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POINTWISE SEMI-SLANT SUBMERSIONS WHOSE TOTAL MANIFOLDS ARE LOCALLY PRODUCT RIEMANNIAN MANIFOLDS

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Abstract. In this paper, we study pointwise semi-slant submersions from locally product Riemannian manifolds onto Riemannian manifolds. We give example and necessary and sufficient conditions for the integrability and totally geodesicness of all distributions which are mentioned in the definition of the pointwise semi-slant submersion. Moreover, we give a characterization theorem for the proper pointwise semi-slant submersions with totally umbilical fibers and first variational formula on the fibers of a pointwise semi-slant submersion. In the view of that formula, finally we obtain necessary and sufficient condition which is new approach to check the harmonicity of a pointwise semi-slant submersion.

1. Introduction

The theory of submanifolds is a very productive area in differential geometry. In the virtue of a smooth map between Riemannian manifolds, a submersion is one of the some ways to get a submanifold. Riemannian submersions were studied first by O'Neill [19] and Gray [11]. Later Riemannian submersions considered with differentiable structures of manifolds. Watson [32] defined submersions between almost Hermitian manifolds by taking account of almost complex structure of total manifold.

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In this case, the vertical and horizontal distributions are invariant. Afterwards, almost Hermitian submersions have been extensively studied different subclasses of almost Hermitian manifolds, for example; see [9].

The notion of anti-invariant submersion from an almost Hermitian manifold onto a Riemannian manifold was first defined by Şahin [22]. He also studied such submersions from Kählerian manifolds onto Riemannian manifolds. In this case, the fibers are anti-invariant with respect to the almost complex structure of the total manifold of the submersion. Moreover, he studied slant [24] and semi-invariant submersions [26] under new conditions. A Lagrangian submersion [22, 28] is a special case of an anti-invariant Riemannian submersion such that the complex or almost complex structure of the total manifold reverses the vertical and horizontal distributions to each other.

Recently, it has been defined and studied that there are several new Riemannian submersions between different types of manifolds; such as slant submersions [24, 14], semi-invariant submersions [21, 26], generic submersions [2, 5], semi-slant submersions [20], pointwise slant submersions [18], anti-invariant submersions [13, 30], hemi-slant submersions [4, 29], para-contact para-complex semi-Riemannian submersions [15, 16], conformal semi-slant submersions [1], semi-slant $\xi^{\perp}-$ Riemannian submersions [3]. We note that some of these submersions have been extended to the subclasses of almost contact manifolds, for example; see [8, 27]. Recent developments on the theory of submersion could be found in the book, [23].

In the present paper, we consider pointwise semi-slant Riemannian submersions from locally product Riemannian manifolds onto Riemannian manifolds. The paper is organized as follows. In section 2, we recall the fundamental equations and notions of a Riemannian submersion. In section 3, we will provide a brief view of locally product Riemannian manifolds. We study on pointwise semi-slant submersions and give necessary and sufficient conditions for the integrability and geodesicness of the distributions which are mentioned in section 4. In particular, we give a characterization theorem for the totally umbilical fibers of the pointwise semi-slant submersions and some results for pointwise semi-slant submersions with parallel canonical structures in section 5. The last section of this paper includes a new notion. We define the first variational formula on the fibers of the pointwise semi-slant submersions. By the virtue of this formula, we prove a theorem for the harmonicity of such submersions and give some interesting results.

2. Riemannian submersions

In this section, we give necessary background for Riemannian submersions.

Let (M, g) and (N, g_N) be Riemannian manifolds, where $dim(M) > dim(N)$. A surjective mapping $\pi : (M, g) \to (N, g_N)$ is called a *Riemannian submersion* [19] if

(S1) π has maximal rank, and

(S2) π_*, restricted to $ker\pi_*^{\perp}$, is a linear isometry.

In this case, for each $q \in N$, $\pi_q^{-1} = \pi^{-1}(q)$ is a k-dimensional submanifold of M and called a *fiber*, where $k = dim(M) - dim(N)$. A vector field on M is called *vertical* (resp. *horizontal*) if it is always tangent (resp. orthogonal) to fibers. A vector field X on M is called *basic* if X is horizontal and π-related to a vector field X_* on N, i.e., $\pi_* X_p = X_{*\pi(p)}$ for all $p \in M$. We will denote by $\mathcal{V}$ and $\mathcal{H}$ the projections on the vertical distribution $ker\pi_*$, and the horizontal distribution $(ker\pi_*)^{\perp}$, respectively. As usual, the manifold (M, g) is called *total manifold* and the manifold (N, g_N) is called *base manifold* of the submersion $\pi : (M, g) \to (N, g_N)$. The geometry of Riemannian submersions is characterized by O'Neill's tensors $\mathcal{T}$ and $\mathcal{A}$, defined as follows:

$$\mathcal{T}_{\bar{U}}\bar{V} = \mathcal{V}\nabla_{\mathcal{V}\bar{U}}\mathcal{H}\bar{V} + \mathcal{H}\nabla_{\mathcal{V}\bar{U}}\mathcal{V}\bar{V}, \tag{2.1}$$

$$\mathcal{A}_{\bar{U}}\bar{V} = \mathcal{V}\nabla_{\mathcal{H}\bar{U}}\mathcal{H}\bar{V} + \mathcal{H}\nabla_{\mathcal{H}\bar{U}}\mathcal{V}\bar{V} \tag{2.2}$$

for any vector fields $\bar{U}$ and $\bar{V}$ on M, where ∇ is the Levi-Civita connection of g. It is easy to see that $\mathcal{T}_{\bar{U}}$ and $\mathcal{A}_{\bar{V}}$ are skew-symmetric operators on the tangent bundle of M reversing the vertical and the horizontal distributions. We now summarize the properties of the tensor fields $\mathcal{T}$ and $\mathcal{A}$. Let V, W be vertical and X, Y be horizontal vector fields on M, then we have

$$\mathcal{T}_V W = \mathcal{T}_W V, \tag{2.3}$$

$$\mathcal{A}_X Y = -\mathcal{A}_Y X = \frac{1}{2}\mathcal{V}[X, Y]. \tag{2.4}$$

On the other hand, from (2.1) and (2.4), we obtain

$$\nabla_V W = \mathcal{T}_V W + \hat{\nabla}_V W, \tag{2.5}$$

$$\nabla_V X = \mathrm{T}_V X + \mathcal{H}\nabla_V X, \tag{2.6}$$

$$\nabla_X V = \mathcal{A}_X V + \mathcal{V}\nabla_X V, \tag{2.7}$$

$$\nabla_X Y = \mathcal{H}\nabla_X Y + \mathcal{A}_X Y, \tag{2.8}$$

where $\hat{\nabla}_V W = \mathcal{V}\nabla_V W$. Moreover, if X is basic, then we have

$$\mathcal{H}\nabla_V X = \mathcal{A}_X V. \tag{2.9}$$

Remark 2.1. *In this paper, we accept all horizontal vector fields as basic vector fields.*

It is not difficult to observe that $\mathcal{T}$ acts on the fibers as the second fundamental form while $\mathcal{A}$ acts on the horizontal distribution and measures of the obstruction to the integrability of this distribution. For details on the Riemannian submersions, we refer to O'Neill's paper [19] and to the book [9].

Finally, we recall that the notion of the second fundamental form of a map between Riemannian manifolds. Let (M, g) and (N, g_N) be Riemannian manifolds and $\varphi : (M, g) \to (N, g_N)$ be a smooth map. Then, the second fundamental form of φ is given by

$$(\nabla\varphi_*)(E, F) = \nabla^{\varphi}_E \varphi_* F - \varphi_*(\nabla_E F)$$

for $E, F \in \Gamma(TM)$, where ∇^{φ} is the pull back connection and we denote for convenience by ∇ the Riemannian connections of the metrics g and g_N. It is well known that the second fundamental form is symmetric [6]. Moreover, φ is said to be *totally geodesic* if $(\nabla\varphi_*)(E, F) = 0$ for all $E, F \in \Gamma(TM)$, and φ is called a *harmonic* map if $trace(\nabla\varphi_*) = 0$ [6].

