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HARMONICITY OF MUS-GRADIENT METRIC

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Abstract. Let (M^m, g) be an m-dimensional Riemannian manifold. In this paper, we introduce an other class of metric on (M^m, g) called Mus-gradient metric. First we investigate the Levi-Civita connection of this metric. Secondly we study some properties of harmonicity with respect to the Mus-gradient metric. In the last section, we investigate the harmonicity of Mus-gradient metric on product manifolds. Also, we construct some examples of harmonic maps.

Keywords: Levi-Civita conexion, Mus-Gradient metric, Harmonic maps

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1. Introduction

The theory of harmonic maps studies the mapping between different metric manifolds from the energy-minimization point of view (solutions to a natural geometrical variational problem). This concept has several applications such as geodesics, minimal surfaces and harmonic functions. Harmonic maps are also closely related to holomorphic maps in several complex variables, to the theory of stochastic processes, to nonlinear field theory in theoretical physics, and to the theory of liquid crystals in materials science. The last years this subject has been developed extensively by several authors (for example see [1], [3], [4], [5], [7], [8], [12], [10], [11], [12] etc...).

The main idea in this note consists in the modification of the metric of the Riemannian

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manifold (M^m, g). Firstly we introduce the Mus-gradient metric on M noted by $\tilde{g}$ and we investigate the Levi-Civita connection of this metric (Theorem 2.1). Secondly we study the harmonicity with respect to the Mus-gradient metric, then we establish necessary and sufficient conditions under which the Identity Map is harmonic with respect to this metric (Theorem 3.2 and Theorem 3.4). Next we study the harmonicity of the map $\sigma : (M, \tilde{g}) \longrightarrow (N, h)$ (Theorem 3.6) and the map $\sigma : (M, g) \longrightarrow (N, \tilde{h})$ (Theorem 3.8). In the last section, we investigate the harmonicity of Mus-gradient metric on product manifolds (Theorem 4.1 to Theorem 4.7). We also construct some examples of harmonic maps.

2. Mus-gradient metric

Definition 2.1. *Let (M^m, g) be a Riemannian manifold and $f : M \to]0, +\infty[$ be a strictly positive smooth function. We define the Mus-gradient metric on M noted $\tilde{g}$ by*

$$\tilde{g}(X, Y)_x = f(x) g(X, Y)_x + X_x(f) Y_x(f), \tag{2.1}$$

where $x \in M$ and $X, Y \in \Im^1_0(M)$, f is called twisting function.

In the following, we consider $\|grad\, f\| = 1$, where $\|.\|$ denote the norm with respect to (M^m, g).

Lemma 2.1. *Let $grad\, f$ (resp. $\widetilde{grad}\, f$) denote the gradient of f with respect to g (resp. $\tilde{g}$), then we have*

$$\widetilde{grad}\, f = \frac{1}{f+1}\, grad\, f. \tag{2.2}$$

Proof. We have

$$\begin{aligned} X(f) &= g(X, grad\, f) \\ &= \frac{1}{f}\big(\widetilde{g}(X, grad\, f) - X(f)(grad\, f)(f)\big) \\ &= \frac{1}{f}\big(\widetilde{g}(X, grad\, f) - X(f)\big) \end{aligned}$$

on the other hand, we have $X(f) = \widetilde{g}(X, \widetilde{grad}\, f)$, then

$$\begin{aligned} \widetilde{g}(X, \widetilde{grad}\, f) &= \frac{1}{f}\big(\widetilde{g}(X, grad\, f) - \widetilde{g}(X, \widetilde{grad}\, f)\big) \\ &= \frac{1}{f+1}\widetilde{g}(X, grad\, f) \end{aligned}$$

so, thus $\widetilde{grad}\, f = \dfrac{1}{f+1}\, grad\, f.$

We shall calculate the Levi-Civita connection $\widetilde{\nabla}$ of $(M^m, \tilde{g})$, as follows.

Theorem 2.1. *Let (M^m, g) be a Riemannian manifold, the Levi-Civita connection $\widetilde{\nabla}$ of $(M^m, \tilde{g})$, is given by*

$$\begin{aligned}\widetilde{\nabla}_X Y &= \nabla_X Y + \frac{X(f)}{2f}Y + \frac{Y(f)}{2f}X \\ &+ \Big(\frac{Hess_f(X,Y)}{f+1} - \frac{X(f)Y(f)}{f(f+1)} - \frac{g(X,Y)}{2(f+1)}\Big) grad\, f \qquad (2.3)\end{aligned}$$

for all vector fields $X, Y \in \Im_0^1(M)$, where ∇ denote the Levi-Civita connection of (M^m, g) and $Hess_f(X,Y) = g(\nabla_X grad\, f, Y)$ is the Hessian of f with respect to g.

Proof. From Kozul formula and Lemma 2.1, we have

$$\begin{aligned}
2\tilde{g}(\widetilde{\nabla}_X Y, Z) &= X\tilde{g}(Y,Z) + Y\tilde{g}(Z,X) - Z\tilde{g}(X,Y) + \tilde{g}(Z,[X,Y]) \\
&+ \tilde{g}(Y,[Z,X]) - \tilde{g}(X,[Y,Z]) \\
&= X\big(fg(Y,Z) + Y(f)Z(f)\big) + Y\big(fg(Z,X) + Z(f)X(f)\big) \\
&- Z\big(fg(X,Y) + X(f)Y(f)\big) + fg(Z,[X,Y]) + Z(f)[X,Y](f) \\
&+ fg(Y,[Z,X]) + Y(f)[Z,X](f) - fg(X,[Y,Z]) \\
&- X(f)[Y,Z](f) \\
&= X(f)g(Y,Z) + fXg(Y,Z) + X(Y(f))Z(f) + Y(f)X(Z(f)) \\
&+ Y(f)g(Z,X) + fYg(Z,X) + Y(Z(f))X(f) + Z(f)Y(X(f)) \\
&- Z(f)g(X,Y) - fZg(X,Y) - Z(X(f))Y(f) - X(f)Z(Y(f)) \\
&+ fg(Z,[X,Y]) + Z(f)\big(X(Y(f)) - Y(X(f))\big) + fg(Y,[Z,X]) \\
&+ Y(f)\big(Z(X(f)) - X(Z(f))\big) - fg(X,[Y,Z]) \\
&- X(f)\big(Y(Z(f)) - Z(Y(f))\big) \\
&= 2fg(\nabla_X Y.Z) + X(f)g(Y,Z) + Y(f)g(Z,X) - Z(f)g(X,Y) \\
&+ 2X(Y(f))Z(f) \\
&= 2\tilde{g}(\nabla_X Y, Z) - 2(\nabla_X Y)(f)Z(f) + 2X(Y(f))Z(f) \\
&+ \frac{X(f)}{f}\big(\tilde{g}(Y,Z) - Y(f)Z(f)\big) + \frac{Y(f)}{f}\big(\tilde{g}(Z,X) - Z(f)X(f)\big) \\
&- Z(f)g(X,Y).
\end{aligned}$$

