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REMARKS ON COMBINATORIAL SUMS ASSOCIATED WITH SPECIAL NUMBERS AND POLYNOMIALS WITH THEIR GENERATING FUNCTIONS

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Abstract. The purpose of this article is to give some novel identities and inequalities associated with combinatorial sums involving special numbers and polynomials. In particular, by using the method of generating functions and their functional equations, we derive not only some inequalities, but also many formulas, identities, and relations for the parametrically generalized polynomials, special numbers and special polynomials. Our identities, relations, inequalities and combinatorial sums are related to the Bernoulli numbers and polynomials of negative order, the Euler numbers and polynomials of negative order, the Stirling numbers, the Daehee numbers, the Changhee numbers, the Bernoulli polynomials, the Euler polynomials, the parametrically generalized polynomials, and other well-known special numbers and polynomials. Moreover, using Mathematica with the help of the Wolfram programming language, we illustrate some plots of the parametrically generalized polynomials under some of their randomly selected special conditions. Finally, we give some remarks and observations on our results.

Keywords: Bernoulli numbers and polynomials, Euler numbers and polynomials, Stirling numbers, Daehee numbers, Changhee numbers, Parametrically generalized polynomials, Generating functions, Special functions, Special numbers and polynomials.

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1. Introduction and preliminaries

Combinatorial sums and combinatorial numbers and polynomials have many applications in mathematics and other applied sciences. These numbers are related to the special functions and also some classes of special numbers and polynomials. The motivation of this paper is to use not only generating functions, but also their functional equations, we give many new formulas and combinatorial sums involving the Bernoulli numbers and polynomials, the Euler numbers and polynomials, the Stirling numbers, and also combinatorial numbers and polynomials such as the Daehee numbers, the Changhee numbers, and the parametrically generalized polynomials. By using these formulas and combinatorial sums, we provide some inequalities applications. In order to illustrate graph and plots of special polynomials, here we use Mathematica with the help of the Wolfram programming language.

Throughout of this paper, we use the following notations and definitions. Let

$$\mathbb{N}=\{1,2,3,\ldots\} \qquad \text{and} \qquad \mathbb{N}_0=\mathbb{N}\cup\{0\}.$$

As usual, $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ denote the set of integer numbers, the set of real numbers, the set of complex numbers, respectively. We assume that:

$$0^n=\begin{cases}1, & n=0\\ 0, & n\in\mathbb{N}.\end{cases}$$

Furthermore,

$$\binom{\lambda}{0}=1 \qquad \text{and} \qquad \binom{\lambda}{n}=\frac{(\lambda)_n}{n!} \quad (n\in\mathbb{N};\ \lambda\in\mathbb{C}),$$

where $(\lambda)_n$ is the falling factorial defined by

$$(\lambda)_n=\lambda(\lambda-1)(\lambda-2)\ldots(\lambda-n+1),$$

with $(\lambda)_0=1$ (*cf.* [1–34]; and references therein).

The Stirling numbers of the second kind are defined by means of the following generating function:

$$F_S(t,k)=\frac{\left(e^t-1\right)^k}{k!}=\sum_{n=0}^{\infty}S_2(n,k)\frac{t^n}{n!}, \tag{1.1}$$

(*cf.* [1–34]; and references therein).

The Stirling numbers of the second kind are also given by the falling factorial polynomials:

$$x^n=\sum_{j=0}^{n}S_2(n,j)(x)_j, \tag{1.2}$$

(*cf.* [1–34]; and references therein).

By using (1.1), an explicit formula for the numbers $S_2(n,k)$ is given as follows:

$$S_2(n,k) = \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}j^n, \tag{1.3}$$

where $n, k \in \mathbb{N}_0$ and for $k > n$,

$$S_2(n,k) = 0,$$

(*cf.* [1–34]; and references therein).

Let $v \in \mathbb{Z}$. The Bernoulli numbers and polynomials of higher order are defined by means of the following generating functions:

$$F_B(t,v) = \left(\frac{t}{e^t - 1}\right)^v = \sum_{n=0}^{\infty} B_n^{(v)}\frac{t^n}{n!}, \tag{1.4}$$

and

$$G_B(t,x,v) = F_B(t,v)\,e^{xt} = \sum_{n=0}^{\infty} B_n^{(v)}(x)\frac{t^n}{n!}, \tag{1.5}$$

such that $v = 0$,

$$B_n^{(0)}(x) = x^n,$$

(*cf.* [13, 23, 29, 30, 34]; and references therein).

Substituting $v = 1$ and $x = 0$ into (1.4) and (1.5), the Bernoulli numbers and polynomials are derived, respectively,

$$B_n = B_n^{(1)},$$

and

$$B_n(x) = B_n^{(1)}(x),$$

(*cf.* [1–34]; and references therein).

By using (1.5), an explicit formula for the polynomials $B_n^{(-k)}(x)$ is given as follows:

$$B_n^{(-k)}(x) = \frac{1}{\binom{n+k}{k}k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}(x+j)^{n+k}, \tag{1.6}$$

where $n \in \mathbb{N}_0$ and $k \in \mathbb{N}$ (*cf.* [5, Equation (3.20)]).

Putting $n = x = k$ in (1.6), we have the following presumably known result:

$$B_n^{(-n)}(n) = \frac{n!}{(2n)!}\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}(n+j)^{2n}.$$

Substituting $x = 0$ into the above equation, and using (1.3), we have the following well-known identity:

$$B_n^{(-k)} = \frac{1}{\binom{n+k}{k}}S_2(n+k,k) \tag{1.7}$$

(*cf.* [33, Equation (7.17)]).