3. Locally Product Riemannian manifolds

Let M be an m-dimensional manifold with a tensor field of type (1,1) such that

$$F^2 = I, (F \neq \pm I)\ , \tag{3.10}$$

where I is the identity morphism on the tangent bundle TM of M. Then we say that M is an *almost product manifold* with almost product structure F. If an almost product manifold denoted by (M, F) admits a Riemannian metric g such that

$$g(F\bar{U}, F\bar{V}) = g(\bar{U}, \bar{V}) \tag{3.11}$$

for all $\bar{U}, \bar{V} \in \Gamma(TM)$, then M is called an *almost product Riemannian manifold.*

Next, we denote by ∇ the Riemannian connection with respect to g on M. We say that M is a *locally product Riemannian manifold*, (briefly, *l.p.R. manifold*) if we have

$$(\nabla_{\bar{U}}\ F)\bar{V} = 0 \tag{3.12}$$

for all $\bar{U}, \bar{V} \in \Gamma(TM)$ [33].

Finally, recall that, if a manifold M can be written as a product of two totally geodesic submanifolds of it, then M is called a *locally product* of two submanifolds.

4. Pointwise Semi-Slant submersions

In this section, we will define pointwise semi-slant submersion and study on geometry of it. Before we start, we remind the definition of pointwise slant submersion.

Definition 4.1. *(*[18]*) Let π be a Riemannian submersion from an almost Hermitian manifold (M, g, J) onto a Riemannian manifold (N, g_N). If, at each given point $p \in M$, the Wirtinger angle $\theta(V)$ between JV and the space $(ker\pi_*)_p$ is independent of the choice of the nonzero vector $V \in (ker\pi_*)$, then we say that π is a* pointwise slant submersion*. In this case, the angle θ can be regarded as a function on M, which is called the* slant function *of the pointwise slant submersion.*

Now, we define a new kind of submersion as in the following.

Definition 4.2. *Let (M, g, F) be a l.p.R. manifold and (N, g_N) be a Riemannian manifold. A Riemannian submersion $\pi : (M, g, F) \to (N, g_N)$ is called a* pointwise semi-slant Riemannian submersion*, if there is a distribution $\mathcal{D} \subset ker\pi_*$ such that*

$$ker\pi_* = \mathcal{D} \oplus \mathcal{D}_\theta\,, \quad F\mathcal{D} = \mathcal{D}\,, \tag{4.13}$$

where $\mathcal{D}_\theta$ is orthogonal complement of $\mathcal{D}$ in $ker\pi_*$ and the angle $\theta = \theta(X)$ between FX and the space $(\mathcal{D}_\theta)_p$ is independent of the choice of nonzero vector $X \in \Gamma((\mathcal{D}_\theta)_p)$ for $p \in M$ i.e.

θ is a function on M, which is called slant function of the pointwise semi-slant submersion. We say that π is proper *if the slant function is $\theta \neq 0$ and $\theta \neq \pi/2$.*

Remark 4.1. *From now on, instead of pointwise semi-slant Riemannian submersion, we will use briefly pointwise semi-slant submersion.*

In this case, for any $V \in \Gamma(ker\pi_*)$, we have

$$V = \mathcal{P}V + \mathcal{Q}V, \tag{4.14}$$

where $\mathcal{P}V \in \Gamma(\mathcal{D})$ and $\mathcal{Q}V \in \Gamma(\mathcal{D}_\theta)$.
For $V \in \Gamma(ker\pi_*)$, we have

$$FV = \phi V + \omega V, \tag{4.15}$$

where $\phi V \in \Gamma(ker\pi_*)$ and $\omega V \in \Gamma(ker\pi_*^\perp)$.
For $\xi \in \Gamma(ker\pi_*^\perp)$, we have

$$F\xi = \mathcal{B}\xi + \mathcal{C}\xi, \tag{4.16}$$

where $\mathcal{B}\xi \in ker\pi_*$ and $\mathcal{C}\xi \in (ker\pi_*^\perp)$.
For any $E \in \Gamma(TM)$, we obtain

$$E = \mathcal{V}E + \mathcal{H}E, \tag{4.17}$$

where $\mathcal{V}E \in \Gamma(ker\pi_*)$ and $\mathcal{H}E \in \Gamma(ker\pi_*^\perp)$.
Therefore, the horizontal distribution $(ker\pi_*)^\perp$ is decomposed as

$$ker\pi_*^\perp = \omega\mathcal{D}_\theta \oplus \mu\,, \tag{4.18}$$

where μ is the orthogonal complementary distribution of $\omega\mathcal{D}_\theta$ in $(ker\pi_*^\perp)$, and it is invariant with respect to F.

Example. Consider the Euclidean 6-space $\mathbb{R}^6$ with usual metric g. Define the almost product structure F on $(\mathbb{R}^6, g)$ by

$$F\partial_1 = \partial_2, \quad F\partial_2 = \partial_1, \quad F\partial_3 = \partial_4, \quad F\partial_4 = \partial_3, \quad F\partial_5 = \partial_5, \quad F\partial_6 = -\partial_6,$$

where $\partial_i = \frac{\partial}{\partial x_i}$, $i = 1, \cdots, 6$ and $(x_1, x_2, \cdots, x_6)$ are natural coordinates of $\mathbb{R}^6$. Now, we define a map $\pi : \mathbb{R}^6 \to \mathbb{R}^3$ by

$$\pi(x_1, \cdots, x_6) = \big(f_1, f_2, f_3\big),$$

where

$$f_1 = (x_1 + (\sqrt{2}-1)x_2 - x_3 + x_4 + x_6),$$
$$f_2 = (\frac{(x_1)^2}{2} + (\sqrt{2}-1)x_2 - \frac{(x_3)^2}{2} + x_4 - x_6),$$
$$f_3 = (x_1 + (\sqrt{2}-1)x_2 - x_3 - x_4 + x_6),$$

and $x_1 \neq x_3$. Then, the Jacobian matrix of π is:

$$\begin{pmatrix} 1 & \sqrt{2}-1 & -1 & 1 & 0 & 1 \\ x_1 & \sqrt{2}-1 & -x_3 & 1 & 0 & -1 \\ 1 & \sqrt{2}-1 & -1 & -1 & 0 & 1 \end{pmatrix}. \tag{4.19}$$

Since the rank of this matris is equal to 3, the map π is a submersion. After some calculations, we see that

$$ker\pi_* = \mathcal{D} \oplus \mathcal{D}_\theta \ ,$$

where

$$\mathcal{D} = span\{\partial_5\},$$

and

$$\mathcal{D}_\theta = span\{\frac{1}{\sqrt{2}}\partial_1 + \frac{1}{\sqrt{2}}\partial_2 + \partial_3, x_3\partial_1 + x_1\partial_3\}.$$

Moreover, the slant function of $\mathcal{D}_\theta$ is $\theta = \arccos(\frac{1}{2}\frac{x_3}{\sqrt{(x_1)^2+(x_3)^2}})$. By direct calculation, we see that π satisfies the condition **(S2)**. Hence the map π is a proper pointwise semi-slant submersion with the slant function θ.

Using (3.10), (4.15) and (4.16), we get the following useful facts.

Lemma 4.1. *Let π be a pointwise semi-slant submersion from a l.p.R. manifold (M, g, F) onto a Riemannian manifold (N, g_N). Then, we have*

$$\textbf{(a) } \phi^2 + \mathcal{B}\omega = I, \qquad \textbf{(b) } \omega\phi + \mathcal{C}\omega = 0,$$
$$\textbf{(c) } \phi\mathcal{B} + \mathcal{B}\mathcal{C} = 0, \qquad \textbf{(d) } \omega\mathcal{B} + \mathcal{C}^2 = I,$$

where I is the identity operator on TM.

By using (4.13)∼(4.18), we get the following two results.

