)\|^2}$$

where $\kappa(t)$ and $\tau(t)$ are curvature functions. Also Frenet formulas are well known as

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \eta\kappa & 0 \\ -\eta\kappa & 0 & \eta\tau \\ 0 & -\eta\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}.$$

Generally, Béziers curve can be defined by $n+1$ control points $P_0, P_1, ..., P_n$ with the parametrization

$$\mathbf{B}(t) = \sum_{i=0}^{n} \binom{n}{i} t^i (1-t)^{n-i} [P_i],$$

where $\binom{n}{i} = \dfrac{n!}{i!(n-i)!}$ is known as the usual binomial coefficients. In this study we will define and work on cubic Bézier curves in E^3. For more detail see [1, 8].

Definition 1.1. *A cubic Bézier curve is a special Bézier curve and it has only four points P_0, P_1, P_2 and P_3, its parametrization is*

$$\alpha(t) = (1-t)^3 P_0 + 3t(1-t)^2 P_1 + 3t^2(1-t) P_2 + t^3 P_3$$

and matrix form of the cubic Bezier curve with control points P_0, P_1, P_2, P_3, is

$$\alpha(t)=\begin{bmatrix} t^3\\ t^2\\ t\\ 1\end{bmatrix}^T\begin{bmatrix} -1 & 3 & -3 & 1\\ 3 & -6 & 3 & 0\\ -3 & 3 & 0 & 0\\ 1 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix} P_0\\ P_1\\ P_2\\ P_3\end{bmatrix}.$$

Also using the derivatives of a cubic Bézier curve Frenet apparatus $\{T, N, B, \kappa, \tau\}$ have already been given as in the following theorems by using matrix representation. For more detail see in [8].

The first derivative of a cubic Bézier curve by using matrix representation is

$$\alpha'(t)=\begin{bmatrix} t^2\\ t\\ 1\end{bmatrix}^T\begin{bmatrix} 1 & -2 & 1\\ -2 & 2 & 0\\ 1 & 0 & 0\end{bmatrix}\begin{bmatrix} Q_0\\ Q_1\\ Q_2\end{bmatrix}$$

where $Q_0 = 3(P_1 - P_0) = (x_0, y_0, z_0)$, $Q_1 = 3(P_2 - P_1) = (x_1, y_1, z_1)$, $Q_2 = 3(P_3 - P_2) = (x_2, y_2, z_2)$ are control points.

The second derivative of a cubic Bézier curve by using matrix representation is

$$\alpha''(t)=\begin{bmatrix} t\\ 1\end{bmatrix}^T\begin{bmatrix} -1 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix} R_0\\ R_1\end{bmatrix}$$

where $R_0 = 6(P_2 - 2P_1 + P_0)$, $R_1 = 6(P_3 - 2P_2 + P_1)$ are control points.

The third derivative of a cubic Bézier curve is constant by using matrix representation is

$$\alpha'''(t) = [R_0R_1]$$

with the control point $[R_0R_1] = R_1 - R_0 = 2[Q_1Q_2] - 2[Q_0Q_1]$.

Frenet apparatus $\{T(t), N(t), B(t), \kappa(t), \tau(t)\}$ of a cubic Bézier curve have already been given as in the following theorems by using the matrix representation. For more detail see in [9].

Tangent vector field of a cubic Bezier curve α with, $\|\alpha'\| = \eta$ has the following the matrix representation

$$T(t) = \frac{1}{\eta}\begin{bmatrix} t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_0 & y_0 & z_0 \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{bmatrix}$$

$$= \frac{1}{\eta}\begin{bmatrix} t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} Q_0 \\ Q_1 \\ Q_2 \end{bmatrix}$$

$$= \frac{1}{\eta}\left(Q_0\left(t^2 - 2t + 1\right) - Q_1\left(2t^2 - 2t\right) + t^2 Q_2\right).$$

Binormal vector field of a cubic Bezier curve by using the matrix representation is

$$B(t) = \frac{6}{m}\begin{bmatrix} t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

$$= \frac{6}{m}\begin{bmatrix} t^2 & t & 1 \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix}$$

$$= \frac{6}{m}\left(B_1 t^2 + B_2 t + B_3\right)$$

where $m = \|\alpha' \Lambda \alpha''\|$ and

$$b_{11} = (y_0 z_1 - y_1 z_0 - y_0 z_2 + y_2 z_0 + y_1 z_2 - y_2 z_1),$$

$$b_{12} = (x_1 z_0 - x_0 z_1 + x_0 z_2 - x_2 z_0 - x_1 z_2 + x_2 z_1),$$

$$b_{13} = (x_0 y_1 - x_1 y_0 - x_0 y_2 + x_2 y_0 + x_1 y_2 - x_2 y_1),$$

$$b_{21} = (2 y_1 z_0 + y_0 z_2 - 2 y_0 z_1 - y_2 z_0),$$

$$b_{22} = (2 x_0 z_1 - 2 x_1 z_0 - x_0 z_2 + x_2 z_0),$$

$$b_{23} = (2 x_1 y_0 - 2 x_0 y_1 + x_0 y_2 - x_2 y_0),$$

$$b_{31} = y_0 z_1 - y_1 z_0,$$

$$b_{32} = x_1 z_0 - x_0 z_1,$$

$$b_{33} = x_0 y_1 - x_1 y_0.$$

Normal vector field of a cubic Bezier curve is a $4th$ order Bezier curve and it has the matrix representation as in