From the definition of Hessian, we obtain

$$\begin{aligned} 2\tilde{g}(\widetilde{\nabla}_X Y, Z) &= 2\tilde{g}(\nabla_X Y, Z) + \frac{X(f)}{f}\tilde{g}(Y,Z) + \frac{Y(f)}{f}\tilde{g}(Z,X) \\ &\quad + \big(2Hess_f(X,Y) - \frac{2X(f)Y(f)}{f} - g(X,Y)\big)Z(f) \\ &= 2\tilde{g}(\nabla_X Y + \frac{X(f)}{2f}Y + \frac{Y(f)}{2f}X, Z) \\ &\quad + 2\big(Hess_f(X,Y) - \frac{X(f)Y(f)}{f} - \frac{1}{2}g(X,Y)\big)\tilde{g}(\widetilde{grad}\, f, Z). \end{aligned}$$

From the formula (2.2), we get

$$\begin{aligned} \widetilde{\nabla}_X Y &= \nabla_X Y + \frac{X(f)}{2f}Y + \frac{Y(f)}{2f}X \\ &\quad + \big(\frac{Hess_f(X,Y)}{f+1} - \frac{X(f)Y(f)}{f(f+1)} - \frac{g(X,Y)}{2(f+1)}\big) grad\, f. \end{aligned}$$

Lemma 2.2. *Let (M^m, g) be a Riemannian manifold, then for all vector field $X \in \Im^1_0(M)$, we have*

$$\widetilde{\nabla}_X grad\, f = \nabla_X grad\, f + \frac{1}{2f}X - \frac{X(f)}{2f(f+1)} grad\, f. \tag{2.4}$$

Proof. Using the theorem 2.1, we have

$$\begin{aligned} \widetilde{\nabla}_X grad\, f &= \nabla_X grad\, f + \frac{X(f)}{2f} grad\, f + \frac{(grad\, f)(f)}{2f} X \\ &\quad + \big(\frac{Hess_f(X, grad\, f)}{f+1} - \frac{X(f)(grad\, f)(f)}{f(f+1)} - \frac{g(X, grad\, f)}{2(f+1)}\big) grad\, f. \end{aligned}$$

Since $\|grad\, f\| = 1$, we obtain $(grad\, f)(f) = 1$ and $Hess_f(X, grad\, f) = 0$. then we get

$$\widetilde{\nabla}_X grad\, f = \nabla_X grad\, f + \frac{1}{2f}X - \frac{X(f)}{2f(f+1)} grad\, f.$$

3. Harmonicity of Mus-gradient metric

Consider a smooth map $\phi : (M^m, g) \to (N^n, h)$ between two Riemannian manifolds, then the second fundamental form of ϕ is defined by

$$(\nabla d\phi)(X,Y) = \nabla^\phi_X d\phi(Y) - d\phi(\nabla_X Y). \tag{3.5}$$

Here ∇ is the Riemannian connection on M and ∇^ϕ is the pull-back connection on the pull-back bundle $\phi^{-1}TN$. The tension field of ϕ is defined by

$$\tau(\phi) = trace_g \nabla d\phi = \sum_{i=1}^{m} \big(\nabla^\phi_{E_i} d\phi(E_i) - d\phi(\nabla_{E_i} E_i)\big), \tag{3.6}$$

where $\{E_i\}_{i=\overline{1,m}}$ is an orthonormal frame on (M^m, g). A map ϕ is called harmonic if and only if $\tau(\phi) = 0$.

Remark 3.1. *Let (M^m, g) be a Riemannian manifold and $\tilde{g}$ the Mus-gradient metric on M. If $\{E_i\}_{i=\overline{1,m}}$ be an orthonormal frame on (M^m, g), such that $E_1 = grad\, f$, the set $\{\widetilde{E}_i\}_{i=\overline{1,m}}$, which is defined as below, is an orthonomal frame on $(M^m, \tilde{g})$, then*

$$\widetilde{E}_1 = \frac{1}{\sqrt{f+1}} E_1, \widetilde{E}_i = \frac{1}{\sqrt{f}} E_i,\ i = \overline{2,m}, \tag{3.7}$$

where $f : M \to]0, +\infty[$ be a strictly positive smooth function.

Theorem 3.1. *The tension field of the Identity Map $I : (M^m, \tilde{g}) \to (M^m, g)$ is given by*

$$\tau(I) \quad = \quad \frac{1}{f(f+1)}\Big(\frac{(m-2)f + m - 1}{2(f+1)} - \Delta(f)\Big) grad\, f, \tag{3.8}$$

where $\Delta(f) = trace_g Hess_f = \sum_{i=1}^{m} g(\nabla_{E_i} grad\, f, E_i)$.