Lemma 4.2. *Let π be a pointwise semi-slant Riemannian submersion from a l.p.R. manifold (M, g, F) onto a Riemannian manifold (N, g_N). Then, we have*

$$\textbf{(a)} \quad \phi\mathcal{D} = \mathcal{D} \qquad \textbf{(b)} \quad \phi\mathcal{D}_\theta \subset \mathcal{D}_\theta \quad \textbf{(c)} \quad \omega\mathcal{D} = \{0\}.$$

Lemma 4.3. *Let π be a pointwise semi-slant Riemannian submersion from a l.p.R. manifold (M, g, F) onto a Riemannian manifold (N, g_N). Then, we have*

$$\textbf{(a)} \quad \mathcal{B}(F\mathcal{D}_\theta) = \mathcal{D}_\theta \quad \textbf{(b)} \quad \mathcal{B}\mu = \{0\} \quad \textbf{(c)} \quad \mathcal{C}(F\mathcal{D}_\theta) = \omega\mathcal{D}_\theta \quad \textbf{(d)} \quad \mathcal{C}\mu = \mu.$$

Now we investigate the effect of the almost product structure F on the O'Neill's tensors $\mathcal{T}$ and $\mathcal{A}$ of a pointwise semi-slant Riemannian submersion $\pi : (M, g, F) \to (N, g_N)$.

Lemma 4.4. *Let π be a pointwise semi-slant submersion from a l.p.R. manifold (M, g, F) onto a Riemannian manifold (N, g_N). Then, we have*

$$\hat{\nabla}_V \phi W + \mathcal{T}_V \omega W = \phi \hat{\nabla}_V W + \mathcal{B}\mathcal{T}_V W, \tag{4.20}$$

$$\mathcal{T}_V \phi W + \mathcal{H}\nabla_V \omega W = \omega \hat{\nabla}_V W + \mathcal{C}\mathcal{T}_V W, \tag{4.21}$$

$$\mathcal{V}\nabla_\xi \mathcal{B}\eta + \mathcal{A}_\xi \mathcal{C}\eta = \phi\mathcal{A}_\xi \eta + \mathcal{B}\mathcal{H}\nabla_\xi \eta, \tag{4.22}$$

$$\mathcal{A}_\xi \mathcal{B}\eta + \mathcal{H}\nabla_\xi \mathcal{C}\eta = \omega\mathcal{A}_\xi \eta + \mathcal{C}\mathcal{H}\nabla_\xi \eta, \tag{4.23}$$

$$\hat{\nabla}_V \mathcal{B}\xi + \mathcal{T}_V \mathcal{C}\xi = \phi\mathcal{T}_V \xi + \mathcal{B}\mathcal{H}\nabla_V \xi, \tag{4.24}$$

$$\mathcal{T}_V \mathcal{B}\xi + \mathcal{H}\nabla_V \mathcal{C}\xi = \omega\mathcal{T}_V \xi + \mathcal{C}\mathcal{H}\nabla_V \xi, \tag{4.25}$$

$$\mathcal{V}\nabla_\xi \phi V + \mathcal{A}_\xi \omega V = \mathcal{B}\mathcal{A}_\xi V + \phi\mathcal{V}\nabla_\xi V, \tag{4.26}$$

$$\mathcal{A}_\xi \phi V + \mathcal{H}\nabla_\xi \omega V = \mathcal{C}\mathcal{A}_\xi V + \omega\mathcal{V}\nabla_\xi V, \tag{4.27}$$

where $V, W \in \Gamma(ker\pi_*)$, and $\xi, \eta \in \Gamma(ker\pi_*^\perp)$.

Proof. For any $V \in \Gamma(ker\pi_*)$ and $\xi \in \Gamma(ker\pi_*^{\perp})$, using (3.12), we have

$$F\nabla_\xi V = \nabla_\xi FV.$$

Hence, using (2.7), (2.8), (4.15) and (4.16), we obtain

$$\mathcal{B}\mathcal{A}_\xi V + \mathcal{C}\mathcal{A}_\xi V + \phi\mathcal{V}\nabla_\xi V + \omega\mathcal{V}\nabla_\xi V = \mathcal{A}_\xi\phi V + \mathcal{V}\nabla_\xi\phi V + \mathcal{A}_\xi\omega V + \mathcal{H}\nabla_\xi\omega V.$$

Taking the vertical and horizontal parts of this equation, we get (4.26) and (4.27). The other assertions can be obtained by using (2.5)∼(2.8), (4.15) and (4.16).

Proposition 4.1. *Let π be a pointwise semi-slant submersion from a l.p.R. manifold (M, g, F) onto a Riemannian manifold (N, g_N). Then, we obtain*

$$\phi^2 X = \cos^2\theta X, \tag{4.28}$$

for $X \in \Gamma(\mathcal{D}_\theta)$, where θ denotes the slant function.

Proof. For any non-zero $X \in \Gamma(\mathcal{D}_\theta)$ we can write following equations:

$$\cos\theta = \frac{g(FX, \phi X)}{|FX||\phi X|} = \frac{g(X, \phi^2 X)}{|X||\phi X|} \text{ and } \cos\theta = \frac{|\phi X|}{|FX|}.$$

Then, we obtain

$$\cos^2\theta = \frac{g(X, \phi^2 X)}{|X||\phi X|}\frac{|\phi X|}{|FX|}.$$

Therefore, we get the equality

$$g(\cos^2\theta X, X) = g(X, \phi^2 X),$$

which gives the assertion.

Remark 4.2. *We easily observe that the converse of the Proposition 4.1 also holds.*

Now we give a theorem for pointwise semi-slant submersions, which has similar idea with the Theorem 4.2. in [25].

Theorem 4.1. *Let π be a pointwise semi-slant Riemannian submersion from a l.p.R. manifold (M, g, F) onto a Riemannian manifold (N, g_N). Then, π is a pointwise semi-slant submersion if and only if there exists a constant $\lambda \in [0, 1]$ such that*

(a) $\mathcal{D}' = \{x \in \mathcal{D}' \,|\, \phi^2 X = \lambda X\}$,

(b) *For any* $X \in \Gamma(TM)$, *orthogonal to* $\mathcal{D}^{'}, \omega X = 0$.

Moreover, in this case $\lambda = \cos^2\theta$, *where* θ *denotes the slant function.*

Proof. Let $\pi : (M, g, F) \to (N, g_N)$ be a pointwise semi-slant submersion. Then, $\lambda = \cos^2\theta$ and $\mathcal{D}^{'} = \mathcal{D}_\theta$. By the definition of the pointwise semi-slant submersion, $\omega X = 0$, where X belongs to orthogonal complement of $\mathcal{D}^{'}$.

Conversely, **(a)** and **(b)** imply that $TM = \mathcal{D} \oplus \mathcal{D}^{'}$. Since $\phi\mathcal{D}^{'} \subseteq \mathcal{D}^{'}$, from **(b)**, $\mathcal{D}$ is an invariant distribution. Thus, π is a pointwise semi-slant submersion.

Now, we investigate the integrability conditions for invariant and slant distributions.

Theorem 4.2. *Let* π *be a pointwise semi-slant Riemannian submersion from an almost product Riemannian manifold* (M, g, F) *onto a Riemannian manifold* (N, g_N). *Then, the invariant distribution* $\mathcal{D}$ *is integrable if and only if*

$$\phi(\hat{\nabla}_V W - \hat{\nabla}_W V) \in \mathcal{D}, \tag{4.29}$$

for $V, W \in \Gamma(\mathcal{D})$.

Proof. For $V, W \in \Gamma(\mathcal{D})$ and $X \in \Gamma(\mathcal{D}_\theta)$, we know $[V, W] \in \mathcal{D}$ if and only if $F[V, W] \in \mathcal{D}$. So by (4.15) we obtain,

$$\begin{aligned} g(F[V, W], X) =& g(F(\nabla_V W - \nabla_W V), X) \\ =& g(F(\mathcal{T}_V W + \hat{\nabla}_V W - \mathcal{T}_W V - \hat{\nabla}_W V), X) \\ =& g(\phi(\hat{\nabla}_V W - \hat{\nabla}_W V), X). \end{aligned}$$

Thus, $[V, W] \in \mathcal{D}$ if and only if $\phi(\hat{\nabla}_V W - \hat{\nabla}_W V) \in \mathcal{D}$.