$$
\begin{aligned}
N(t) &= \frac{6}{\eta m}\begin{bmatrix} t^4 \\ t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} n_{11} & n_{12} & n_{13} \\ n_{21} & n_{22} & n_{23} \\ n_{31} & n_{32} & n_{33} \\ n_{41} & n_{42} & n_{43} \\ n_{51} & n_{52} & n_{53} \end{bmatrix} \\
&= \frac{6}{\eta m}\begin{bmatrix} t^4 \\ t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} N_0 \\ N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix} \\
&= \frac{6}{\eta m}\left(N_0 t^4 + N_1 t^3 + N_2 t^2 + N_3 t + N_4\right)
\end{aligned}
$$

where

$$n_{11} = b_{12}d_{13} - b_{13}d_{12},$$

$$n_{21} = b_{12}d_{23} - b_{13}d_{22} + b_{22}d_{13} - b_{23}d_{12},$$

$$n_{31} = b_{12}d_{33} - b_{13}d_{32} + b_{22}d_{23} - b_{23}d_{22} + b_{32}d_{13} - b_{33}d_{12},$$

$$n_{41} = b_{22}d_{33} - b_{23}d_{32} + b_{32}d_{23} - b_{33}d_{22},$$

$$n_{51} = b_{32}d_{33} - b_{33}d_{32},$$

$$n_{12} = b_{11}d_{13} - b_{13}d_{11},$$

$$n_{22} = -b_{11}d_{23} - b_{21}d_{13} + b_{13}d_{21} + b_{23}d_{11},$$

$$n_{32} = b_{23}d_{21} + b_{33}d_{11} - b_{11}d_{33} - b_{21}d_{23} + b_{13}d_{31} - b_{31}d_{13},$$

$$n_{42} = -b_{21}d_{33} - b_{31}d_{23} + b_{23}d_{31} + b_{33}d_{21},$$

$$n_{52} = -b_{31}d_{33} + b_{33}d_{31},$$

$$n_{13} = b_{11}d_{12} - b_{12}d_{11},$$

$$n_{23} = b_{11}d_{22} - b_{12}d_{21} + b_{21}d_{12} - b_{22}d_{11},$$

$$n_{33} = b_{11}d_{32} - b_{12}d_{31} + b_{21}d_{22} - b_{22}d_{21} + b_{31}d_{12} - b_{32}d_{11},$$

$$n_{43} = b_{21}d_{32} - b_{22}d_{31} + b_{31}d_{22} - b_{32}d_{21},$$

$$n_{53} = b_{31}d_{32} - b_{32}d_{31}.$$

The first and second curvatures of a cubic Bezier curve by using the matrix representation are

$$\begin{aligned}
\kappa(t) &= \frac{6}{\eta^3}\begin{bmatrix} t^4 \\ t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} b_{11}^2 + b_{12}^2 + b_{13}^2 \\ 2b_{11}b_{21} + 2b_{12}b_{22} + 2b_{13}b_{23} \\ 2b_{11}b_{31} + 2b_{12}b_{32} + 2b_{13}b_{33} + b_{21}^2 + b_{22}^2 + b_{23}^2 \\ 2b_{21}b_{31} + 2b_{22}b_{32} + 2b_{23}b_{33} \\ b_{31}^2 + b_{32}^2 + b_{33}^2 \end{bmatrix} \\
&= \frac{6}{\eta^3}\begin{bmatrix} t^4 \\ t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \end{bmatrix} \\
&= \frac{6}{\eta^3}\left(C_1t^4 + C_2t^3 + C_3t^2 + C_4t + C_5\right)
\end{aligned}$$

where

$C_1 = b_{11}^2 + b_{12}^2 + b_{13}^2,$

$C_2 = 2b_{11}b_{21} + 2b_{12}b_{22} + 2b_{13}b_{23},$

$C_3 = 2b_{11}b_{31} + 2b_{12}b_{32} + 2b_{13}b_{33} + b_{21}^2 + b_{22}^2 + b_{23}^2,$

$C_4 = 2b_{21}b_{31} + 2b_{22}b_{32} + 2b_{23}b_{33},$

$C_5 = b_{31}^2 + b_{32}^2 + b_{33}^2,$

and

$$\tau(t) = \frac{x_0y_1z_2 - x_0y_2z_1 - x_1y_0z_2 + x_1y_2z_0 + x_2y_0z_1 - x_2y_1z_0}{m^2}.$$