Proof. Let $\{\widetilde{E}_i\}_{i=\overline{1,m}}$ be a locale orthonormal frame on $(M^m, \tilde{g})$ defined by (3.10), then

$$\begin{aligned}
\tau(I) &= \sum_{i=1}^{m} \big(\nabla^I_{\widetilde{E}_i} dI(\widetilde{E}_i) - dI(\widetilde{\nabla}_{\widetilde{E}_i} \widetilde{E}_i)\big) \\
&= \sum_{i=1}^{m} \big(\nabla_{\widetilde{E}_i} \widetilde{E}_i - \widetilde{\nabla}_{\widetilde{E}_i} \widetilde{E}_i\big) \\
&= \sum_{i=1}^{m} \Big(- \frac{\widetilde{E}_i(f)}{f} \widetilde{E}_i - \big(\frac{Hess_f(\widetilde{E}_i, \widetilde{E}_i)}{f+1} - \frac{\widetilde{E}_i(f)^2}{f(f+1)} - \frac{g(\widetilde{E}_i, \widetilde{E}_i)}{2(f+1)}\big) grad\, f\Big) \\
&= \big(\frac{-1}{f(f+1)} - \frac{\Delta(f)}{f(f+1)} + \frac{1}{f(f+1)^2} + \frac{1}{2(f+1)^2} + \frac{m-1}{2f(f+1)}\big) grad\, f \\
&= \frac{1}{f(f+1)}\big(\frac{(m-2)f + m - 1}{2(f+1)} - \Delta(f)\big) grad\, f.
\end{aligned}$$

From the Theorem 3.1 we obtain

Theorem 3.2. *The Identity Map $I : (M^m, \tilde{g}) \to (M^m, g)$ is harmonic if and only if $f =$ const or*

$$\Delta(f) = \frac{(m-2)f + m - 1}{2(f+1)}. \tag{3.9}$$

Example 3.1. *Let $M =]0, +\infty[\times_F \mathbb{R}^{m-1}$ be the Riemannian twisted product manifold equipped with the Riemannian metric g defined by*

$$g = dx_1^2 + F(x_1) g_{\mathbb{R}^{m-1}}$$

were $g_{\mathbb{R}^{m-1}}$ is the standard metric and

$$F(x_1) = e^{\frac{m-2}{m-1} x_1} (x_1 + 1)^{\frac{1}{m-1}}.$$

Let $f(x_1,\cdots,x_m)=x_1$*, it's clear that* $\|gradf\|=1$

as we have

$$\Delta(f)=\frac{(m-2)f+m-1}{2(f+1)}.$$

So, thus the Identity Map $I:(M^m,\tilde{g})\to(M^m,g)$ *is harmonic.*

Example 3.2. *Let* $m=2$ *and* $f(x,y)=F_1(y-Ix)+F_2(y+Ix)+\frac{1}{2}x^2+\frac{1}{2}y^2$*, where* $F_1,F_2:\mathbb{C}\to\mathbb{R}_+^*$ *and* $I^2=-1$*. Then the Identity Map* $I:(M^m,\tilde{g})\to(M^m,g)$ *is harmonic.*

Theorem 3.3. *The tension field of the Identity Map* $I:(M^m,g)\to(M^m,\tilde{g})$ *is given by*

$$\tau(I) \;=\; \frac{1}{f+1}\big(\Delta(f)+\frac{2-m}{2}\big)grad\,f. \tag{3.10}$$

Proof. Let $\{E_i\}_{i=\overline{1,2m}}$ be a locale orthonormal frame on M, then

$$\begin{aligned}
\tau(I) &= \sum_{i=1}^{m}\big(\nabla^I_{E_i}dI(E_i)-dI(\nabla_{E_i}E_i)\big)\\
&= \sum_{i=1}^{m}\widetilde{\nabla}_{dI(E_i)}dI(E_i)-\nabla_{E_i}E_i\\
&= \sum_{i=1}^{m}\widetilde{\nabla}_{E_i}E_i-\nabla_{E_i}E_i\\
&= \sum_{i=1}^{m}\Big(\frac{E_i(f)}{f}E_i+\big(\frac{Hess_f(E_i,E_i)}{f+1}-\frac{E_i(f)^2}{f(f+1)}-\frac{g(E_i,E_i)}{2(f+1)}\big)grad\,f\Big)\\
&= \frac{1}{f}grad\,f+\big(\frac{\Delta(f)}{f+1}-\frac{1}{f(f+1)}-\frac{m}{2(f+1)}\big)grad\,f\\
&= \frac{1}{f+1}\big(\frac{2-m}{2}+\Delta(f)\big)grad\,f.
\end{aligned}$$

From the Theorem 3.3 we obtain

Theorem 3.4. *The Identity Map* $I:(M^m,g)\to(M^m,\tilde{g})$ *is harmonic if and only if*

$$\Delta(f)=\frac{m-2}{2}. \tag{3.11}$$

Example 3.3. *The Identity Map* $I:(IR^2,g=dx^2)\to(IR^2,\tilde{g})$ *is harmonic if and only if*

$$\Delta(f)=\frac{\partial^2 f}{(\partial x)^2}+\frac{\partial^2 f}{(\partial y)^2}=0. \tag{3.12}$$

Example 3.4. *Let* $M=]0,+\infty[\times]\frac{-\pi}{4},\frac{3\pi}{4}[$ *be endowed with the Riemannian metric* g *in polar coordinate defined by*

$$g=dr^2+r^2d\theta^2.$$

The non-null Christoffel symbols of the Riemannian connection are:

$$\Gamma_{12}^2 = \Gamma_{21}^2 = \frac{1}{r},\ \Gamma_{22}^1 = -r.$$

Relatively to the orthonormal frame

$$e_1 = \frac{\partial}{\partial r},\ e_2 = \frac{1}{r}\frac{\partial}{\partial \theta},$$

we have

$$\nabla_{e_1} e_1 = \nabla_{e_1} e_2 = 0\ ,\ \nabla_{e_2} e_1 = \frac{1}{r^2}\frac{\partial}{\partial \theta},\ \nabla_{e_2} e_2 = \frac{-1}{r}\frac{\partial}{\partial r}.$$