In a similar way, we get the following theorem.

Theorem 4.3. *Let* π *be a pointwise semi-slant Riemannian submersion from an almost product Riemannian manifold* (M, g, F) *onto a Riemannian manifold* (N, g_N). *Then, the slant distribution* $\mathcal{D}_\theta$ *is integrable if and only if*

$$\phi(\hat{\nabla}_X Y - \hat{\nabla}_Y X) \in \mathcal{D}_\theta,$$

for $X, Y \in \Gamma(\mathcal{D}_\theta)$.

If we consider the total manifold l.p.R. instead of almost product Riemannian, we obtain the following results.

Lemma 4.5. *Let π be a proper pointwise semi-slant submersion from a l.p.R. manifold (M, g, F) onto a Riemannian manifold (N, g_N). Then, we have the followings*

$$i)\ g(\nabla_V W, X) = \csc^2\theta\{g(\mathcal{T}_V W, \omega\phi X) + g(\mathcal{T}_V \phi W, \omega X)\} \tag{4.30}$$

$$ii)\ g(\nabla_X Y, V) = \csc^2\theta\{g(\mathcal{T}_X \omega\phi Y, V) + g(\mathcal{T}_X \omega Y, \phi V)\} \tag{4.31}$$

where θ is the slant function, $V, W \in \Gamma(\mathcal{D})$ and $X, Y \in \Gamma(\mathcal{D}_\theta)$.

Proof. Let $V, W \in \Gamma(\mathcal{D})$ and $X, Y \in \Gamma(\mathcal{D}_\theta)$. Then, by using (3.11) and (4.15), we obtain

$$\begin{aligned} g(\nabla_V W, X) =& g(\nabla_V FW, FX) \\ =& g(\nabla_V FW, \phi X) + g(\nabla_V FW, \omega X) \\ =& g(\nabla_V W, \phi^2 X) + g(\nabla_V W, \omega\phi X) + g(\nabla_V \phi W, \omega X). \end{aligned}$$

If we regard (4.28),(2.5) and (2.6) for the last expression, we get the following equality

$$(1 - \cos^2\theta) g(\nabla_V W, X) = g(\mathcal{T}_V W, \omega\phi X) + g(\mathcal{T}_V \phi X, \omega X).$$

So, that is what we needed.

For the second equation we apply the same idea. Let $X, Y \in \Gamma(\mathcal{D}_\theta)$ and $V \in \Gamma(\mathcal{D})$. Then by using (3.11) and (4.15), we get

$$\begin{aligned} g(\nabla_X Y, V) =& g(\nabla_X FY, FV) \\ =& g(\nabla_X \phi Y, FV) + g(\nabla_X \omega Y, FV) \\ =& g(\nabla_X \phi^2 Y, V) + g(\nabla_X \omega\phi Y, V) + g(\nabla_X \omega Y, FV). \end{aligned}$$

If we consider (4.28),(2.5) and (2.6) with the last equation, we get the following

$$\begin{aligned} g(\nabla_X Y, V) =& g(\nabla_X (\cos^2\theta) Y, V) + g(\nabla_X \omega\phi Y, V) + g(\nabla_X \omega Y, FV) \\ =& g(-(\sin 2\theta)(X\theta) Y, V) + g(\cos^2\theta \nabla_X Y, V) + g(\mathcal{T}_X \omega\phi Y, V) \\ &+ g(\mathcal{T}_X \omega Y, \phi V). \end{aligned}$$

Therefore, since $g(-(\sin 2\theta)(X\theta) Y, V) = 0$, we get the assertion.

Theorem 4.4. *Let π be a proper pointwise semi-slant submersion from a l.p.R. manifold (M, g, F) onto a Riemannian manifold (N, g_N). Then, the invariant distribution $\mathcal{D}$ is integrable if and only if*

$$g(\mathcal{T}_V \phi W - \mathcal{T}_W \phi V, \omega X) = 0,$$

for $V, W \in \Gamma(\mathcal{D})$ and $X \in \Gamma(\mathcal{D}_\theta)$.

Proof. Let $V, W \in \Gamma(\mathcal{D})$ and $X \in \Gamma(\mathcal{D}_\theta)$. Then, by Lemma 4.5 and (2.3), we have

$$\begin{aligned} g([V,W],X) =& g(\nabla_V W, X) - g(\nabla_W V, X) \\ &= \csc^2\theta\{g(\mathcal{T}_V W, \omega\phi X) + g(\mathcal{T}_V \phi W, \omega X) \\ &- g(\mathcal{T}_W V, \omega\phi X) + g(\mathcal{T}_W \phi V, \omega X)\} \\ &= \csc^2\theta\{g(\mathcal{T}_V \phi W, \omega X) - g(\mathcal{T}_W \phi V, \omega X)\}. \end{aligned}$$

Therefore, $\mathcal{D}$ is integrable if and only if $g(\mathcal{T}_V \phi W - \mathcal{T}_W \phi V, \omega X) = 0$.

In the same way, we examine the slant distribution.

Theorem 4.5. *Let π be a proper pointwise semi-slant submersion from a l.p.R. manifold (M, g, F) onto a Riemannian manifold (N, g_N). Then, the slant distribution $\mathcal{D}_\theta$ is integrable if and only if*

$$g(\mathcal{T}_X \omega\phi Y - \mathcal{T}_Y \omega\phi X, V) = g(\mathcal{T}_Y \omega X - \mathcal{T}_X \omega Y, \phi V)$$

for $X, Y \in \Gamma(\mathcal{D}_\theta)$ and $V \in \Gamma\mathcal{D}_\theta$.

Proof. Let $X, Y \in \Gamma(\mathcal{D}_\theta)$ and $V \in \Gamma(\mathcal{D})$. By using Lemma 4.5, we obtain

$$\begin{aligned} g([X,Y],V) = \csc^2\theta\{& g(\mathcal{T}_X \omega\phi Y, V) + g(\mathcal{T}_X \omega Y, \phi V) \\ & - g(\mathcal{T}_Y \omega\phi X, V) + g(\mathcal{T}_Y \omega X, \phi V)\}. \end{aligned}$$

Thus, slant distribution $\mathcal{D}_\theta$ is integrable if and only if
$g(\mathcal{T}_X \omega\phi Y - \mathcal{T}_Y \omega\phi X, V) = g(\mathcal{T}_Y \omega X - \mathcal{T}_X \omega Y, \phi V)$.

Now, we focus on that in which conditions the distributions, which we study on, define totally geodesic foliation.

Proposition 4.2. *Let π be a pointwise semi-slant submersion from a l.p.R. manifold (M, g, F) onto a Riemannian manifold (N, g_N). Then, $ker\pi_*$ defines a totally geodesic foliation if and only if*

$$\mathcal{C}(\mathcal{T}_V \phi W + \mathcal{H}\nabla_V \omega W) + \omega(\hat{\nabla}_V \phi W + \mathcal{T}_V \omega W) = 0, \tag{4.32}$$

for $V, W \in \Gamma(ker\pi_*)$.

Proof. For $V, W \in \Gamma(ker\pi_*)$, by using (2.5), (2.6) and (4.15), we get

$$\begin{aligned}\nabla_V W =& F\nabla_V FW = F(\nabla_V \phi W + \nabla_V \omega W)\\ =& F(\mathcal{T}_V \phi W + \hat{\nabla}_V \phi W + \mathcal{T}_V \omega W + \mathcal{H}\nabla_V \omega W)\\ =& \mathcal{B}\mathcal{T}_V \phi W + \mathcal{C}\mathcal{T}_V \phi W + \phi\hat{\nabla}_V \phi W + \omega\hat{\nabla}_V \phi W\\ &+ \phi\mathcal{T}_V \omega W + \omega\mathcal{T}_V \omega W + \mathcal{B}\mathcal{H}\nabla_V \omega W + \mathcal{C}\mathcal{H}\nabla_V \omega W.\end{aligned}$$

Therefore, $ker\pi_*$ defines a totally geodesic foliation if and only if
$\mathcal{C}(\mathcal{T}_V \phi W + \mathcal{H}\nabla_V \omega W) + \omega(\hat{\nabla}_V \phi W + \mathcal{T}_V \omega W) = 0$.