2. Mannheim partner of a cubic Bezier curve

Mannheim curve was firstly defined by A. Mannheim in 1878. A curve is called a Mannheim curve if and only if $\frac{\kappa}{\kappa^2 + \tau^2}$ is a nonzero constant, κ is the curvature and τ is the torsion. Mannheim curve was redefined as; if the principal normal vector of first curve and binormal vector of second curve are linearly dependent, then first curve is called Mannheim curve, and the second curve is called Mannheim partner curve by Liu and Wang. As a result they called these new curves as Mannheim partner curves. For more detail see

[10]. $\alpha^*(t) = \alpha(t) + \mu(t) B^*(t)$, $N = B^*$. Hence $\alpha^*(t) = \alpha(t) + \mu(t) N(t)$. We know for a Mannheim curve α, that μ is constant.

Since $\dfrac{d\alpha^*}{dt} = \eta T + \dot{\mu}(t) N(t) + \eta\mu(-\kappa T + \tau B)$, $\dfrac{d\alpha^*}{dt} \perp B^*$ and $\dfrac{d\alpha^*}{dt} \perp N$, we get μ is constant. Also $dtds^* = \dfrac{1}{\cos\theta}$ and $|\mu|$ is the distance between the curves α and α^*. Also we can write $\dfrac{dt}{ds^*} = \dfrac{1}{\sqrt{1+\mu\tau}}$.

Theorem 2.1. *The Mannheim partner of a cubic Bezier curve has the following matrix representation*

$$\alpha^* = \begin{bmatrix} t^4 \\ t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} \frac{6\mu}{\eta m} N_0 \\ \frac{6\mu}{\eta m} N_1 + P_3 + 3P_1 - 3P_2 - P_0 \\ \frac{6\mu}{\eta m} N_2 + 3P_2 - 6P_1 + 3P_0 \\ \frac{6\mu}{\eta m} N_3 + 3P_1 - 3P_0 \\ \frac{6\mu}{\eta m} N_4 + P_0 \end{bmatrix}.$$

Proof. Let $\alpha^* = \alpha(t) + \mu N$ be Mannheim partner of a cubic Bezier curve $\alpha(t)$, hence

$$\alpha^* = \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix} + \frac{6\mu}{\eta m} \begin{bmatrix} t^4 \\ t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} N_0 \\ N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix}$$

$$\begin{aligned} &= P_2\left(3t^2 - 3t^3\right) + t^3 P_3 + P_0\left(-t^3 + 3t^2 - 3t + 1\right) + P_1\left(3t^3 - 6t^2 + 3t\right) \\ &\quad + \frac{6}{m}\frac{\mu}{\eta} N_4 + \frac{6}{m} t \frac{\mu}{\eta} N_3 + \frac{6}{m} t^2 \frac{\mu}{\eta} N_2 + \frac{6}{m} t^3 \frac{\mu}{\eta} N_1 + \frac{6}{m} t^4 \frac{\mu}{\eta} N_0 \end{aligned}$$

$$\begin{aligned} &= t^4 \frac{6\mu}{m\eta} N_0 + t^3\left(\frac{6\mu}{m\eta} N_1 + P_3 + 3P_1 - 3P_2 - P_0\right) \\ &\quad + t^2\left(\frac{6\mu}{m\eta} N_2 + 3P_2 - 6P_1 + 3P_0\right) + t\left(\frac{6\mu}{m\eta} N_3 + 3P_1 - 3P_0\right) \\ &\quad + \frac{6\mu}{\eta m} N_4 + P_0. \end{aligned}$$

So we can write this as in the following matrix form

$$\alpha^* = \begin{bmatrix} t^4 \\ t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} \frac{6\mu}{\eta m} N_0 \\ \frac{6\mu}{\eta m} N_1 + P_3 + 3P_1 - 3P_2 - P_0 \\ \frac{6\mu}{\eta m} N_2 + 3P_2 - 6P_1 + 3P_0 \\ \frac{6\mu}{\eta m} N_3 + 3P_1 - 3P_0 \\ \frac{6\mu}{\eta m} N_4 + P_0 \end{bmatrix}.$$

Theorem 2.2. *The Mannheim partner of a cubic Bezier curve is a 4^{th} order Bezier curve with constant speed. It has the control points P_0^*, P_1^*, P_2^*, P_3^* and P_4^* based on the control points of the cubic Bezier curve, as in the following way, where η, m are constants,*

$$\begin{bmatrix} P_0^* \\ P_1^* \\ P_2^* \\ P_3^* \\ P_4^* \end{bmatrix} = \begin{bmatrix} P_0 + \frac{6\mu}{m\eta} N_4 \\ \frac{1}{4}P_0 + \frac{3}{4}P_1 + \frac{3\mu}{2m\eta} N_3 + \frac{6\mu}{m\eta} N_4 \\ \frac{1}{2}P_1 + \frac{1}{2}P_2 + \frac{\mu}{m\eta} N_2 + \frac{3\mu}{m\eta} N_3 + \frac{6\mu}{m\eta} N_4 \\ \frac{3}{4}P_2 + \frac{1}{4}P_3 + \frac{3\mu}{2m\eta} N_1 + \frac{3\mu}{m\eta} N_2 + \frac{9\mu}{2m\eta} N_3 + \frac{6\mu}{m\eta} N_4 \\ P_3 + \frac{6\mu}{m\eta} N_0 + \frac{6\mu}{m\eta} N_1 + \frac{6\mu}{m\eta} N_2 + \frac{6\mu}{m\eta} N_3 + \frac{6\mu}{m\eta} N_4 \end{bmatrix}.$$