Let $v \in \mathbb{Z}$. The Euler numbers and polynomials of higher order are defined by means of the following generating functions:

$$F_E(t,v) = \left(\frac{2}{e^t+1}\right)^v = \sum_{n=0}^{\infty} E_n^{(v)} \frac{t^n}{n!} \tag{1.8}$$

and

$$G_E(t,x,v) = F_E(t,v)\, e^{xt} = \sum_{n=0}^{\infty} E_n^{(v)}(x) \frac{t^n}{n!}, \tag{1.9}$$

such that $v=0$,

$$E_n^{(0)}(x) = x^n$$

(*cf.* [13, 23, 28, 29, 34]; and references therein).

Substituting $v=1$ and $x=0$ into (1.8) and (1.9), the Euler numbers and polynomials are derived, respectively,

$$E_n = E_n^{(1)}(0)$$

and

$$E_n(x) = E_n^{(1)}(x)$$

(*cf.* [1–34]; and references therein).

By using (1.9), we have

$$E_n^{(-k)}(x) = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} \sum_{d=0}^{j} \binom{d-k-1}{d} \frac{d!(-1)^d}{2^d} S_2(j,d), \tag{1.10}$$

where $n \in \mathbb{N}_0$ and $k \in \mathbb{N}$ (*cf.* [23, 28, 29, 34]).

Putting $n = x = k$ in (1.10), we have the following presumably known result:

$$E_n^{(-n)}(n) = \sum_{j=0}^{n} \binom{n}{j} n^{n-j} \sum_{d=0}^{j} \binom{d-n-1}{d} \frac{d!(-1)^d}{2^d} S_2(j,d).$$

By using (1.4) and (1.8), a relation between the numbers $E_n^{(-k)}$ and the numbers $B_n^{(-k)}$ is given as follows:

$$B_n^{(-k)} = \frac{1}{2^n} \sum_{j=0}^{n} \binom{n}{j} B_j^{(-k)} E_{n-j}^{(-k)}, \tag{1.11}$$

where $n \in \mathbb{N}_0$ and $k \in \mathbb{N}$ (*cf.* [13, Equation (3.1)]).

By using (1.1) and (1.8), a relation between the numbers $E_n^{(-k)}$ and the numbers $S_2(n,k)$ is given as follows:

$$S_2(n,k) = \frac{2^{k-n}}{k!} \sum_{m=0}^{n} \sum_{j=0}^{k} (-1)^{k-j} \binom{n}{m} \binom{k}{j} j^m E_{n-m}^{(-k)}, \tag{1.12}$$

(*cf.* [13, Theorem 2.14]).

The Euler numbers of the second kind, E_n^*, are defined by means of the following generating function:

$$\frac{2}{e^t+e^{-t}}=\sum_{n=0}^{\infty}E_n^*\frac{t^n}{n!} \tag{1.13}$$

(*cf.* [19, 26, 28, 30]; and references therein).

By using (1.9) and (1.13), a relation between the Euler numbers E_n^* and the Euler polynomials is given as follows:

$$E_n^*=2^nE_n\left(\frac{1}{2}\right)$$

(*cf.* [19, 21, 28, 30]).

Kilar and Simsek [13, Corollary 3.5] gave the following identity for the numbers $S_2(n,k)$:

$$S_2(n+k,k)=\sum_{j=0}^{n}\frac{\binom{n}{j}\binom{n+k}{k}}{2^{k+n}\binom{j+k}{k}}S_2(j+k,k)\,B(n-j,k)\,, \tag{1.14}$$

where $n,k\in\mathbb{N}_0$ and

$$\begin{aligned}B(n,k) &= \sum_{j=0}^{n}\binom{k}{j}j!2^{k-j}S_2(n,j)\\ &= \sum_{j=0}^{k}\binom{k}{j}j^n\end{aligned}$$

(*cf.* [32, Identity 12.]; see also [7, 29]).

Substituting $n=k$ into (1.14), we have

$$S_2(2n,n)=\binom{2n}{n}\sum_{j=0}^{n}\frac{\binom{n}{j}}{4^n\binom{j+n}{n}}S_2(j+n,n)\,B(n-j,n)\,.$$

The Daehee numbers, D_n, are defined by means of the following generating function:

$$\frac{\log(1+t)}{t}=\sum_{n=0}^{\infty}D_n\frac{t^n}{n!} \tag{1.15}$$

(*cf.* [17, 25, 30]).

By using (1.15), an explicit formula for the Daehee numbers is given by

$$D_n=(-1)^n\frac{n!}{n+1} \tag{1.16}$$

(*cf.* [17, 25, 30]).

The Changhee numbers, Ch_n, are defined by means of the following generating function:

$$\frac{2}{2+t}=\sum_{n=0}^{\infty}Ch_n\frac{t^n}{n!} \tag{1.17}$$

(*cf.* [18, 30]).

By using (1.17), an explicit formula for the Changhee numbers is given by

$$Ch_n = (-1)^n \frac{n!}{2^n} \tag{1.18}$$

(*cf.* [18, 30]).

Kucukoglu and Simsek [22] defined the numbers $\beta_n(k)$ by means of the following generating function:

$$\left(1-\frac{z}{2}\right)^k = \sum_{n=0}^{\infty} \beta_n(k) \frac{z^n}{n!}, \tag{1.19}$$

where $k \in \mathbb{N}_0$, $z \in \mathbb{C}$ with $|z| < 2$.

By using (1.19), we have

$$\beta_n(k) = \frac{(-1)^n n!}{2^n} \binom{k}{n} = \binom{k}{n} Ch_n \tag{1.20}$$

where $n, k \in \mathbb{N}_0$ (*cf.* [22, Equations (4.9) and (4.10)]).

The polynomials $C_n(x, y)$ and $S_n(x, y)$ are defined by means of the following generating functions:

$$G_C(t, x, y) = e^{xt} \cos(yt) = \sum_{n=0}^{\infty} C_n(x, y) \frac{t^n}{n!} \tag{1.21}$$

and

$$G_S(t, x, y) = e^{xt} \sin(yt) = \sum_{n=0}^{\infty} S_n(x, y) \frac{t^n}{n!}, \tag{1.22}$$

(*cf.* [9–12, 14–16, 20, 24]).