Proposition 4.3. *Let π be a pointwise semi-slant submersion from a l.p.R. manifold (M, g, F) onto a Riemannian manifold (N, g_N). Then, $ker\pi_*^\perp$ defines a totally geodesic foliation if and only if*

$$\mathcal{B}(\mathcal{A}_\xi \mathcal{B}\eta + \mathcal{H}\nabla_\xi \mathcal{C}\eta) + \phi(\mathcal{V}\nabla_\xi \mathcal{B}\eta + \mathcal{A}_\xi \mathcal{C}\eta) = 0, \tag{4.33}$$

for $\xi, \eta \in \Gamma(ker\pi_*^\perp)$.

Proof. This proof can likewise be done using the techniques of the proof of Proposition 4.2 .

In the view of Proposition 4.2 and Proposition 4.3, we obtain the following result.

Corollary 4.1. *Let π be a pointwise semi-slant submersion from a l.p.R. manifold (M, g, F) onto a Riemannian manifold (N, g_N). Then, M is a locally product $M_{ker\pi_*} \times M_{ker\pi_*^\perp}$ if and only if (4.32) and (4.33) hold, where $M_{ker\pi_*}$ and $M_{ker\pi_*^\perp}$ are integral manifolds of the distributions $ker\pi_*$ and $ker\pi_*^\perp$, respectively.*

Proposition 4.4. *Let π be a pointwise semi-slant submersion from a l.p.R. manifold (M, g, F) onto a Riemannian manifold (N, g_N). Then, the invariant distribution $\mathcal{D}$ defines a totally geodesic foliation on $ker\pi_*$ if and only if for $U, V \in \Gamma(\mathcal{D})$,*

$$Q(\mathcal{B}\mathcal{T}_U \phi V + \phi\hat{\nabla}_U \phi V) = 0 \ \textit{and} \ (\mathcal{C}\mathcal{T}_U \phi V + \omega\hat{\nabla}_U \phi V) = 0. \tag{4.34}$$

Proof. For $U, V \in \Gamma(\mathcal{D})$, from (2.5), (2.6), (4.15) and (4.16) we obtain

$$\begin{aligned}\nabla_U V =& F\nabla_U FV = F(\nabla_U \phi V + \nabla_U \omega W)\\ =& F(\nabla_U \phi V) = F(\mathcal{T}_U \phi V + \hat{\nabla}_U \omega V)\\ =& \mathcal{B}\mathcal{T}_U \phi V + \mathcal{C}\mathcal{T}_U \phi V + \phi\hat{\nabla}_U \omega V + \omega\hat{\nabla}_U \omega V.\end{aligned}$$

Therefore, we obtain the assertion.

Proposition 4.5. *Let π be a pointwise semi-slant submersion from a l.p.R. manifold (M, g, F) onto a Riemannian manifold (N, g_N). Then, the slant distribution $\mathcal{D}_\theta$ defines a totally geodesic foliation on $ker\pi_*$ if and only if for*
$X, Y \in \Gamma(\mathcal{D}_\theta)$,

$$P(\mathcal{B}(\mathcal{T}_X\phi Y + \mathcal{H}\nabla_X\omega Y) + \phi(\mathcal{T}_X\omega Y + \hat{\nabla}_X\phi Y)) = 0 \tag{4.35}$$

and

$$\omega(\hat{\nabla}_X\phi Y + \mathcal{T}_X\omega Y) + \mathcal{C}(\mathcal{T}_X\phi Y + \mathcal{H}\nabla_X\omega Y) = 0. \tag{4.36}$$

Proof. The argument is same with the proof of Proposition 4.4.

By Proposition 4.4 and Proposition 4.5 we have the following result.

Corollary 4.2. *Let π be a pointwise semi-slant submersion from a l.p.R. manifold (M, g, F) onto a Riemannian manifold (N, g_N). Then, the vertical distribution $ker\pi_*$ is a locally product $M_{\mathcal{D}} \times M_{\mathcal{D}_\theta}$ if and only if (4.34) and (4.35) hold, where $M_{\mathcal{D}}$ and $M_{\mathcal{D}_\theta}$ are intergral manifolds of $\mathcal{D}$ and $\mathcal{D}_\theta$, respectively.*

Theorem 4.6. *Let π be a pointwise semi-slant submersion from a l.p.R. manifold (M, g, F) onto a Riemannian manifold (N, g_N). Then, π is a totally geodesic map if and only if*

$$\omega(\hat{\nabla}_V\phi W + \mathcal{T}_U\omega W) + \mathcal{C}(\mathcal{T}_V\phi W + \mathcal{H}\nabla_V\omega W) = 0 \tag{4.37}$$

and

$$\omega(\hat{\nabla}_V\mathcal{B}\xi + \mathcal{T}_V\mathcal{C}\xi) + \mathcal{C}(\mathcal{T}_V\mathcal{B}\xi + \mathcal{H}\nabla_V\mathcal{C}\xi) = 0 \tag{4.38}$$

for $V, W \in \Gamma(ker\pi_*)$ and $\xi \in \Gamma(ker\pi_*^\perp)$.

Proof. Since π is a Riemannian submersion, we have

$$(\nabla\pi_*)(\xi, \eta) = 0, \ for \ \xi, \eta \in \Gamma(ker\pi_*^\perp).$$

For $V, W \in \Gamma(ker\pi_*)$, we obtain

$$\begin{aligned}(\nabla\pi_*)(V, W) = & \nabla^{\pi}_V(\pi_* W) - \pi_* \nabla_V W \\ = & -\pi_*(F\nabla_V FW) = -\pi_*(F(\nabla_V \phi W + \nabla_V \omega W) \\ = & -\pi_*(F(\mathcal{T}_V \phi W + \hat{\nabla}_V \phi W + \mathcal{T}_V \omega W + \mathcal{H}\nabla_V \omega W) \\ = & -\pi_*(\mathcal{B}\mathcal{T}_V \phi W + \mathcal{C}\mathcal{T}_V \phi W + \phi\hat{\nabla}_V \phi W + \omega\hat{\nabla}_V \phi W \\ & +\phi\mathcal{T}_V \omega W + \omega\mathcal{T}_V \omega W + \mathcal{B}\mathcal{H}\nabla_V \omega W + \mathcal{C}\mathcal{H}\nabla_V \omega W) \\ = & -\pi_*(\mathcal{C}\mathcal{T}_V \phi W + \omega\hat{\nabla}_V \phi W + \omega\mathcal{T}_V \omega W + \mathcal{C}\mathcal{H}\nabla_V \omega W).\end{aligned}$$

Thus,

$(\nabla\pi_*)(V, W) = 0 \Leftrightarrow \omega(\hat{\nabla}_V \phi W + \mathcal{T}_V \omega W) + \mathcal{C}(\mathcal{T}_V \phi W + \mathcal{H}\nabla_V \omega W) = 0$. By a similar way above, for $V \in \Gamma(ker\pi_*)$ and $\xi \in \Gamma(ker\pi_*^{\perp})$, we get

$$(\nabla\pi_*)(V, \xi) = 0 \Leftrightarrow \omega(\hat{\nabla}_V \mathcal{B}\xi + \mathcal{T}_V \mathcal{C}\xi) + \mathcal{C}(\mathcal{T}_V \mathcal{B}\xi + \mathcal{H}\nabla_V \mathcal{C}\xi) = 0.$$

Recall that the fibers of a Riemannian submersion $\pi : (M, g) \to (N, g_N)$ is called *totally umbilical* if

$$T_U V = g(U, V)H \tag{4.39}$$

for any $U, V \in \Gamma(ker\pi_*)$, where H is the mean curvature vector field of the fiber.
Let π be a pointwise semi-slant submersion from a l.p.R. manifold (M, g, F) onto a Riemannian manifold (N, g_N) . We can define

$$(\nabla_U \phi)V = \hat{\nabla}_U \phi V - \phi\hat{\nabla}_U V, \tag{4.40}$$

$$(\nabla_U \omega)V = \mathcal{H}\nabla_U \omega V - \omega\hat{\nabla}_U V, \tag{4.41}$$

$$(\nabla_U \mathcal{B})\xi = \hat{\nabla}_U \mathcal{B}\xi - \mathcal{B}\mathcal{H}\nabla_U \xi, \tag{4.42}$$

$$(\nabla_U \mathcal{C})\xi = \mathcal{H}\nabla_U \mathcal{C}\xi - \mathcal{C}\mathcal{H}\nabla_U \xi, \tag{4.43}$$

where $U, V \in \Gamma(ker\pi_*)$ and $\xi \in \Gamma(ker\pi_*^{\perp})$.