Proof. Let P_0^*, P_1^*, P_2^*, P_3^* and P_4^* be the control points of 4th order Bezier curve which is Mannheim partner of a cubic Bezier curve, so we can write

$$\begin{bmatrix} 1 & -4 & 6 & -4 & 1 \\ -4 & 12 & -12 & 4 & 0 \\ 6 & -12 & 6 & 0 & 0 \\ -4 & 4 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_0^* \\ P_1^* \\ P_2^* \\ P_3^* \\ P_4^* \end{bmatrix} = \begin{bmatrix} \frac{6\mu}{m\eta} N_0 \\ +\frac{6\mu}{m\eta} N_1 + P_3 - 3P_2 - P_0 + 3P_1 \\ +\frac{6\mu}{m\eta} N_2 + 3P_2 + 3P_0 - 6P_1 \\ +\frac{6\mu}{m\eta} N_3 + 3P_1 - 3P_0 \\ +\frac{6\mu}{m\eta} N_4 + P_0 \end{bmatrix}.$$

By using the following inverse matrix

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & \frac{1}{4} & 1 \\ 0 & 0 & \frac{1}{6} & \frac{1}{2} & 1 \\ 0 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

we obtain

$$\begin{bmatrix} P_0^* \\ P_1^* \\ P_2^* \\ P_3^* \\ P_4^* \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & \frac{1}{4} & 1 \\ 0 & 0 & \frac{1}{6} & \frac{1}{2} & 1 \\ 0 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{6\mu}{m\eta} N_0 \\ +\frac{6\mu}{m\eta} N_1 + P_3 - 3P_2 - P_0 + 3P_1 \\ +\frac{6\mu}{m\eta} N_2 + 3P_2 + 3P_0 - 6P_1 \\ +\frac{6\mu}{m\eta} N_3 + 3P_1 - 3P_0 \\ +\frac{6\mu}{m\eta} N_4 + P_0 \end{bmatrix}$$

which completes the proof.

Furthermore, the equality $\dfrac{\kappa}{\kappa^2+\tau^2}$ =constant is known as the offset property, for some non-zero constant. For some function μ, since N and B^* are linearly dependent, equation can be rewritten as $\alpha^*(t) = \alpha(t) - \mu N(t)$ where $\mu = \dfrac{-\kappa}{\kappa^2+\tau^2}$. Frenet-Serret apparatus of Mannheim partner curve α^*, based on Frenet-Serret vectors of Mannheim curve α are

$$\begin{aligned} T^* &= \cos\theta\ T - \sin\theta\ B, \\ N^* &= \sin\theta\ T + \cos\theta B, \\ B^* &= N, \\ \mu &= \frac{-\kappa}{\kappa^2+\tau^2} \end{aligned}$$

where $\theta = \sphericalangle(T, T^*)$.

Theorem 2.3. *Tangent vector field of Mannheim partner of a cubic Bezier curve based on the angle θ is*

$$T^* = \begin{bmatrix} t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} \frac{1}{\eta}(9P_1 - 3P_0 - 9P_2 + 3P_3)\cos\theta - \frac{6}{m} B_1 \sin\theta \\ \frac{1}{\eta}(6P_0 - 12P_1 + 6P_2)\cos\theta - \frac{6}{m} B_2 \sin\theta \\ \frac{1}{\eta} 3(P_1 - P_0)\cos\theta - \frac{6}{m} B_3 \sin\theta \end{bmatrix}.$$

Proof. Since $T^* = \cos\theta\ T - \sin\theta\ B$, we have

$$\begin{aligned} T^* &= \frac{1}{\eta}\left(Q_0\left(t^2 - 2t + 1\right) - Q_1\left(2t^2 - 2t\right) + t^2 Q_2\right)\cos\theta\ - \left(\frac{6}{m}\left(B_1 t^2 + B_2 t + B_3\right)\right)\sin\theta \\ &= \frac{1}{\eta}\left(t^2 Q_0 \cos\theta - 2t^2 Q_1 \cos\theta + t^2 Q_2 \cos\theta\right) - \frac{6}{m} t^2 B_1 \sin\theta \\ &\quad + \frac{1}{\eta}\left(-2Q_0 t\cos\theta + 2Q_1 t\cos\theta\right) - \frac{6}{m} t B_2 \sin\theta \\ &\quad + \frac{1}{\eta} Q_0 \cos\theta - \frac{6}{m} B_3 \sin\theta. \end{aligned}$$