By using (1.21) and (1.22), the polynomials $C_n(x, y)$ and $S_n(x, y)$ are computed by the following formulas:

$$C_n(x, y) = \sum_{j=0}^{\left[\frac{n}{2}\right]} (-1)^j \binom{n}{2j} x^{n-2j} y^{2j}$$

and

$$S_n(x, y) = \sum_{j=0}^{\left[\frac{n-1}{2}\right]} (-1)^j \binom{n}{2j+1} x^{n-2j-1} y^{2j+1},$$

respectively (*cf.* [9–12, 14–16, 20, 24]).

By using (1.21) and (1.22), the polynomials $C_n(x, y)$ and $S_n(x, y)$ are also computed by the following formulas:

$$C_n(x, y) = \sum_{j=0}^{\left[\frac{n}{2}\right]} \sum_{d=0}^{n-2j} (-1)^j \binom{n}{2j} S_2(n-2j, d)\, y^{2j} (x)_d \tag{1.23}$$

and

$$S_n(x, y) = \sum_{j=0}^{\left[\frac{n-1}{2}\right]} \sum_{d=0}^{n-2j-1} (-1)^j \binom{n}{2j+1} S_2(n-2j-1, d)\, y^{2j+1} (x)_d \tag{1.24}$$

(*cf.* [2]).

Simsek [31] defined new classes of special numbers and polynomials by means of the following generating functions:

$$F_{\mathcal{Y}}(t,k,a)=\frac{at}{4\sinh\left(\frac{(k+2)t}{2}\right)\cosh\left(\frac{kt}{2}\right)}=\sum_{n=0}^{\infty}\mathcal{Y}_n(k,a)\frac{t^n}{n!} \tag{1.25}$$

and

$$G_{\mathcal{Y}}(t,x,k,a)=e^{xt}F_{\mathcal{Y}}(t,k,a)=\sum_{n=0}^{\infty}Q_n(x,k,a)\frac{w^n}{n!}, \tag{1.26}$$

where $k\in\mathbb{Z}$ and $a\in\mathbb{R}$ (or $\mathbb{C}$).

Substituting $x=0$ into (1.26), we have

$$\mathcal{Y}_n(k,a)=Q_n(0,k,a).$$

Simsek also gave the representation of equation (1.25) as follows:

$$F_{\mathcal{Y}}(t,k,a)=\frac{tae^{(k+1)t}}{\left(e^{(k+2)t}-1\right)\left(e^{kt}+1\right)}.$$

(*cf.* [31]).

By using (1.25) and (1.26), a relation between the polynomials $Q_n(x,k,a)$ and the numbers $\mathcal{Y}_n(k,a)$ is given as follows:

$$Q_n(x,k,a)=\sum_{j=0}^{n}\binom{n}{j}x^{n-j}\mathcal{Y}_j(k,a)$$

(*cf.* [31]).

By using (1.5), (1.8) and (1.25), we have the following identity:

$$\mathcal{Y}_n(k,a)=\frac{a}{2(k+2)}\sum_{s=0}^{n}\binom{n}{s}k^{n-s}(k+2)^sE_{n-s}B_s\left(\frac{k+1}{k+2}\right), \tag{1.27}$$

where $n\in\mathbb{N}_0$ (*cf.* [31, Equation (15)]).

Recently, Bayad and Simsek [2] defined new classes of the parametrically generalized polynomials, the polynomials $Q_n^{(C)}(x,y,k,a)$ and $Q_n^{(S)}(x,y,k,a)$, by means of the following generating functions, respectively:

$$H_C(t,x,y,a,k)=\frac{e^{xt}\cos(yt)\,at}{4\sinh\left(\frac{(k+2)t}{2}\right)\cosh\left(\frac{kt}{2}\right)}=\sum_{n=0}^{\infty}Q_n^{(C)}(x,y,k,a)\frac{t^n}{n!} \tag{1.28}$$

and

$$H_S(t,x,y,a,k)=\frac{e^{xt}\sin(yt)\,at}{4\sinh\left(\frac{(k+2)t}{2}\right)\cosh\left(\frac{kt}{2}\right)}=\sum_{n=0}^{\infty}Q_n^{(S)}(x,y,k,a)\frac{t^n}{n!}, \tag{1.29}$$

where $k\in\mathbb{Z}$ and $a\in\mathbb{R}$ (or $\mathbb{C}$).

By using (1.28) and (1.29), the polynomials $Q_n^{(C)}(x,y,k,a)$ and $Q_n^{(S)}(x,y,k,a)$ are computed by the following formulas:

$$Q_n^{(C)}(x,y,k,a) = \sum_{j=0}^{n} \binom{n}{j} \mathcal{Y}_j(k,a) C_{n-j}(x,y) \tag{1.30}$$

and

$$Q_n^{(S)}(x,y,k,a) = \sum_{j=0}^{n} \binom{n}{j} \mathcal{Y}_j(k,a) S_{n-j}(x,y) \tag{1.31}$$

(*cf.* [2]).

The rest of this article is briefly summarized as follows:

In Section 2, by using generating functions and functional equations techniques, we derive some formulas, combinatorial sums and relations including the parametrically generalized polynomials, the Bernoulli numbers and polynomials of higher order, the Euler numbers and polynomials of higher order, the Euler numbers of the second kind, the polynomials $C_n(x,y)$, and the polynomials $S_n(x,y)$.

In Section 3, we give many inequalities for combinatorial sums including the Bernoulli numbers of negative order, the Euler numbers of negative order, the Bernoulli polynomials, the Changhee numbers, the Daehee numbers, the Stirling numbers, the numbers $B(n,k)$ and the numbers $\beta_n(k)$.