We say that ϕ (resp. ω, $\mathcal{B}$ or $\mathcal{C}$) is *parallel* if $\nabla\phi = 0$ (resp. $\nabla\omega = 0$, $\nabla\mathcal{B} = 0$ or $\nabla\mathcal{C} = 0$).

Lemma 4.6. *Let π be a pointwise semi-slant submersion with parallel canonical structures from a l.p.R. manifold (M,g,F) onto a Riemannian manifold (N,g_N). Then for any $U,V\in\Gamma(ker\pi_*)$ and $\xi\in\Gamma(ker\pi_*^{\perp})$, we have*

$$(\nabla_U\phi)V=\mathcal{B}\mathcal{T}_UV-\mathcal{T}_U\omega V, \tag{4.44}$$

$$(\nabla_U\omega)V=\mathcal{C}\mathcal{T}_UV-\mathcal{T}_U\phi V, \tag{4.45}$$

$$(\nabla_U\mathcal{B})\xi=\phi\mathcal{T}_U\xi-\mathcal{T}_U\mathcal{C}\xi, \tag{4.46}$$

$$(\nabla_U\mathcal{C})\xi=\omega\mathcal{T}_U\xi-\mathcal{T}_U\mathcal{B}\xi. \tag{4.47}$$

Proof. All of the equations follow from Lemma 4.4 and (4.40)∼(4.43).

Theorem 4.7. *Let π be a proper pointwise semi-slant submersion with totally umbilical fibers from a l.p.R. manifold (M,g,F) onto a Riemannian manifold (N,g_N). If $dim(D_\theta)\geq 2$ and ϕ is parallel, then the fibers of π are totally geodesic or the mean curvature vector field H belongs to μ.*

Proof. If the fibers of π are totally geodesic, it is obvious. Let us assume the other case. Since $dim(D_\theta)\geq 2$, then we can choose $X,Y\in\Gamma(\mathcal{D}_\theta)$ such that the set $\{X,\ Y\}$ is orthonormal. By using (3.11), (3.12), (4.15), (4.16), (2.5) and (2.6), we have

$$\begin{aligned}\nabla_XFY=&F\nabla_XY\\ \nabla_X\phi Y+\nabla_X\omega Y=&F(\mathcal{T}_XY+\hat{\nabla}_XY)\\ \mathcal{T}_X\phi Y+\hat{\nabla}_X\phi Y+\mathcal{T}_X\omega Y+\mathcal{H}\nabla_X\omega Y=&\mathcal{B}\mathcal{T}_XY+\mathcal{C}\mathcal{T}_XY+\phi\hat{\nabla}_XY+\omega\hat{\nabla}_XY.\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}g(\hat{\nabla}_X\phi Y+\mathcal{T}_X\omega Y,X)=&g(\mathcal{B}\mathcal{T}_XY+\phi\hat{\nabla}_XY,X)\\ g(\phi\hat{\nabla}_XY-\hat{\nabla}_X\phi Y,X)=&g(\mathcal{T}_X\omega Y-\mathcal{B}\mathcal{T}_XY,X)\\ g((\nabla_X\phi)Y,X)=&g(F\mathcal{T}_XY-\mathcal{T}_XFY,X).\end{aligned}$$

Since ϕ is parallel, we get

$$g(F\mathcal{T}_XY,X)=g(\mathcal{T}_XFY,X). \tag{4.48}$$

Thus, using (4.39) and (4.48), we have

$$g(H,FY)=g(\mathcal{T}_XX,FY)=-g(\mathcal{T}_XFY,X)=-g(F\mathcal{T}_XY,X)$$
$$=-g(\mathcal{T}_XY,FX)=-g(X,Y)g(H,FX)=0,$$

since $g(X,Y)=0$. So, we deduce that $H\perp\omega\mathcal{D}_\theta$. Therefore, it follows $H\in\mu$ from (4.18).

Corollary 4.3. *Let π be a proper pointwise semi-slant submersion with totally umbilical fibers from a l.p.R. manifold (M,g,F) onto a Riemannian manifold (N,g_N). If $(ker\pi_*)^\perp=\omega\mathcal{D}_\theta$, i.e. $\mu=\{0\}$ and ϕ is parallel, then the fibers of π are totally geodesic.*

Let π be a pointwise semi-slant submersion from a l.p.R. manifold (M,g,F) onto a Riemannian manifold (N,g_N). Then, we say that the fibers of π are *mixed geodesic*, if $\mathcal{T}_XW=0$, for all $X\in\Gamma(\mathcal{D}_\theta)$, $W\in\Gamma(\mathcal{D})$, [26].

Theorem 4.8. *Let π be a proper pointwise semi-slant submersion from a l.p.R. manifold (M,g,F) onto a Riemannian manifold (N,g_N). If ω is parallel, i.e. $\nabla\omega=0$, then the fibers of π are mixed geodesic.*

Proof. Let ω be parallel, then for any $U,V\in\Gamma(ker\pi_*)$ from (4.45), we have

$$\mathcal{C}\mathcal{T}_UV=\mathcal{T}_U\phi V. \tag{4.49}$$

Using (4.49), we obtain

$$\mathcal{C}^2\mathcal{T}_UV=\mathcal{T}_U\phi^2V. \tag{4.50}$$

If we put $U=W\in\Gamma(\mathcal{D})$ and $V=X\in\Gamma(\mathcal{D}_\theta)$ in (4.50) and using (4.28), we get

$$\mathcal{C}^2\mathcal{T}_WX=\cos^2\theta\mathcal{T}_WX. \tag{4.51}$$

On the other hand, using the symmetry property of $\mathcal{T}$ on $\Gamma(ker\pi_*)$ and (4.49), we have

$$\mathcal{C}^2\mathcal{T}_WX=\mathcal{C}^2\mathcal{T}_XW=\mathcal{T}_X\phi^2W=\mathcal{T}_XW, \tag{4.52}$$

that is

$$\mathcal{C}^2\mathcal{T}_WX=\mathcal{T}_XW. \tag{4.53}$$

Since submersion π is proper, from (4.51) and (4.53), it follows that

$$\mathcal{T}_XW=0. \tag{4.54}$$

Remark 4.3. *Most of our results for pointwise semi-slant submersion is similar to semi-slant case, see* [12].

5. The first variational form of a pointwise semi-slant submersion

In this section, we give a different approach to check whether a submersion is harmonic and define the first variational form of a semi-slant submersion.

Let π be a pointwise semi-slant submersion from a l.p.R. manifold (M, g, F) onto a Riemannian manifold (N, g_N). Then, we can define the 1-form dual to the vector field $F\xi$, for $\xi \in \Gamma(ker\pi_*^{\perp})$, such that

$$\sigma_\xi : \Gamma(ker\pi_*) \mapsto \mathcal{F}(\pi_q^{-1}), q \in N$$

$$V \mapsto \sigma_\xi(V) = g(F\xi, V),$$

for all $V \in \Gamma(ker\pi_*)$. In the view of [31] and [7], we define the followings.