Therefore, based on the control points Q_0, Q_1, Q_2, the following matrix representation can be written as

$$T^* = \begin{bmatrix} t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} \frac{1}{\eta}(Q_0 - 2Q_1 + Q_2)\cos\theta - \frac{6}{m}B_1 \sin\theta \\ \frac{1}{\eta}(-2Q_0 + 2Q_1)\cos\theta - \frac{6}{m}B_2 \sin\theta \\ \frac{1}{\eta}Q_0 \cos\theta - \frac{6}{m}B_3 \sin\theta \end{bmatrix}.$$

Also it can be written in the following matrix representation, based on the control points P_0, P_1, P_2, P_3

$$T^* = \begin{bmatrix} t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} \frac{1}{\eta}(3(P_1 - P_0) - 6(P_2 - P_1) + 3(P_3 - P_2))\cos\theta - \frac{6}{m}B_1 \sin\theta \\ \frac{1}{\eta}(-6(P_1 - P_0) + 6(P_2 - P_1))\cos\theta - \frac{6}{m}B_2 \sin\theta \\ \frac{1}{\eta}3(P_1 - P_0)\cos\theta - \frac{6}{m}B_3 \sin\theta \end{bmatrix}$$

$$= \begin{bmatrix} t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} \frac{1}{\eta}(9P_1 - 3P_0 - 9P_2 + 3P_3)\cos\theta - \frac{6}{m}B_1 \sin\theta \\ \frac{1}{\eta}(6P_0 - 12P_1 + 6P_2)\cos\theta - \frac{6}{m}B_2 \sin\theta \\ \frac{1}{\eta}3(P_1 - P_0)\cos\theta - \frac{6}{m}B_3 \sin\theta \end{bmatrix}.$$

Corollary 2.1. *Tangent vector field of Mannheim partner can be written as in the following way where η, m are constants*

$$T^* = \begin{bmatrix} t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{m\eta}(mQ_0 \cos\theta - 6\eta B_3 \sin\theta) \\ -\frac{1}{m\eta}(3\eta B_2 \sin\theta - mQ_1 \cos\theta + 6\eta B_3 \sin\theta) \\ -\frac{1}{m\eta}(6\eta B_1 \sin\theta - mQ_2 \cos\theta + 6\eta B_2 \sin\theta + 6\eta B_3 \sin\theta) \end{bmatrix}.$$

Proof. As a quadratic Bezier curve, tangent vector field of Mannheim partner of a cubic Bezier curve with the control points Q_0^*, Q_1^*, Q_2^* is

$$T^* = \begin{bmatrix} t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} Q_0^* \\ Q_1^* \\ Q_2^* \end{bmatrix}.$$

Hence, by using the inverse matrix the control points are

$$\begin{bmatrix} Q_0^* \\ Q_1^* \\ Q_2^* \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & \frac{1}{2} & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\eta}(Q_0 - 2Q_1 + Q_2)\cos\theta - \frac{6}{m}B_1 \sin\theta \\ \frac{1}{\eta}(-2Q_0 + 2Q_1)\cos\theta - \frac{6}{m}B_2 \sin\theta \\ \frac{1}{\eta}Q_0 \cos\theta - \frac{6}{m}B_3 \sin\theta \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{m\eta}(mQ_0 \cos\theta - 6\eta B_3 \sin\theta) \\ -\frac{1}{m\eta}(3\eta B_2 \sin\theta - mQ_1 \cos\theta + 6\eta B_3 \sin\theta) \\ -\frac{1}{m\eta}(6\eta B_1 \sin\theta - mQ_2 \cos\theta + 6\eta B_2 \sin\theta + 6\eta B_3 \sin\theta) \end{bmatrix}.$$

Theorem 2.4. *Normal vector field of Mannheim partner of a cubic Bezier curve based on the angle θ is*

$$N^* = \begin{bmatrix} t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} \frac{1}{\eta}(9P_1 - 3P_0 - 9P_2 + 3P_3)\sin\theta + \frac{6}{m}B_1\cos\theta \\ +\frac{1}{\eta}(6P_0 - 12P_1 + 6P_2)\sin\theta + \frac{6}{m}B_2\cos\theta \\ +\frac{1}{\eta}3(P_1 - P_0)\sin\theta + \frac{6}{m}B_3\cos\theta \end{bmatrix}.$$