Let $f(r,\theta) = r\sin(\theta + \frac{\pi}{4})$, *for all* $(r,\theta) \in M$.
By direct computations we obtain

$$\begin{aligned} grad\, f &= \sin(\theta + \frac{\pi}{4})\frac{\partial}{\partial r} + \frac{1}{r}\cos(\theta + \frac{\pi}{4})\frac{\partial}{\partial \theta}, \\ \|grad\, f\| &= 1, \\ \Delta(f) &= 0. \end{aligned}$$

By virtue of the Theorem 3.4 *the identity map* $I : (M^m, g) \to (M^m, \tilde{g})$ *is harmonic, where*

$$\tilde{g} = \big(r\sin(\theta + \frac{\pi}{4}) + \sin^2(\theta + \frac{\pi}{4})\big)dr^2 + r^2\big(r\sin(\theta + \frac{\pi}{4}) + \cos^2(\theta + \frac{\pi}{4})\big)d\theta^2 + r\cos(2\theta)drd\theta.$$

Theorem 3.5. *The tension field of the map* $\sigma : (M^m, \widetilde{g}) \longrightarrow (N^n, h)$ *is given by*

$$\begin{aligned} \widetilde{\tau}(\sigma) &= \frac{1}{f}\tau(\sigma) + \frac{1}{f(f+1)}\big(\frac{(m-2)f + m - 1}{2(f+1)} - \Delta(f)\big)d\sigma(grad\, f) \\ &\quad - \frac{1}{f(f+1)}\nabla^N_{d\sigma(grad\, f)} d\sigma(grad\, f), \end{aligned} \tag{3.13}$$

where $f : M \to]0, +\infty[$ *be a strictly positive smooth function and* $\tau(\sigma)$ *is the tension field of* $\sigma : (M, g) \longrightarrow (N, h)$.

Proof. Let $\{\widetilde{E}_i\}_{i=\overline{1,m}}$ be a locale orthonormal frame on $(M^m, \tilde{g})$ defined by (3.10), then

$$\begin{aligned} \tau(I) &= \sum_{i=1}^m \big(\nabla^\sigma_{\widetilde{E}_i} d\sigma(\widetilde{E}_i) - d\sigma(\widetilde{\nabla}^M_{\widetilde{E}_i}\widetilde{E}_i)\big) \\ &= \sum_{i=1}^m \nabla^\sigma_{\widetilde{E}_i} d\sigma(\widetilde{E}_i) - \sum_{i=1}^m d\sigma(\widetilde{\nabla}^M_{\widetilde{E}_i}\widetilde{E}_i). \end{aligned}$$

By direct computations we obtain

$$\begin{aligned}\sum_{i=1}^{m}\nabla^{\sigma}_{\widetilde{E}_i}d\sigma(\widetilde{E}_i) &= \nabla^{\sigma}_{\widetilde{E}_1}\widetilde{E}_1+\sum_{i=2}^{m}\nabla^{\sigma}_{\widetilde{E}_i}\widetilde{E}_i\\ &= \frac{1}{\sqrt{f+1}}\nabla^{\sigma}_{E_1}\frac{1}{\sqrt{f+1}}E_1+\sum_{i=2}^{m}\frac{1}{\sqrt{f}}\nabla^{\sigma}_{E_i}\frac{1}{\sqrt{f}}E_i\\ &= \frac{-1}{2(f+1)^2}d\sigma(grad\, f)-\frac{1}{f(f+1)}\nabla^{N}_{d\sigma(grad\, f)}d\sigma(grad\, f)\\ &\quad +\frac{1}{f}\sum_{i=1}^{m}\nabla^{\sigma}_{E_i}d\sigma(E_i),\end{aligned}$$

and

$$\begin{aligned}\sum_{i=1}^{m}d\sigma(\widetilde{\nabla}^{M}_{\widetilde{E}_i}\widetilde{E}_i) &= d\sigma\big(\sum_{i=1}^{m}\widetilde{\nabla}^{M}_{\widetilde{E}_i}\widetilde{E}_i\big)\\ &= d\sigma\big(\widetilde{\nabla}^{M}_{\widetilde{E}_1}\widetilde{E}_1+\sum_{i=2}^{m}\widetilde{\nabla}^{M}_{\widetilde{E}_i}\widetilde{E}_i\big)\\ &= d\sigma\big(\frac{1}{\sqrt{f+1}}\widetilde{\nabla}^{M}_{E_1}\frac{1}{\sqrt{f+1}}E_1+\sum_{i=2}^{m}\frac{1}{\sqrt{f}}\widetilde{\nabla}^{M}_{E_i}\frac{1}{\sqrt{f}}E_i\big)\\ &= \frac{1}{f}\sum_{i=1}^{m}d\sigma(\nabla^{\sigma}_{E_i}E_i)+\big(\frac{\Delta(f)}{f(f+1)}-\frac{m-1}{2f(f+1)}\big)d\sigma(grad\, f),\end{aligned}$$

hence we get

$$\begin{aligned}\widetilde{\tau}(\sigma) &= \frac{1}{f}\tau(\sigma)+\frac{1}{f(f+1)}\big(\frac{(m-2)f+m-1}{2(f+1)}-\Delta(f)\big)d\sigma(grad\, f)\\ &\quad -\frac{1}{f(f+1)}\nabla^{N}_{d\sigma(grad\, f)}d\sigma(grad\, f).\end{aligned}$$

From the Theorem 3.5 we obtain

Theorem 3.6. *Let $\sigma:(M^m,g)\longrightarrow(N^n,h)$ be harmonic. Then the map $\sigma:(M^m,\widetilde{g})\longrightarrow(N^n,h)$ is harmonic if and only if*

$$\begin{aligned}\tau(\sigma) &= \frac{1}{f+1}\big(\Delta(f)-\frac{(m-2)f+m-1}{2(f+1)}\big)d\sigma(grad\, f)\\ &\quad +\frac{1}{f+1}\nabla^{N}_{d\sigma(grad\, f)}d\sigma(grad\, f). \qquad (3.14)\end{aligned}$$