In Section 4, using Mathematica with the help of the Wolfram programming language, we present some plots of the parametrically generalized polynomials under some of their randomly selected special cases.

Finally, in Section 5, we give remarks and observations on our results.

2. Combinatorial sums and identities for the parametrically generalized polynomials, and special numbers and polynomials

In this section, using generating functions and functional equations, we give some interesting identities and combinatorial sums related to the parametrically generalized polynomials, the polynomials $C_n(x,y)$, the polynomials $S_n(x,y)$, the Bernoulli numbers and polynomials of higher order, the Euler numbers and polynomials of higher order and the Euler numbers of the second kind.

Theorem 2.1. *Let $n \in \mathbb{N}_0$ and $a \neq 0$. Then we have*

$$C_n(x+k+1,y) = \sum_{d=0}^{n} \sum_{j=0}^{d} \binom{d}{j} \binom{n}{d} \frac{2(k+2)^{j+1} k^{d-j}}{a(j+1)} E_{d-j}^{(-1)} Q_{n-d}^{(C)}(x,y,k,a).$$

Proof. Combining (1.28) with (1.4), (1.8) and (1.21), we get the following functional equation:

$$\frac{a}{2(k+2)}G_C(t,x+k+1,y)=F_B((k+2)t,-1)F_E(kt,-1)H_C(t,x,y,a,k).$$

From the above equation, we obtain

$$\begin{aligned}&\frac{a}{2(k+2)}\sum_{n=0}^{\infty}C_n(x+k+1,y)\frac{t^n}{n!}\\ =&\sum_{n=0}^{\infty}(k+2)^n B_n^{(-1)}\frac{t^n}{n!}\sum_{n=0}^{\infty}k^n E_n^{(-1)}\frac{t^n}{n!}\sum_{n=0}^{\infty}Q_n^{(C)}(x,y,k,a)\frac{t^n}{n!}.\end{aligned}$$

Therefore

$$\begin{aligned}&\frac{a}{2(k+2)}\sum_{n=0}^{\infty}C_n(x+k+1,y)\frac{t^n}{n!}\\ =&\sum_{n=0}^{\infty}\sum_{d=0}^{n}\sum_{j=0}^{d}\binom{d}{j}\binom{n}{d}(k+2)^j k^{d-j}B_j^{(-1)}E_{d-j}^{(-1)}Q_{n-d}^{(C)}(x,y,k,a)\frac{t^n}{n!}.\end{aligned}$$

Comparing coefficient of $\frac{t^n}{n!}$ on both sides of the above equation, and combining with following well-known formula

$$B_n^{(-1)}=\frac{1}{n+1},$$

we arrive at the desired result.

Theorem 2.2. *Let $n\in\mathbb{N}_0$ and $a\neq 0$. Then we have*

$$S_n(x+k+1,y)=\sum_{d=0}^{n}\sum_{j=0}^{d}\binom{d}{j}\binom{n}{d}\frac{2(k+2)^{j+1}k^{d-j}}{a(j+1)}E_{d-j}^{(-1)}Q_{n-d}^{(S)}(x,y,k,a).$$

Proof. Combining (1.29) with (1.4), (1.8) and (1.22), we have

$$\frac{a}{2(k+2)}G_S(t,x+k+1,y)=F_B((k+2)t,-1)F_E(kt,-1)H_S(t,x,y,a,k).$$

From the above functional equation, we obtain

$$\begin{aligned}&\frac{a}{2(k+2)}\sum_{n=0}^{\infty}S_n(x+k+1,y)\frac{t^n}{n!}\\ =&\sum_{n=0}^{\infty}(k+2)^n B_n^{(-1)}\frac{t^n}{n!}\sum_{n=0}^{\infty}k^n E_n^{(-1)}\frac{t^n}{n!}\sum_{n=0}^{\infty}Q_n^{(S)}(x,y,k,a)\frac{t^n}{n!}.\end{aligned}$$

Therefore

$$\begin{aligned}&\frac{a}{2(k+2)}\sum_{n=0}^{\infty}S_n(x+k+1,y)\frac{t^n}{n!}\\ =&\sum_{n=0}^{\infty}\sum_{d=0}^{n}\sum_{j=0}^{d}\binom{d}{j}\binom{n}{d}(k+2)^j k^{d-j}B_j^{(-1)}E_{d-j}^{(-1)}Q_{n-d}^{(S)}(x,y,k,a)\frac{t^n}{n!}.\end{aligned}$$

Comparing coefficient of $\frac{t^n}{n!}$ on both sides of the above equation, after some elementary calculations, we arrive at the desired result.

Theorem 2.3. *Let $n \in \mathbb{N}_0$ and $a \neq 0$. Then we have*

$$B_n = \frac{1}{a(n+1)(k+2)^{n-1}} \sum_{d=0}^{\left[\frac{n+1}{2}\right]} \sum_{j=0}^{n+1-2d} (-1)^d \binom{n+1-2d}{j} \binom{n+1}{2d} 2^{2d+1} y^{2d-1} k^j \quad (2.32)$$
$$\times E_j^{(-1)}\left(\frac{-x-k-1}{k}\right) B_{2d}\left(\frac{1}{2}\right) Q_{n+1-2d-j}^{(S)}(x,y,k,a).$$

Proof. By using (1.4), (1.8) and (1.29), we get the following functional equation:

$$\frac{a}{(k+2)} F_B((k+2)t, 1) = \frac{2}{\sin(yt)} G_E\left(kt, \frac{-x-k-1}{k}, -1\right) H_S(t,x,y,a,k).$$

Combining above equation with the following well-known identity:

$$\frac{t}{\sin(t)} = \sum_{n=0}^{\infty} (-1)^n 2^{2n} B_{2n}\left(\frac{1}{2}\right) \frac{t^{2n}}{(2n)!} \quad (2.33)$$