Proof. Since $N^* = \sin\theta\, T + \cos\theta\, B$, we have

$$\begin{aligned} N^* &= \frac{1}{\eta}\left(Q_0\left(t^2 - 2t + 1\right) - Q_1\left(2t^2 - 2t\right) + t^2 Q_2\right)\sin\theta + \frac{6}{m}\left(B_1 t^2 + B_2 t + B_3\right)\cos\theta \\ &= \frac{1}{\eta}\left(t^2 Q_0 \sin\theta - 2t^2 Q_1 \sin\theta + t^2 Q_2 \sin\theta\right) + \frac{6}{m}t^2 B_1\cos\theta \\ &+ \frac{1}{\eta}\left(-2tQ_0\sin\theta + 2tQ_1\sin\theta\right) + \frac{6}{m}tB_2\cos\theta \\ &+ \frac{1}{\eta}Q_0\sin\theta + \frac{6}{m}B_3\cos\theta. \end{aligned}$$

It can be written in the following matrix representation, based on the control points Q_0, Q_1, Q_2

$$N^* = \begin{bmatrix} t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} \frac{1}{\eta}(Q_0 - 2Q_1 + Q_2)\sin\theta + \frac{6}{m}B_1\cos\theta \\ +\frac{1}{\eta}(-2Q_0 + 2Q_1)\sin\theta + \frac{6}{m}B_2\cos\theta \\ +\frac{1}{\eta}Q_0\sin\theta + \frac{6}{m}B_3\cos\theta \end{bmatrix}.$$

Also it can be written in the following matrix representation, based on the control points P_0, P_1, P_2, P_3

$$N^* = \begin{bmatrix} t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} \frac{1}{\eta}(9P_1 - 3P_0 - 9P_2 + 3P_3)\sin\theta + \frac{6}{m}B_1\cos\theta \\ +\frac{1}{\eta}(6P_0 - 12P_1 + 6P_2)\sin\theta + \frac{6}{m}B_2\cos\theta \\ +\frac{1}{\eta}3(P_1 - P_0)\sin\theta + \frac{6}{m}B_3\cos\theta \end{bmatrix}.$$

This completes the proof.

Corollary 2.2. *Normal vector field of Mannheim partner of a cubic Bezier can be written as in the following way, where η, m are constants*

$$N^* = \begin{bmatrix} t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{m\eta}(mQ_0\sin\theta + 6\eta B_3\cos\theta) \\ \frac{1}{m\eta}(mQ_1\sin\theta + 3\eta B_2\cos\theta + 6\eta B_3\cos\theta) \\ \frac{1}{m\eta}(mQ_2\sin\theta + 6\eta B_1\cos\theta + 6\eta B_2\cos\theta + 6\eta B_3\cos\theta) \end{bmatrix}.$$

Proof. As a quadratic Bezier curve normal vector field of Mannheim partner of a cubic Bezier curve with the control points N_0^*, N_1^*, N_2^* is

$$N^* = \begin{bmatrix} t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} N_0^* \\ N_1^* \\ N_2^* \end{bmatrix}.$$

Hence, using the inverse matrix the control points are

$$\begin{bmatrix} N_0^* \\ N_1^* \\ N_2^* \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & \frac{1}{2} & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\eta}\left(Q_0 - 2Q_1 + Q_2\right)\sin\theta + \frac{6}{m}B_1\cos\theta \\ +\frac{1}{\eta}\left(-2Q_0 + 2Q_1\right)\sin\theta + \frac{6}{m}B_2\cos\theta \\ +\frac{1}{\eta}Q_0\sin\theta + \frac{6}{m}B_3\cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{m\eta}\left(mQ_0\sin\theta + 6\eta B_3\cos\theta\right) \\ \frac{1}{m\eta}\left(mQ_1\sin\theta + 3\eta B_2\cos\theta + 6\eta B_3\cos\theta\right) \\ \frac{1}{m\eta}\left(mQ_2\sin\theta + 6\eta B_1\cos\theta + 6\eta B_2\cos\theta + 6\eta B_3\cos\theta\right) \end{bmatrix}.$$

This completes the proof.

Theorem 2.5. *Binormal vector field of Mannheim partner of a cubic Bezier curve based on the angle θ are*

$$\begin{aligned} B^* &= N \\ &= \frac{6\mu}{\eta m}\left(N_0t^4 + N_1t^3 + N_2t^2 + N_3t + N_4\right). \end{aligned}$$

Theorem 2.6. *The curvature and the torsion of Mannheim partner of a cubic Bezier curve based on the angle θ are have the following equalities,*

Proof. Since

$$\kappa(t) = \frac{6}{\eta^3}\left(C_1t^4 + C_2t^3 + C_3t^2 + C_4t + C_5\right)$$

where

$C_1 = b_{11}^2 + b_{12}^2 + b_{13}^2,$

$C_2 = 2b_{11}b_{21} + 2b_{12}b_{22} + 2b_{13}b_{23},$

$C_3 = 2b_{11}b_{31} + 2b_{12}b_{32} + 2b_{13}b_{33} + b_{21}^2 + b_{22}^2 + b_{23}^2,$