Example 3.5. *If we set $\sigma=Id_M$ and $f=const$ then $\sigma:(M^m,\widetilde{g})\longrightarrow(N^n,h)$ is harmonic.*

Lemma 3.1. [1] *Given a smooth map $\sigma:(M^m,g)\longrightarrow(N^n,h)$ between two Riemannian manifolds and $f\in C^{\infty}(N)$, then we have*

$$\Delta(f\circ\sigma) = trace_g Hess_f(d\sigma,d\sigma)+df(\tau(\sigma)). \qquad (3.15)$$

Proof. Let $X, Y \in \Im_0^1(M)$, we have $f \circ \sigma \in C^\infty(M)$ then

$$\begin{aligned}\nabla d(f\circ\sigma)(X,Y) &= \nabla_X^{f\circ\sigma} d(f\circ\sigma)(Y) - d(f\circ\sigma)(\nabla_X^M Y)\\ &= \nabla_{d\sigma(X)}^{f} df(d\sigma(Y)) - df(d\sigma(\nabla_X^M Y))\\ &= \nabla df(d\sigma(X), d\sigma(Y)) + df(\nabla_{d\sigma(X)}^N d\sigma(Y)) - df(d\sigma(\nabla_X^M Y))\\ &= \nabla df(d\sigma(X), d\sigma(Y)) + df(\nabla d\sigma(X,Y)).\end{aligned}$$

By passing to the trace in the last equation and using

$$trace_g \nabla df = trace_g Hess_f$$

we get

$$\Delta(f\circ\sigma) = trace_g Hess_f(d\sigma, d\sigma) + df(\tau(\sigma)).$$

Theorem 3.7. *The tension field of the map* $\sigma : (M^m, g) \longrightarrow (N^n, \widetilde{h})$ *is given by*

$$\begin{aligned}\widetilde{\tau}(\sigma) &= \tau(\sigma) + \frac{1}{f} d\sigma(grad(f\circ\sigma))\\ &\quad + \frac{1}{f+1}\big(\Delta(f\circ\sigma) - df(\tau(\sigma)) - \frac{\|grad\,(f\circ\sigma)\|^2}{f} - \frac{\|d\sigma\|^2}{2}\big)(grad\, f)\circ\sigma, \quad (3.16)\end{aligned}$$

where $f : N \to]0, +\infty[$ *be a strictly positive smooth function and* $\tau(\sigma)$ *is the tension field of* $\sigma : (M, g) \longrightarrow (N, h)$.

Proof. Let $\{E_i\}_{i=\overline{1,m}}$ be a locale orthonormal frame on (M^m, g), then

$$\begin{aligned}
\widetilde{\tau}(\sigma) &= \sum_{i=1}^{m}\big(\widetilde{\nabla}^{\sigma}_{E_i} d\sigma(E_i) - d\sigma(\nabla^M_{E_i}E_i)\big)\\
&= \sum_{i=1}^{m}\big(\widetilde{\nabla}^{N}_{d\sigma(E_i)} d\sigma(E_i) - d\sigma(\nabla^M_{E_i}E_i)\big)\\
&= \sum_{i=1}^{m}\big(\nabla^{N}_{d\sigma(E_i)} d\sigma(E_i) + \frac{d\sigma(E_i)(f)}{f} d\sigma(E_i)\\
&\quad +\big(\frac{Hess_f(d\sigma(E_i), d\sigma(E_i))}{f+1} - \frac{(d\sigma(E_i)(f))^2}{f(f+1)}\\
&\quad -\frac{h(d\sigma(E_i), d\sigma(E_i))}{2(f+1)}\big)(grad\, f)\circ\sigma - d\sigma(\nabla^M_{E_i}E_i)\big)\\
&= \sum_{i=1}^{m}\big(\nabla^{\sigma}_{E_i} d\sigma(E_i) - d\sigma(\nabla^M_{E_i}E_i) + \frac{E_i(f\circ\sigma)}{f} d\sigma(E_i)\\
&\quad +\big(\frac{Hess_f(d\sigma(E_i), d\sigma(E_i))}{f+1} - \frac{(E_i(f\circ\sigma))^2}{f(f+1)}\\
&\quad -\frac{h(d\sigma(E_i), d\sigma(E_i))}{2(f+1)}\big)(grad\, f)\circ\sigma\big)\\
&= \tau(\sigma) + \frac{1}{f} d\sigma(grad(f\circ\sigma))\\
&\quad +\big(\frac{trace\, Hess_f(d\sigma, d\sigma)}{f+1} - \frac{\|grad\,(f\circ\sigma)\|^2}{f(f+1)} - \frac{\|d\sigma\|^2}{2(f+1)}\big)(grad\, f)\circ\sigma\\
&= \tau(\sigma) + \frac{1}{f} d\sigma(grad(f\circ\sigma))\\
&\quad +\big(\frac{\Delta(f\circ\sigma) - df(\tau(\sigma))}{f+1} - \frac{\|grad\,(f\circ\sigma)\|^2}{f(f+1)} - \frac{\|d\sigma\|^2}{2(f+1)}\big)(grad\, f)\circ\sigma\\
&= \tau(\sigma) + \frac{1}{f} d\sigma(grad(f\circ\sigma))\\
&\quad +\frac{1}{f+1}\big(\Delta(f\circ\sigma) - df(\tau(\sigma)) - \frac{\|grad\,(f\circ\sigma)\|^2}{f} - \frac{\|d\sigma\|^2}{2}\big)(grad\, f)\circ\sigma.
\end{aligned}$$

From the Theorem 3.7 we obtain

Theorem 3.8. *The map $\sigma : (M^m, g) \longrightarrow (N^n, \widetilde{h})$ is harmonic if and only if*

$$\begin{aligned}
\tau(\sigma) &= \frac{-1}{f+1}\big(\Delta(f\circ\sigma) - df(\tau(\sigma)) - \frac{\|grad\,(f\circ\sigma)\|^2}{f} - \frac{\|d\sigma\|^2}{2}\big)(grad\, f)\circ\sigma\\
&\quad -\frac{1}{f} d\sigma(grad(f\circ\sigma)).
\end{aligned} \tag{3.17}$$

4. Harmonicity on product manifold

Let (M, g) and (N, h) be a Riemannian manifolds.