The *Legendre variations* of any fiber of π, denoted by the set $\mathbb{L}$, where

$$\mathbb{L} = \{\xi \in \Gamma(ker\pi_*^{\perp}) : d\sigma_\xi = 0,\ i.e.\ \sigma_\xi\ is\ closed\},$$

the *Hamiltonian variations* of any fiber of π, denoted by the set $\mathbb{E}$,

$$\mathbb{E} = \{\xi \in \Gamma(ker\pi_*)^{\perp} : \exists f \in \mathcal{F}(\pi_q^{-1}) \Rightarrow \sigma_\xi = df,\ i.e.\ \sigma_\xi\ is\ exact\}$$

and the *harmonic variations* of any fiber of π are given by the set

$$\mathbb{H} = \{\xi \in \Gamma(ker\pi_*)^{\perp} : \Delta\sigma_\xi = 0;\ i.e.\ \sigma_\xi\ is\ harmonic\}.$$

By the definitions of differential and co-differential operators, we observe that

$$\mathbb{E} \subset \mathbb{L}\ ,\ \ \mathbb{H} \subset \mathbb{L}\ \ and\ \ \mathbb{E} \cap \mathbb{H} = 0. \tag{5.55}$$

Now, we examine that in which conditions the 1-form σ_ξ defined above is a Legendre variation.

Lemma 5.1. *Let π be a pointwise semi-slant submersion from a l.p.R. manifold (M, g, F) onto a Riemannian manifold (N, g_N). The 1-form σ_ξ is a Legendre variation if and only if*

$$g(\mathcal{T}_U\xi, \phi V) - g(\mathcal{T}_V\xi, \phi U) = g(\mathcal{A}_\xi U, \omega V) - g(\mathcal{A}_\xi V, \omega U), \tag{5.56}$$

for all $U, V \in \Gamma(ker\pi_*)$.

Proof. Let U, V be in $ker\pi_*$. Then, by the definition of differential, (2.6) and (3.11), we obtain

$$\begin{aligned}(d\sigma_\xi)(U,V) =& Ug(F\xi,V) - Vg(F\xi,U) - g(F\xi,[U,V])\\ =& Ug(\xi,FV) - Vg(\xi,FU) - g(\xi,F[U,V])\\ =& g(\nabla_U\xi,FV) + g(\xi,\nabla_U FV)\\ & -g(\nabla_V\xi,FU) - g(\xi,\nabla_V FU)\\ & -g(\xi,F\nabla_U V) + g(\xi,F\nabla_V U)\\ =& g(\nabla_U\xi,\phi V+\omega V) - g(\nabla_V\xi,\phi U+\omega U)\\ =& g(\nabla_U\xi,\phi V) + g(\nabla_U\xi,\omega V)\\ & -g(\nabla_V\xi,\phi U) + g(\nabla_V\xi,\omega U)\\ =& g(\mathcal{T}_U\xi,\phi V) + g(\mathcal{H}\nabla_U\xi,\omega V)\\ & -g(\mathcal{T}_V\xi,\phi U) + g(\mathcal{H}\nabla_V\xi,\omega U).\end{aligned}$$

Since we may assume ξ is basic, we have

$$\begin{aligned}(d\sigma_\xi)(U,V) =& g(\mathcal{T}_U\xi,\phi V) + g(\mathcal{A}_\xi U,\omega V)\\ & -g(\mathcal{T}_V\xi,\phi U) + g(\mathcal{A}_\xi V,\omega U).\end{aligned}$$

Thus, the assertion follows.

Lemma 5.2. *For $\xi \in \Gamma(\mu)$, $\sigma_\xi \equiv 0$.*

Proof. Let $\xi \in \Gamma(\mu)$. Then, $F\xi \in \Gamma(\mu)$. For any $V \in \Gamma(ker\pi_*)$, we get

$$\sigma_\xi(V) = g(F\xi,V) = 0.$$

So, $\sigma_\xi \equiv 0$, for all $V \in \Gamma(ker\pi_*)$.

Remark 5.1. *Because of Lemma 5.2, throughout this paper, we can assume that H belongs to $\Gamma(\omega\mathcal{D}_\theta)$.*

Proposition 5.1. *Let π be a pointwise semi-slant submersion from a l.p.R. manifold (M,g,F) onto a Riemannian manifold (N,g_N) and f be a smooth function on a fiber. Then, $F(grad(f)|_{\omega\mathcal{D}_\theta}) \in \mathbb{E}$.*

Proof. Let f be a smooth function on a fiber. For $\xi = F(grad(f)|_{\omega\mathcal{D}_\theta})$, and any $V \in \Gamma(ker\pi_*)$, we obtain

$$\sigma_\xi(V) = g(F\xi, V) = g(grad(f), V) = V[f] = df(V).$$

Thus, we get $\sigma_\xi = df$, i.e. $\xi \in \mathbb{E}$.

Let π be a pointwise semi-slant submersion with compact fibers from a l.p.R. manifold (M, g, F) onto a Riemannian manifold (N, g_N) and $\xi \in \Gamma(ker\pi_*^\perp)$. The first variation of the volume form of a fiber π_q^{-1}, for $q \in N$, is defined as follows [17]

$$\mathbf{V}'(\xi) = -k \int_{\pi_q^{-1}} g(\xi, H) {*}1. \tag{5.57}$$

where $k = dim(\pi_q^{-1})$. We call the fibers;

- If $\mathbf{V}'(\xi) = 0$, for all $\xi \in \mathbb{L}$, then π_q^{-1} is $\mathbb{L} - minimal$,
- If $\mathbf{V}'(\xi) = 0$, for all $\xi \in \mathbb{E}$, then π_q^{-1} is $\mathbb{E} - minimal$,
- If $\mathbf{V}'(\xi) = 0$, for all $\xi \in \mathbb{H}$, then π_q^{-1} is $\mathbb{H} - minimal$.

Remark 5.2. *One can easily see that if the fiber is minimal, then the fiber is $\mathbb{L}, \mathbb{E}$ and $\mathbb{H} - minimal$. On the other hand, because of the facts that $\mathbb{E} \subset \mathbb{L}$ and $\mathbb{H} \subset \mathbb{L}$, the fiber is $\mathbb{E} - minimal$ and $\mathbb{H} - minimal$ if it is $\mathbb{L} - minimal$.*

Theorem 5.1. *Let π be a pointwise semi-slant submersion with compact fibers from a l.p.R. manifold (M, g, F) onto a Riemannian manifold (N, g_N). Then,*

(a) *The fiber π_q^{-1} is $\mathbb{L} - minimal$ if and only if σ_H is co-exact.*
(b) *The fiber π_q^{-1} is $\mathbb{E} - minimal$ if and only if σ_H is co-closed.*
(c) *The fiber π_q^{-1} is $\mathbb{H} - minimal$ if and only if σ_H is the sum of an exact and a co-exact 1-form.*

Proof.

(a) $\Rightarrow:$ Let the fiber π_q^{-1} is $\mathbb{L} - minimal$, then for any $\xi \in \mathbb{L}$, we have $g(H, \xi) = 0$ from (5.57). By the definition of the Hodge star operator [10], we have

$$\sigma_\xi \wedge \sigma_H(V_1, V_2, ..., V_k) = g(\xi, H){*}1(V_1, V_2, ..., V_k),$$

for $V_1, V_2, ..., V_k \in \Gamma(ker\pi_*)$. From the definition of the global scalar product $(.|.)$ (see [10]) on the module of all forms on the fiber, we get

$$(\sigma_\xi|\sigma_H) = \int_{\pi_q^{-1}} \sigma_\xi \wedge *\sigma_H = 0. \tag{5.58}$$

Denote by δ the codifferential operator [10] on the fiber π_q^{-1}. Since σ_ξ is closed, for any 2-form β on π_q^{-1}, we have

$$0 = (d\sigma_\xi|\beta) = (\sigma_\xi|\delta\beta). \tag{5.59}$$

Since π_q^{-1} is compact, by (5.58) and (5.59) we conclude that σ_H is co-exact.