$C_4 = 2b_{21}b_{31} + 2b_{22}b_{32} + 2b_{23}b_{33},$

$C_5 = b_{31}^2 + b_{32}^2 + b_{33}^2$

and

$$\tau(t) = \frac{x_0y_1z_2 - x_0y_2z_1 - x_1y_0z_2 + x_1y_2z_0 + x_2y_0z_1 - x_2y_1z_0}{m^2}.$$

The curvature and the torsion have the following equalities of Mannheim partner of a cubic Bezier curve;

$$\kappa^* = -\frac{d\theta}{ds^*} = \frac{\dot{\theta}}{\cos\theta},$$

$$\begin{aligned}\tau^* &= \frac{\kappa}{\mu\tau} \\ &= \frac{\frac{6}{\eta^3}\left(C_1t^4 + C_2t^3 + C_3t^2 + C_4t + C_5\right)}{\mu\left(\frac{x_0y_1z_2 - x_0y_2z_1 - x_1y_0z_2 + x_1y_2z_0 + x_2y_0z_1 - x_2y_1z_0}{m^2}\right)} \\ &= \frac{6m^2}{\mu\eta^3}\frac{C_1t^4 + C_2t^3 + C_3t^2 + C_4t + C_5}{x_0y_1z_2 - x_0y_2z_1 - x_1y_0z_2 + x_1y_2z_0 + x_2y_0z_1 - x_2y_1z_0}.\end{aligned}$$

Theorem 2.7. *Frenet vector fields $\{T^*, N^*, B^*\}$ of Mannheim partner of any cubic Bezier curve in E^3 are*

$$T^* = \frac{\begin{bmatrix} t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} \frac{(1-\mu\kappa)}{\eta}\left(9P_1 - 3P_0 - 9P_2 + 3P_3\right) + \frac{6\mu\tau}{m}B_1 \\ \frac{(1-\mu\kappa)}{\eta}\left(6P_0 - 12P_1 + 6P_2\right) + \frac{6\mu\tau}{m}B_2 \\ \frac{(1-\mu\kappa)}{\eta}3\left(P_1 - P_0\right) + \frac{6\mu\tau}{m}B_3 \end{bmatrix}}{\sqrt{(1-\mu\kappa)^2 + (\mu\tau)^2}},$$

$$N^* = \frac{\begin{bmatrix} t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} \frac{\mu\tau}{\eta}\left(9P_1 - 3P_0 - 9P_2 + 3P_3\right) - \frac{6(1-\mu\kappa)}{m}B_1 \\ \frac{\mu\tau}{\eta}\left(6P_0 - 12P_1 + 6P_2\right) - \frac{6(1-\mu\kappa)}{m}B_2 \\ \frac{\mu\tau}{\eta}3\left(P_1 - P_0\right) - \frac{6(1-\mu\kappa)}{m}B_3 \end{bmatrix}}{\sqrt{(1-\mu\kappa)^2 + (\mu\tau)^2}},$$

$$B^* = \frac{6}{\eta m}\begin{bmatrix} t^4 \\ t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} N_0 \\ N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix}.$$

Proof. Let a curve α^* be a Mannheim partner of α with Frenet-Serret apparatus, then

$$T^* = \frac{(1-\mu\kappa)\,T + \mu\tau B}{\sqrt{(1-\mu\kappa)^2 + (\mu\tau)^2}},$$

$$N^* = \frac{\mu\tau T - (1-\mu\kappa)\,B}{\sqrt{(1-\mu\kappa)^2 + (\mu\tau)^2}},$$

$$B^* = N,$$

$$\frac{dt}{ds^*} = \frac{1}{\eta\sqrt{(1-\mu\kappa)^2 + (\mu\tau)^2}}.$$

Tangent vector field of Mannheim partner of a cubic Bezier curve is

$$\begin{aligned} T^* &= \frac{\frac{(1-\mu\kappa)}{\eta}\left(Q_0\left(t^2-2t+1\right) - Q_1\left(2t^2-2t\right) + t^2Q_2\right) + \mu\tau\frac{6}{m}\left(B_1t^2 + B_2t + B_3\right)}{\sqrt{(1-\mu\kappa)^2 + (\mu\tau)^2}} \\ &= \frac{\frac{(1-\mu\kappa)}{\eta}\left(Q_0 - 2tQ_0 + 2tQ_1 + t^2Q_0 - 2t^2Q_1 + t^2Q_2\right) + \left(\frac{6\mu\tau}{m}B_1t^2 + \frac{6\mu\tau}{m}B_2t + \frac{6\mu\tau}{m}B_3\right)}{\sqrt{(1-\mu\kappa)^2 + (\mu\tau)^2}}. \end{aligned}$$