Definition 4.1. *Let (M, g) and (N, h) be two Riemannian manifolds of dimension m and n respectively. We define the product metric on $M \times N$ by*

$$G = \pi^* g + \eta^* h$$

where $\pi : M \times N \longrightarrow M$ and $\eta : M \times N \longrightarrow N$ denote the first and the second canonical projection.

Proposition 4.1. *For all vector fields $X_1, X_2 \in \mathcal{H}(M)$ and $Y_1, Y_2 \in \mathcal{H}(N)$ we have*

$$\begin{aligned} G\big((X_1, Y_1), (X_2, Y_2)\big) &= g(X_1, X_2) + h(Y_1, Y_2) \\ G\big((X_1, 0), (X_2, 0)\big) &= g(X_1, X_2) \\ G\big((0, Y_1), (0, Y_2)\big) &= h(Y_1, Y_2) \\ G\big((X_1, 0), (0, Y_2)\big) &= 0. \end{aligned}$$

Subsequently, if $X \in \mathcal{H}(M)$ and $Y \in \mathcal{H}(N)$, then we denote (X, Y) by $X + Y$.

Remark 4.1. • *Any vector field of $\mathcal{H}(M)$ is orthogonal to all vector fields of $\mathcal{H}(N)$.*

• *Let $(E_1, ..., E_m)$ (resp $(E_{m+1}, ..., E_{m+n})$) is an orthonormal basis of $\mathcal{H}(M)$ (resp $\mathcal{H}(N)$) then $(E_1, ..., E_{m+n})$ is an orthonormal basis of $\mathcal{H}(M \times N)$.*

• *Let $f \in \mathbf{C}^\infty(M)$, then $\triangle(f) = \sum_{i=1}^m Hess_f(E_i, E_i)$.*

Proposition 4.2. *Let (M, g) and (N, h) be two Riemannian manifolds. If ${}^M\nabla$ (resp ${}^N\nabla$) denote the connection of Levi-Civita on M (resp N), then the levi-civita connection ∇ on the manifold $M \times N$ associated with the product metric $G = \pi^* g + \eta^* h$ is verifies the following properties:*

$$\begin{cases} \nabla_{X_1} X_2 =^M \nabla_{X_1} X_2 \\ \nabla_{Y_1} Y_2 =^N \nabla_{Y_1} Y_2 \\ \nabla_{X_1} Y_1 = \nabla_{Y_2} X_2 = 0 \\ \nabla_{\left(X_1 + Y_1\right)} \big(X_2 + Y_2\big) =^M \nabla_{X_1} X_2 +^N \nabla_{Y_1} Y_2 \end{cases}$$

for any $X_1, Y_1 \in \mathcal{H}(M)$ *and* $X_2, Y_2 \in \mathcal{H}(N)$.

Lemma 4.1. *Let* (M^m, g) *and* (N^n, h) *be two Riemannian manifolds and* $f \in C^\infty(M)$. *If* $P : (x, y) \in M \times N \to y \in N$ *(resp* $P : (x, y) \in M \times N \to (0, y) \in M \times N$*) is the second projection, then we have*

$$\begin{aligned} grad(f) &= grad_G(f) = grad_g(f), \\ d\,P(grad(f)) &= 0, \\ d\,P(\widetilde{\nabla}_X X) &= d\,P(\nabla_X X) + \frac{X(f)}{f} d\,P(X) \end{aligned} \tag{4.18}$$

where $X \in \mathcal{H}(M \times N)$.

Proof. The proof of the formula (4.18) is a direct consequence of Theorem 2.1.

Theorem 4.1. *Let* (M^m, g) *be a Riemannian manifolds and* (N^n, h) *be an Euclidian manifold. If* $f \in C^\infty(M)$ *is a smooth positif function, then the second projection*

$$\begin{aligned} P : (M \times N, \widetilde{G}) &\to (N, h) \\ (x, y) &\mapsto y \end{aligned}$$

is harmonic map. where $G = g + h$.

Proof. Let $(E_1, ..., E_m)$ be an orthonormal basis on (M^m, g) such as $E_1 = grad(f)$ and $(E_{m+1}, ..., E_{m+n})$ be an orthonormal basis on (N^n, h) such as ${}^N\nabla_{E_i} E_j = 0, \quad (i, j \geq m+1)$, then $(E_1, ..., E_{m+n})$ is an orthonormal basis on $(M \times N, g + h)$.

From Lemma 4.1, we obtain:

$$\begin{aligned} {}^N\nabla_{d\,P(\widetilde{E_i})} d\,P(\widetilde{E_i}) - d\,P\big(\widetilde{\nabla}_{\widetilde{E_i}} \widetilde{E_i}\big) &= -d\,P\big(\nabla_{\widetilde{E_i}} \widetilde{E_i}\big) \\ &= -d\,P\big({}^M\nabla_{\widetilde{E_i}} \widetilde{E_i}\big) \\ &= 0 \end{aligned}$$

for $1 \leq i \leq m$, and

$$\begin{aligned} {}^N\nabla_{d\,P(\widetilde{E_i})} d\,P(\widetilde{E_i}) - d\,P\big(\widetilde{\nabla}_{\widetilde{E_i}} \widetilde{E_i}\big) &= {}^N\nabla_{\widetilde{E_i}} \widetilde{E_i} - d\,P\big(\nabla_{\widetilde{E_i}} \widetilde{E_i}\big) \\ &= 0 \end{aligned}$$

for $m + 1 \leq i \leq m + n$. We therefore deduce $\tau(P) = 0$.