(*cf.* [19, Equation (2.24)]), we have

$$\frac{ay}{2(k+2)} \sum_{n=0}^{\infty} (k+2)^n B_n \frac{t^{n+1}}{n!} = \sum_{n=0}^{\infty} (-1)^n (2y)^{2n} B_{2n}\left(\frac{1}{2}\right) \frac{t^{2n}}{(2n)!}$$
$$\times \sum_{n=0}^{\infty} E_n^{(-1)}\left(\frac{-x-k-1}{k}\right) \frac{(kt)^n}{n!} \sum_{n=0}^{\infty} Q_n^{(S)}(x,y,k,a) \frac{t^n}{n!}.$$

Therefore

$$\frac{ay}{2} \sum_{n=0}^{\infty} n(k+2)^{n-2} B_{n-1} \frac{t^n}{n!}$$
$$= \sum_{n=0}^{\infty} \sum_{d=0}^{\left[\frac{n}{2}\right]} \binom{n}{2d} \sum_{j=0}^{n-2d} \binom{n-2d}{j} k^j E_j^{(-1)}\left(\frac{-x-k-1}{k}\right)$$
$$\times Q_{n-2d-j}^{(S)}(x,y,k,a)(-1)^d (2y)^{2d} B_{2d}\left(\frac{1}{2}\right) \frac{t^n}{n!}.$$

Comparing coefficient of $\frac{t^n}{n!}$ on both sides of the above equation, after some elementary calculations, we arrive at the desired result.

Theorem 2.4. *Let $n \in \mathbb{N}_0$ and $a \neq 0$. Then we have*

$$E_n = \frac{1}{ak^n(n+1)} \sum_{d=0}^{\left[\frac{n+1}{2}\right]} \sum_{j=0}^{n+1-2d} (-1)^d \binom{n+1-2d}{j} \binom{n+1}{2d} 2^{2d+1} y^{2d-1} (k+2)^{j+1}$$
$$\times B_j^{(-1)}\left(\frac{-x-k-1}{k+2}\right) B_{2d}\left(\frac{1}{2}\right) Q_{n+1-2d-j}^{(S)}(x,y,k,a).$$

Proof. By using (1.5), (1.8) and (1.29), we get the following functional equation:

$$\frac{a}{2}F_E\left(kt,1\right)=\frac{k+2}{\sin\left(yt\right)}G_B\left(\left(k+2\right)t,\frac{-x-k-1}{k+2},-1\right)H_S(t,x,y,a,k).$$

Combining above equation with (2.33), we have

$$\begin{aligned}\frac{ayt}{2}\sum_{n=0}^{\infty}k^nE_n\frac{t^n}{n!} &= \sum_{n=0}^{\infty}\left(-1\right)^n\left(2y\right)^{2n}B_{2n}\left(\frac{1}{2}\right)\frac{t^{2n}}{(2n)!}\\ &\times\sum_{n=0}^{\infty}\left(k+2\right)^nB_n^{(-1)}\left(\frac{-x-k-1}{k+2}\right)\frac{t^n}{n!}\sum_{n=0}^{\infty}Q_n^{(S)}\left(x,y,k,a\right)\frac{t^n}{n!}.\end{aligned}$$

Therefore

$$\begin{aligned}\frac{ay}{2}\sum_{n=0}^{\infty}nk^{n-1}E_{n-1}\frac{t^n}{n!}=&\sum_{n=0}^{\infty}\sum_{d=0}^{\left[\frac{n}{2}\right]}\left(-1\right)^d\binom{n}{2d}\sum_{j=0}^{n-2d}\binom{n-2d}{j}\left(k+2\right)^j\\ &\times B_j^{(-1)}\left(\frac{-x-k-1}{k+2}\right)Q_{n-2d-j}^{(S)}\left(x,y,k,a\right)\left(2y\right)^{2d}B_{2d}\left(\frac{1}{2}\right)\frac{t^n}{n!}.\end{aligned}$$

Comparing coefficient of $\frac{t^n}{n!}$ on both sides of the above equation, after some elementary calculations, we arrive at the desired result.

Theorem 2.5. *Let $n\in\mathbb{N}_0$ and $a\neq 0$. Then we have*

$$\frac{a}{2}\sum_{j=0}^{n}\binom{n}{j}\frac{(k+2)^{j-1}}{k^{j-n}}B_j\left(\frac{x}{k+2}\right)E_{n-j}\left(\frac{k+1}{k}\right)=\sum_{j=0}^{\left[\frac{n}{2}\right]}\left(-1\right)^j\binom{n}{2j}y^{2j}E_{2j}^{*}Q_{n-2j}^{(C)}\left(x,y,k,a\right).$$

Proof. By using (1.5), (1.9) and (1.28), we get the following functional equation:

$$\frac{a}{2\left(k+2\right)}G_B\left(\left(k+2\right)t,\frac{x}{k+2},1\right)G_E\left(kt,\frac{k+1}{k},1\right)=\sec\left(yt\right)H_C(t,x,y,a,k).$$

Combining above equation with the following well-known identity:

$$\sec\left(t\right)=\sum_{n=0}^{\infty}\left(-1\right)^nE_{2n}^{*}\frac{t^{2n}}{(2n)!}\tag{2.34}$$

(*cf.* [19, Equation (2.40)]), we have

$$\begin{aligned}&\frac{a}{2\left(k+2\right)}\sum_{n=0}^{\infty}\left(k+2\right)^nB_n\left(\frac{x}{k+2}\right)\frac{t^n}{n!}\sum_{n=0}^{\infty}k^nE_n\left(\frac{k+1}{k}\right)\frac{t^n}{n!}\\ =&\sum_{n=0}^{\infty}\left(-1\right)^nE_{2n}^{*}\frac{t^{2n}}{(2n)!}\sum_{n=0}^{\infty}Q_n^{(C)}\left(x,y,k,a\right)\frac{t^n}{n!}.\end{aligned}$$