Hence its matrix representation, based on the control points Q_0, Q_1, Q_2 is

$$T^* = \frac{\begin{bmatrix} t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} \frac{(1-\mu\kappa)}{\eta}\left(Q_0 - 2Q_1 + Q_2\right) + \frac{6\mu\tau}{m}B_1 \\ \frac{(1-\mu\kappa)}{\eta}\left(-2Q_0 + 2Q_1\right) + \frac{6\mu\tau}{m}B_2 \\ \frac{(1-\mu\kappa)}{\eta}\left(Q_0\right) + \frac{6\mu\tau}{m}B_3 \end{bmatrix}}{\sqrt{(1-\mu\kappa)^2 + (\mu\tau)^2}},$$

and based on the control points P_0, P_1, P_2, P_3 is

$$T^* = \frac{\begin{bmatrix} t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} \frac{(1-\mu\kappa)}{\eta}\left(9P_1 - 3P_0 - 9P_2 + 3P_3\right) + \frac{6\mu\tau}{m}B_1 \\ \frac{(1-\mu\kappa)}{\eta}\left(6P_0 - 12P_1 + 6P_2\right) + \frac{6\mu\tau}{m}B_2 \\ \frac{(1-\mu\kappa)}{\eta}3\left(P_1 - P_0\right) + \frac{6\mu\tau}{m}B_3 \end{bmatrix}}{\sqrt{(1-\mu\kappa)^2 + (\mu\tau)^2}}.$$

So the normal vector field of Mannheim partner of a cubic Bezier curve is

$$\begin{aligned} N^* &= \frac{\mu\tau T - (1-\mu\kappa)\,B}{\sqrt{(1-\mu\kappa)^2 + (\mu\tau)^2}} \\ &= \frac{\frac{\mu\tau}{\eta}\left(Q_0\left(t^2-2t+1\right) - Q_1\left(2t^2-2t\right) + t^2Q_2\right) - (1-\mu\kappa)\frac{6}{m}\left(B_1t^2 + B_2t + B_3\right)}{\sqrt{(1-\mu\kappa)^2 + (\mu\tau)^2}}. \end{aligned}$$

Hence its matrix representation is

$$N^{*}=\frac{\begin{bmatrix} t^{2}\\ t\\ 1\end{bmatrix}^{T}\begin{bmatrix} \frac{\mu\tau}{\eta}\left(Q_{0}-2Q_{1}+t^{2}Q_{2}\right)-\frac{6\left(1-\mu\kappa\right)}{m}B_{1}\\ \frac{\mu\tau}{\eta}\left(-2Q_{0}+2Q_{1}\right)-\frac{6\left(1-\mu\kappa\right)}{m}B_{2}\\ \frac{\mu\tau}{\eta}Q_{0}-\frac{6\left(1-\mu\kappa\right)}{m}B_{3}\end{bmatrix}}{\sqrt{\left(1-\mu\kappa\right)^{2}+\left(\mu\tau\right)^{2}}},$$

$$N^{*}=\frac{\begin{bmatrix} t^{2}\\ t\\ 1\end{bmatrix}^{T}\begin{bmatrix} \frac{\mu\tau}{\eta}\left(9P_{1}-3P_{0}-9P_{2}+3P_{3}\right)-\frac{6\left(1-\mu\kappa\right)}{m}B_{1}\\ \frac{\mu\tau}{\eta}\left(6P_{0}-12P_{1}+6P_{2}\right)-\frac{6\left(1-\mu\kappa\right)}{m}B_{2}\\ \frac{\mu\tau}{\eta}3\left(P_{1}-P_{0}\right)-\frac{6\left(1-\mu\kappa\right)}{m}B_{3}\end{bmatrix}}{\sqrt{\left(1-\mu\kappa\right)^{2}+\left(\mu\tau\right)^{2}}}.$$

Also, since $B^{*}=N$, its matrix representation is trivial.

Theorem 2.8. *The second curvature τ^{*} of Mannheim partner of any cubic Bezier curve is*

$$\tau^{*}=\frac{\sqrt{\begin{array}{c}\left(\frac{x_{0}y_{1}z_{2}-x_{0}y_{2}z_{1}-x_{1}y_{0}z_{2}+x_{1}y_{2}z_{0}+x_{2}y_{0}z_{1}-x_{2}y_{1}z_{0}}{m^{2}}\right)^{2}\\ -\left(\frac{6}{\eta^{3}}C_{1}t^{4}+C_{2}t^{3}+C_{3}t^{2}+C_{4}t+C_{5}\right)^{2}\end{array}}}{\sqrt{\left(1-\mu\kappa\right)^{2}+\left(\mu\tau\right)^{2}}}.$$

Proof. Since $\frac{dB^{*}}{ds^{*}}=\frac{dB^{*}}{dt}\frac{dt}{ds^{*}}=-\tau^{*}N^{*}$ and $\langle -\tau^{*}N^{*},-\tau^{*}N^{*}\rangle=\tau^{*2}$ we have

$$\tau^{*}=\frac{\sqrt{\tau^{2}-\kappa^{2}}}{\sqrt{\left(1-\mu\kappa\right)^{2}+\left(\mu\tau\right)^{2}}},\quad \tau>\kappa.$$

By using $\kappa\left(t\right)$ and $\tau\left(t\right)$ of any cubic Bezier curve, we get the proof.

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