We find the same result for the following theorem

Theorem 4.2. *Let (M^m, g) be a Riemannian manifolds and (N^n, h) be an Euclidian manifold. If $f \in C^\infty(M)$ is a smooth positif function, then*

$$\begin{aligned} P : (M \times N, \widetilde{G}) &\rightarrow (M \times N, G) \\ (x, y) &\mapsto (0, y) \end{aligned}$$

is harmonic map. where $G = g + h$.

Theorem 4.3. *Let (M^m, g) be a Riemannian manifolds and (N^n, h) be an Euclidian manifold. If $f \in C^\infty(M)$ is a smooth positif function, then the tension field of*

$$\begin{aligned} P : (M \times N, G) &\rightarrow (M \times N, \widetilde{G}) \\ (x, y) &\mapsto (0, y) \end{aligned}$$

is given by

$$\tau(P) = \frac{-n}{2(f+1)} grad(f).$$

Proof. Similarly to the proof of Theorem 4.1, we obtain

$$\begin{aligned} \widetilde{\nabla}_{d\,P(E_i)} d\,P(E_i) - d\,P\big(\nabla_{E_i} E_i\big) &= 0, \quad (i \leq m). \\ \widetilde{\nabla}_{d\,P(E_i)} d\,P(E_i) - d\,P\big(\nabla_{E_i} E_i\big) &= -\frac{1}{2(f+1)} grad(f), \quad (i \geq m+1). \end{aligned}$$

Theorem 4.4. *Let (M^m, g) be a Riemannian manifolds and (N^n, h) be an Euclidian manifold. If $f \in C^\infty(M)$ is a smooth positif function, then the tension field of*

$$\begin{aligned} P : (M \times N, \widetilde{G}) &\rightarrow (M \times N, \widetilde{G}) \\ (x, y) &\mapsto (0, y) \end{aligned}$$

is given by

$$\tau(P) = -\frac{n}{2f(f+1)} grad(f)$$

where $G = g + h$.

Proof. Let $i \in \{m+1, .., n+m\}$, from Theorem 2.1 and Lemma 4.1 we obtain

$$\begin{aligned}
\widetilde{\nabla}_{d\,P(\widetilde{E}_i)} d\,P(\widetilde{E_i}) - d\,P\big(\widetilde{\nabla}_{\widetilde{E_i}}\widetilde{E_i}\big) &= \widetilde{\nabla}_{\widetilde{E}_i}\widetilde{E_i} - d\,P\big(\widetilde{\nabla}_{\widetilde{E_i}}\widetilde{E_i}\big) \\
&= \widetilde{\nabla}_{\widetilde{E}_i}\widetilde{E_i} - d\,P\big(\nabla_{\widetilde{E_i}}\widetilde{E_i}\big) \\
&= \widetilde{\nabla}_{\widetilde{E}_i}\widetilde{E_i} - \nabla_{\widetilde{E_i}}\widetilde{E_i} \\
&= -\frac{G(\widetilde{E_i}, \widetilde{E_i})}{2(f+1)} grad(f) \\
&= -\frac{1}{2f(f+1)} grad(f).
\end{aligned}$$

Example 4.1. *Let* $(M, g) = (IR^m, dx^{2)}$, $(m \geq 3)$ *and* $f(x_1, x_2, x_3......, x_m)$ $= f(x_1, x_2)$ *such that* $\big(\frac{\partial f}{\partial x_1}\big)^2 + \big(\frac{\partial f}{\partial x_2}\big)^2 = 1$. *If we put*

$$\begin{aligned}
\widetilde{P} : (M, \widetilde{g}) &\rightarrow (M, g) \\
(x_1, x_2, x_3......, x_m) &\mapsto (0, 0, x_3......, x_m)
\end{aligned}$$

then we obtain

$$\begin{aligned}
E_1 &= \partial_1(f)\partial_1 + \partial_2(f)\partial_2 \\
E_2 &= \partial_2(f)\partial_1 - \partial_1(f)\partial_2 \\
E_i &= \partial_i, \qquad (i \geq 3) \\
d\,\widetilde{P}(\widetilde{\nabla}_X X) &= d\,\widetilde{P}(\nabla_X X) + \frac{X(f)}{f} d\,P(X).
\end{aligned}$$

So

$$\begin{aligned}
\tau(\widetilde{P}) &= \sum_i \nabla_{d\,\widetilde{P}(\widetilde{E_i})} d\,\widetilde{P}(\widetilde{E_i}) - \sum_i d\,\widetilde{P}\big(\widetilde{\nabla}_{\widetilde{E_i}}\widetilde{E_i}\big) \\
&= 0.
\end{aligned}$$

Then $\widetilde{P}$ *is harmonic.*

On the other hand, the tension field of the projection

$$\begin{aligned}
P : (M, g) &\rightarrow (M, \widetilde{g}) \\
(x_1, x_2, x_3......, x_m) &\mapsto (0, 0, x_3......, x_m)
\end{aligned}$$

is given by the following formula

$$\tau(P) = \frac{2-m}{2(f+1)} grad(f).$$

Therefore, P *is non-harmonic.*

Theorem 4.5. *Let (M^m, g) be a Riemannian manifolds and (N^n, h) be an Euclidian manifold. If $f \in C^\infty(M)$ is a smooth positif function, then the tension field of*

$$\begin{aligned} Q : (M \times N, \widetilde{G}) &\to (M, g) \\ (x, y) &\mapsto x \end{aligned}$$

is given by

$$\tau(Q) = -\frac{1}{f(f+1)}\Big[\triangle(f) + \frac{(n+m-2)(f+1)+1}{2(f+1)}\Big]grad(f).$$

Proof. Let $(E_1, ..., E_m)$ be an orthonormal basis on (M^m, g) such as $E_1 = grad(f)$ and $(E_{m+1}, ..., E_{m+n})$ be an orthonormal basis on (N^n, h) such as ${}^N\nabla_{E_i}E_j = 0, \quad (i, j \geq m+1)$, then $(E_1, ..., E_{m+n})$ is an orthonormal basis on $(M \times N, g + h)$.