$\Leftarrow:$ Suppose that σ_H is co-exact, we have $\sigma_H = \delta\psi$ for some 2-form ψ. Then, for any $\xi \in \mathbb{L}$,

$$(\sigma_\xi|\sigma_H) = (\sigma_\xi|\delta\psi) = (d\sigma_\xi|\psi) = 0$$

and then

$$\mathbf{V}'(\xi) = -k\int_{\pi_q^{-1}} g(H,\xi)*1 = -k\int_{\pi^{-1}(q)} (\sigma_\xi \wedge *\sigma_H) = -k(\sigma_\xi|\sigma_H) = 0,$$

i.e. π_q^{-1} is $\mathbb{L} - minimal$.

(b) $\Rightarrow:$ Let the fiber π_q^{-1} be $\mathbb{E} - minimal$. Then, we have

$$0 = \mathbf{V}'(\xi) = -k\int_{\pi_q^{-1}} g(\xi,H)*1 = -k\int_{\pi_q^{-1}} (\sigma_\xi \wedge *\sigma_H) = -k(\sigma_\xi|\sigma_H),$$

that is, $(\sigma_\xi|\sigma_H) = 0$. Since $\xi \in \mathbb{E}$, $\sigma_\xi = df$ for some function f on the fiber π_q^{-1}. Thus,

$$(df|\sigma_H) = (f|\delta\sigma_H) = 0.$$

Hence it follows that $\delta\sigma_H = 0$, i.e. σ_H is co-closed.

$\Leftarrow:$ Suppose that σ_H is co-closed. Let $\xi \in \mathbb{E}$, then there exists a function $f \in \mathcal{F}(\pi_q^{-1})$ such that $\sigma_\xi = df$. Hence, we have

$$(\sigma_\xi|\sigma_H) = (df|\sigma_H) = (f|\delta\sigma_H) = 0.$$

Therefore,

$$\mathbf{V}'(\xi) = -k\int_{\pi_q^{-1}} g(H,\xi)*1 = -k\int_{\pi_q^{-1}} (\sigma_\xi \wedge *\sigma_H) = -k(\sigma_\xi|\sigma_H) = 0,$$

that is $\mathbf{V}'(\xi) = 0$ for $\xi \in \mathbb{E}$, i.e. π_q^{-1} is $\mathbb{E} - minimal$.

(c) $\Rightarrow:$ If the fiber π_q^{-1} is $\mathbb{H} - minimal$, then for $\xi \in \mathbb{H}$, we have

$$0 = \mathbf{V}'(\xi) = -k\int_{\pi_q^{-1}} g(\xi,H)*1 = -k\int_{\pi_q^{-1}} (\sigma_\xi \wedge *\sigma_H) = -k(\sigma_\xi|\sigma_H).$$

It means that, σ_H is orthogonal to harmonic 1-forms on the fiber π_q^{-1}. Thus, by the Hodge decomposition theorem [10], we conclude that σ_H is the sum of an exact and a co-exact 1-form.

$\Leftarrow$: Let σ_H be the sum of an exact 1-form $\omega_1 = df$ and a co-exact 1-form $\omega_2 = \delta\psi$. For $\xi \in \mathbb{H}$, we have

$$\begin{aligned}(\sigma_\xi|\sigma_H) =&(\sigma_\xi|df + \delta\psi) = (\sigma_\xi|df) + (\sigma_\xi|\delta\psi)\\ =&(\delta\sigma_\xi|f) + (d\sigma_\xi|\psi) = 0,\end{aligned}$$

since $d\sigma_\xi = \delta\sigma_\xi = 0$. Thus,

$$\mathbf{V}'(\xi) = -k\int_{\pi_q^{-1}} g(\xi, H)*1 = -k\int_{\pi_q^{-1}}(\sigma_\xi \wedge *\sigma_H) = -k(\sigma_\xi|\sigma_H),$$

that is, the fiber is $\mathbb{H} - minimal$.

Theorem 5.2. *Let π be a pointwise semi-slant submersion with compact fibers from a l.p.R. manifold (M, g, F) onto a Riemannian manifold (N, g_N). If $H \in \mathbb{L}$, then*

(a) *π_q^{-1} is $\mathbb{L} - minimal$ if and only if π_q^{-1} is minimal.*
(b) *π_q^{-1} is $\mathbb{E} - minimal$ if and only if σ_H is a harmonic variation.*
(c) *π_q^{-1} is $\mathbb{H} - minimal$ if and only if σ_H is an exact 1-form.*

Proof. **(a)** If the fiber π_q^{-1} is $\mathbb{L} - minimal$, then by Theorem 5.1-(a) we have, σ_H is co-exact. Hence σ_H is co-closed. Taking into account the fact that $d\sigma_H = 0$, we deduce that σ_H is harmonic. But this is a contradiction because of Hodge decomposition theorem [10]. So, σ_H must be zero. Hence we conclude that $H = 0$. The converse is clear.

(b) $\Rightarrow$: If the fiber π_q^{-1} is $\mathbb{E} - minimal$, then we have $\delta\sigma_H = 0$ from Theorem 5.1-(b). Since $d\sigma_H = 0$, σ_H is also harmonic, i.e. $\Delta\sigma_H = 0$.

$\Leftarrow$: If σ_H is harmonic, then σ_H is co-closed. By Theorem 5.1-(b), the fiber π_q^{-1} is $\mathbb{E} - minimal$.

(c) $\Rightarrow$: Assume that π_q^{-1} is $\mathbb{H} - minimal$. Then, from Theorem 5.1-(c), σ_H is the sum of an exact 1-form and a co-exact 1-form. On the other hand, the condition $H \in \mathbb{L}$ implies that σ_H is orthogonal to every co-exact 1-form on π_q^{-1}. Thus, σ_H must be exact.

$\Leftarrow$: Let σ_H be an exact 1-form. For $\xi \in \mathbb{H}$, we obtain

$$\begin{aligned}\mathbf{V}'(\xi) =& - k \int_{\pi_q^{-1}} g(\xi, H) {*1} = -k \int_{\pi_q^{-1}} (\sigma_\xi \wedge *\sigma_H) \\ =& - k(\sigma_\xi | \sigma_H) = (\sigma_\xi | df) = (\delta\sigma_\xi | f) = 0,\end{aligned}$$

that is, π_q^{-1} is $\mathbb{H} - minimal$.

Remark 5.3. *It is well known that, the fibers of a submerion is minimal if and only if the submersion is harmonic. Now, we give a new approach for harmonicity of a pointwise semi-slant submersion. By Theorem 5.2-(a), we obtain the following result.*

Corollary 5.1. *Let π be a pointwise semi-slant submersion with compact fibers from a l.p.R. manifold (M, g, F) onto a Riemannian manifold (N, g_N). If $H \in \mathbb{L}$, then π is harmonic if and only if π_q^{-1} is $\mathbb{L} - minimal$.*

Lemma 5.3. *Let π be a pointwise semi-slant submersion with compact fibers from a l.p.R. manifold (M, g, F) onto a Riemannian manifold (N, g_N). Then,*

$$\delta\sigma_H = 0 \Leftrightarrow \Sigma_i g(\mathcal{T}_{\phi E_i} E_i, H) = \Sigma_i g(\mathcal{A}_{\omega E_i} E_i, H), \tag{5.60}$$

where $\{E_1, E_2, ..., E_m\}$ is a local basis of $\mathcal{D}_\theta$.

Proof.

$$\delta\sigma_H = 0 \Leftrightarrow \Sigma_i g(\nabla_{E_i} FH, E_i) = 0.$$

Using (3.12),

$$\begin{aligned}\Rightarrow \delta\sigma_H = 0 \Leftrightarrow& \Sigma_i g(\nabla_{E_i} H, FE_i) \Leftrightarrow \Sigma_i g(\nabla_{E_i} H, \phi E_i + \omega E_i) \\ =& \Sigma_i g(\nabla_{E_i} H, \phi E_i) + \Sigma_i g(\nabla_{E_i} H, \omega E_i) \\ =& \Sigma_i g(\mathcal{T}_{E_i} H, \phi E_i) + \Sigma_i g(\mathcal{A}_H E_i, \omega E_i).\end{aligned}$$

Thus, the assertion follows from the skew-symmetry and symmetry properties of the O'Neill tensors $\mathcal{A}$ and $\mathcal{T}$.

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