Therefore

$$\begin{aligned}&\frac{a}{2\left(k+2\right)}\sum_{n=0}^{\infty}\sum_{j=0}^{n}\binom{n}{j}(k+2)^jk^{n-j}B_j\left(\frac{x}{k+2}\right)E_{n-j}\left(\frac{k+1}{k}\right)\frac{t^n}{n!}\\ =&\sum_{n=0}^{\infty}\sum_{j=0}^{\left[\frac{n}{2}\right]}\binom{n}{2j}\left(-1\right)^jE_{2j}^{*}Q_{n-2j}^{(C)}\left(x,y,k,a\right)\frac{t^n}{n!}.\end{aligned}$$

Comparing coefficient of $\frac{t^n}{n!}$ on both sides of the above equation, after some elementary calculations, we arrive at the desired result.

Combining (1.24) with (1.12), after some elementary calculations, we obtain the following theorem:

Theorem 2.6. *Let $n \in \mathbb{N}$. Then we have*

$$\begin{aligned} S_n(x,y) &= \sum_{j=0}^{\left[\frac{n-1}{2}\right]} \sum_{d=0}^{n-2j-1} (-1)^j \binom{n}{2j+1} \frac{2^{d-n+2j+1} y^{2j+1} (x)_d}{d!} \\ &\quad \times \sum_{m=0}^{n-2j-1} \sum_{v=0}^{d} (-1)^{d-v} \binom{n-2j-1}{m} \binom{d}{v} v^m E_{n-2j-1-m}^{(-d)}. \end{aligned}$$

3. Inequalities applications for combinatorial sums involving special numbers

In this section, we give the upper bound and the lower bound for the special numbers and polynomials, and combinatorial sums involving the Bernoulli numbers of negative order, the Euler numbers of negative order, the Changhee numbers, the Daehee numbers, the Stirling numbers of the second kind, the numbers $B(n,k)$ and the numbers $\beta_n(k)$.

In order to give our results, we need the following inequalities for the special numbers.

Gun and Simsek [8] gave the lower bound and the upper bound for the Bernoulli numbers of negative order $B_n^{(-k)}$ as follows:

$$B_n^{(-k)} \geq \frac{k^n}{\binom{n+k}{k}} \tag{3.35}$$

and

$$B_n^{(-k)} \leq \frac{\binom{n+k-1}{k-1} k^n}{\binom{n+k}{k}}, \tag{3.36}$$

where $n \in \mathbb{N}_0$ and $k \in \mathbb{N}$.

Comtet [6] gave the lower bound and the upper bound for the Stirling numbers of the second kind $S_2(n,k)$ as follows:

$$S_2(n,k) \geq k^{n-k} \tag{3.37}$$

and

$$S_2(n,k) \leq \binom{n-1}{k-1} k^{n-k}. \tag{3.38}$$

Abramowitz and Stegun [1, p. 805] gave the following inequality for the Bernoulli numbers:

$$\frac{2(2n)!}{(2\pi)^{2n}} < (-1)^{n+1} B_{2n} < \frac{2(2n)!}{(2\pi)^{2n}(1-2^{1-2n})}, \tag{3.39}$$

where $n \in \mathbb{N}$.

Combining (1.11) with (3.35), we get the following theorem for the Euler numbers of negative order and the Bernoulli numbers of negative order:

Theorem 3.1. *Let $n \in \mathbb{N}_0$ and $k \in \mathbb{N}$. Then we have*

$$\sum_{j=0}^{n} \binom{n}{j} B_j^{(-k)} E_{n-j}^{(-k)} \geq \frac{2^n k^n}{\binom{k+n}{k}}. \tag{3.40}$$

By using (1.6), (1.10), (1.18) and (3.40), we derive the following corollary:

Corollary 3.1. *Let $n \in \mathbb{N}_0$ and $k \in \mathbb{N}$. Then we have*

$$\sum_{j=0}^{n} \sum_{d=0}^{k} \sum_{m=0}^{n-j} \frac{(-1)^{k-d}}{\binom{j+k}{k} k!} \binom{n}{j} \binom{k}{d} \binom{m-k-1}{m} d^{j+k} Ch_m S_2(n-j, m) \geq \frac{2^n k^n}{\binom{k+n}{k}}.$$

By using (1.18), (1.20) and (3.40), we get the following corollary:

Corollary 3.2. *Let $n \in \mathbb{N}_0$ and $k \in \mathbb{N}$. Then we have*

$$\sum_{j=0}^{n} \binom{n}{j} B_j^{(-k)} E_{n-j}^{(-k)} \geq \frac{(2k)^n Ch_k}{\beta_k (n+k)}.$$

Combining (1.11) with (3.36), we obtain the following theorem:

Theorem 3.2. *Let $n \in \mathbb{N}_0$ and $k \in \mathbb{N}$. Then we have*

$$\sum_{j=0}^{n} \binom{n}{j} B_j^{(-k)} E_{n-j}^{(-k)} \leq \frac{2^n \binom{n+k-1}{k-1} k^n}{\binom{n+k}{k}}. \tag{3.41}$$

Substituting $n = k$ into (3.41), we arrive at the following result:

Corollary 3.3. *Let $n \in \mathbb{N}$. Then we have*

$$\sum_{j=0}^{n} \binom{n}{j} B_j^{(-n)} E_{n-j}^{(-n)} \leq \frac{2^n \binom{2n-1}{n-1} n^n}{\binom{2n}{n}}.$$

By using (1.18), (1.20) and (3.41), we obtain the following corollary:

Corollary 3.4. *Let $n \in \mathbb{N}_0$ and $k \in \mathbb{N}$. Then we have*

$$\sum_{j=0}^{n} \binom{n}{j} B_j^{(-k)} E_{n-j}^{(-k)} \leq (2k)^n \frac{\beta_{k-1}(n+k-1) Ch_k}{\beta_k (n+k) Ch_{k-1}}.$$

Combining (1.14) with (3.37), we get the following theorem:

Theorem 3.3. *Let $n \in \mathbb{N}_0$ and $k \in \mathbb{N}$. Then we have*

$$\sum_{j=0}^{n} \frac{\binom{n}{j}\binom{n+k}{k}}{2^{k+n}\binom{j+k}{k}} S_2(j+k, k) B(n-j, k) \geq k^n. \tag{3.42}$$

By using (1.14), (1.16), (1.20) and (3.37), we have the following corollary:

Corollary 3.5. *Let $n \in \mathbb{N}_0$ and $k \in \mathbb{N}$. Then we have*

$$\sum_{j=0}^{n} \frac{\binom{n}{j}\beta_k(n+k)}{2^n\binom{j+k}{k}D_k} S_2(j+k,k)\,B(n-j,k) \geq (k+1)\,k^n.$$

Combining (1.12) with (3.37), we arrive at the following theorem:

Theorem 3.4. *Let $n \in \mathbb{N}_0$ and $k \in \mathbb{N}$. Then we have*

$$\frac{2^{k-n}}{k!}\sum_{m=0}^{n}\sum_{j=0}^{k}(-1)^{k-j}\binom{n}{m}\binom{k}{j}j^m E_{n-m}^{(-k)} \geq k^{n-k}.$$

Combining (1.12) with (1.20) and (3.38), we get the following theorem for the Euler numbers of negative order:

Theorem 3.5. *Let $n \in \mathbb{N}_0$ and $k \in \mathbb{N}$. Then we have*

$$\frac{2^{k-n}}{k!}\sum_{m=0}^{n}\sum_{j=0}^{k}(-1)^{k-j}\binom{n}{m}\binom{k}{j}j^m E_{n-m}^{(-k)} \leq \frac{k^{n-k}\beta_{k-1}(n-1)}{Ch_{k-1}}.$$

4. Some plots of the parametrically generalized polynomials

In this section, with the help of Wolfram programming language in Mathematica 35, we illustrated the plots of the parametrically generalized polynomials by applying the formulas given by (1.30) and (1.31).

Figure 1 is obtained by $y = 2$, $k = -10$, $a = 2$, and $n \in \{0,1,2,3,4,5\}$ using (1.30) for $x \in [-50, 50]$.

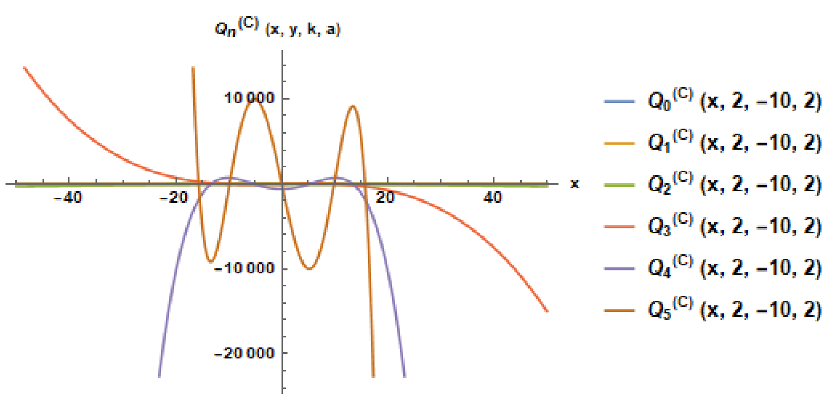

Figure 1. Plots of the polynomials $Q_n^{(C)}(x,2,-10,2)$ for randomly selected special cases when $n \in \{0,1,2,3,4,5\}$ and $x \in [-50, 50]$.

Figure 2 is obtained by $n = 4$, $y = 2$, $a = 2$, and $k \in \{0, 1, 2, 3, 4, 5\}$ using (1.30) for $x \in [-5, 5]$.

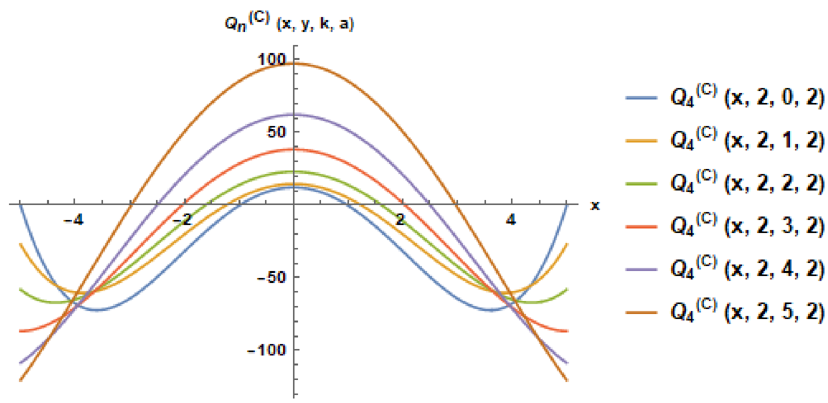

FIGURE 2. Plots of the polynomials $Q_n^{(C)}(x, 2, k, 2)$ for randomly selected special cases when $k \in \{0, 1, 2, 3, 4, 5\}$ with $n = 4$ and $x \in [-5, 5]$.

Figure 3 is obtained by $n = 4$, $k = -8$, $a = 2$, and $y \in \{0, 1, 2, 3, 4, 5\}$ using (1.30) for $x \in [-6, 6]$.