From Remark 3.1 and Theorem 2.1, we have:

$$\begin{aligned} \sum_{i=m+1}^{m+n}\Big[{}^M\nabla_{dQ(\widetilde{E_i})}dQ(\widetilde{E_i}) - dQ\big(\widetilde{\nabla}_{\widetilde{E_i}}\widetilde{E_i}\big)\Big] &= -\sum_{i=m+1}^{m+n} dQ\big(\widetilde{\nabla}_{\widetilde{E_i}}\widetilde{E_i}\big) \\ &= \sum_{i=m+1}^{m+n}\frac{G(\widetilde{E_i}, \widetilde{E_i})}{2(f+1)}grad(f) \\ &= \frac{n}{2f(f+1)}grad(f) \end{aligned}$$

$$\begin{aligned} {}^M\nabla_{dQ(\widetilde{E_1})}dQ(\widetilde{E_1}) \quad - \quad dQ\big(\widetilde{\nabla}_{\widetilde{E_1}}\widetilde{E_1}\big) = & \quad {}^M\nabla_{\widetilde{E_1}}\widetilde{E_1} - \widetilde{\nabla}_{\widetilde{E_1}}\widetilde{E_1} \\ = & \quad -\frac{\widetilde{E_1}(f)}{f}\widetilde{E_1} + \frac{\big(\widetilde{E_1}(f)\big)^2}{f(f+1)}grad(f) + \frac{G(\widetilde{E_1}, \widetilde{E_1})}{2(f+1)}grad(f) \\ = & \quad -\frac{1}{f(f+1)}grad(f) + \frac{1}{f(f+1)^2}grad(f) + \frac{1}{2(f+1)^2}grad(f) \\ = & \quad \Big[\frac{-1}{2(f+1)^2} - \frac{Hess_f(E_1, E_1)}{f(f+1)}\Big]grad(f) \end{aligned}$$

$$\begin{aligned} \sum_{i=2}^{m}\Big[{}^M\nabla_{dQ(\widetilde{E_i})}dQ(\widetilde{E_i}) - dQ\big(\widetilde{\nabla}_{\widetilde{E_i}}\widetilde{E_i}\big)\Big] &= \sum_{i=2}^{m}\Big[{}^M\nabla_{\widetilde{E_i}}\widetilde{E_i} - \widetilde{\nabla}_{\widetilde{E_i}}\widetilde{E_i}\Big] \\ &= \sum_{i=2}^{m}\Big[\frac{G(\widetilde{E_i}, \widetilde{E_i})}{2(f+1)} - \frac{Hess_f(E_i, E_i)}{f(f+1)}\Big]grad(f) \\ &= \Big[\frac{m-1}{2f(f+1)} - \frac{\triangle(f)}{f(f+1)}\Big]grad(f). \end{aligned}$$

Theorem 4.6. *Let (M^m, g) be a Riemannian manifolds and (N^n, h) be an Euclidian manifold. If $f \in C^\infty(M)$ is a smooth positif function, then the tension field of*

$$\begin{aligned} Q : (M \times N, \widetilde{G}) &\to (M \times N, \widetilde{G}) \\ (x, y) &\mapsto (x, 0) \end{aligned}$$

is given by

$$\tau(Q) = -\frac{n}{2f(f+1)} grad(f).$$

The proof of Theorem 4.6 follows immediately from the Remark 3.1, Remark 4.1 and Theorem 2.1.

Theorem 4.7. *Let (M^m, g) be a Riemannian manifolds and (N^n, h) be an Euclidian manifold. If $f \in C^\infty(M)$ is a smooth positif function, then the tension field of*

$$\begin{aligned} Q : (M \times N, G) &\to (M \times N, \widetilde{G}) \\ (x, y) &\mapsto (x, 0) \end{aligned}$$

is given by

$$\tau(Q) = \frac{1}{f(f+1)} \Big[\triangle(f) - \frac{(2-m)f + 1 - m}{2(f+1)} \Big] grad(f).$$

Proof. Let $(E_1, ..., E_m)$ be an orthonormal basis on (M^m, g) such as $E_1 = grad(f)$ and $(E_{m+1}, ..., E_{m+n})$ be an orthonormal basis on (N^n, h) such as ${}^N\nabla_{E_i} E_j = 0, \quad (i, j \geq m+1)$, then $(E_1, ..., E_{m+n})$ is an orthonormal basis on $(M \times N, g + h)$.

From Remark 3.1, Remark 4.1 and Theorem 2.1, we obtain:

$$\widetilde{\nabla}_{dQ(E_i)} dQ(E_i) - dQ\big(\nabla_{E_i} E_i\big) = 0, \qquad (m+1 \leq i \leq m+n).$$

$$\begin{aligned} \widetilde{\nabla}_{dQ(E_1)} dQ(E_1) - dQ\big(\nabla_{E_1} E_1\big) &= \widetilde{\nabla}_{E_1} E_1 - \nabla_{E_1} E_1 \\ &= \Big[\frac{Hess_f(E_1, E_1)}{f(f+1)} + \frac{1}{2(f+1)^2} \Big] grad(f). \end{aligned}$$

$$\begin{aligned} \widetilde{\nabla}_{dQ(E_i)} dQ(E_i) - dQ\big(\nabla_{E_i} E_i\big) &= \widetilde{\nabla}_{E_i} E_i - \nabla_{E_i} E_i \qquad (2 \leq i \leq m) \\ &= \Big[\frac{Hess_f(E_i, E_i)}{f(f+1)} - \frac{1}{2f(f+1)} \Big] grad(f). \end{aligned}$$

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