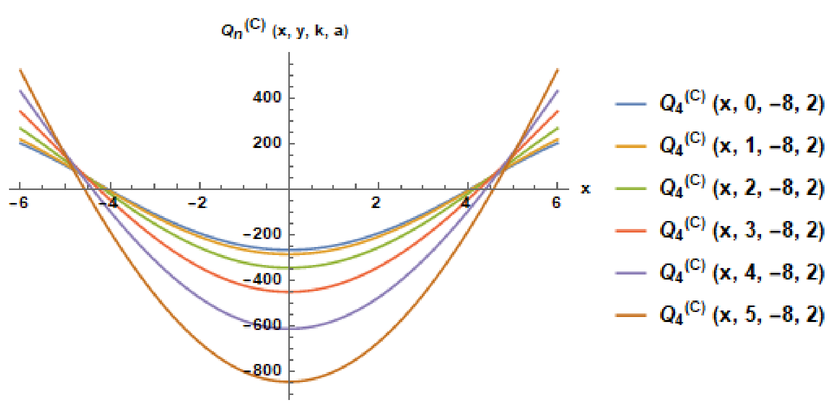

FIGURE 3. Plots of the polynomials $Q_n^{(C)}(x, y, -8, 2)$ for randomly selected special cases when $y \in \{0, 1, 2, 3, 4, 5\}$ with $n = 4$ and $x \in [-6, 6]$.

Figure 4 is obtained by $n = 15$, $k = -8$, $a = 2$, and $y \in \{0, 1, 2, 3, 4, 5\}$ using (1.30) for $x \in [-6, 6]$.

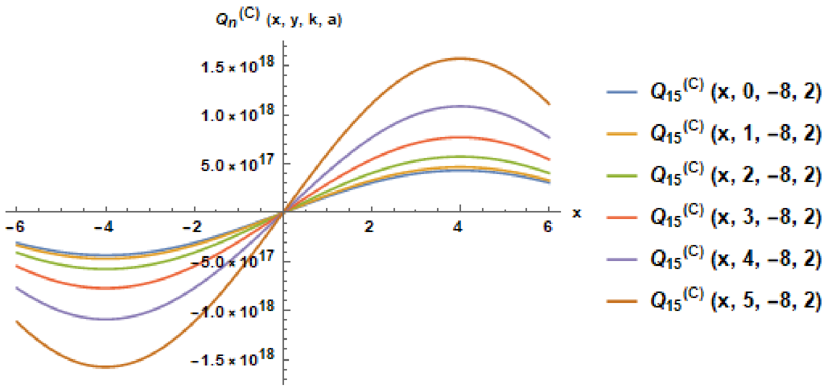

FIGURE 4. Plots of the polynomials $Q_n^{(C)}(x, y, -8, 2)$ for randomly selected special cases when $y \in \{0, 1, 2, 3, 4, 5\}$ with $n = 15$ and $x \in [-6, 6]$.

Figure 5 is obtained by $y = 2$, $k = -10$, $a = 2$, and $n \in \{0, 1, 2, 3, 4, 5\}$ using (1.31) for $x \in [-50, 50]$.

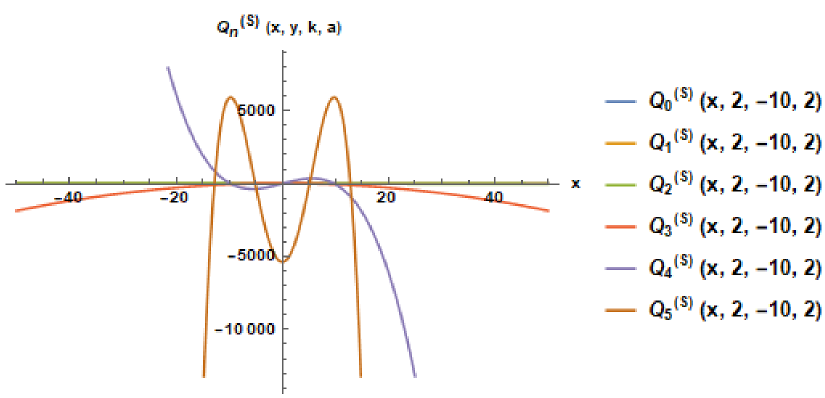

FIGURE 5. Plots of the polynomials $Q_n^{(S)}(x, 2, -10, 2)$ for randomly selected special cases when $n \in \{0, 1, 2, 3, 4, 5\}$ and $x \in [-50, 50]$.

Figure 6 is obtained by $n = 4$, $y = 2$, $a = 2$, and $k \in \{0, 1, 2, 3, 4, 5\}$ using (1.31) for $x \in [-5, 5]$.

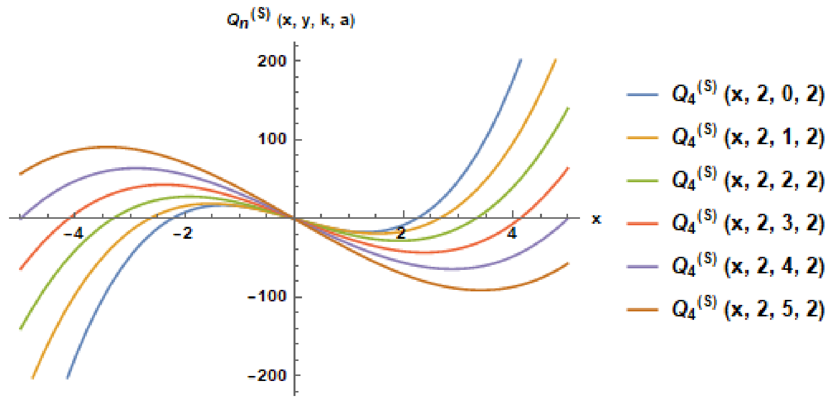

FIGURE 6. Plots of the polynomials $Q_n^{(S)}(x, 2, k, 2)$ for randomly selected special cases when $k \in \{0, 1, 2, 3, 4, 5\}$ with $n = 4$ and $x \in [-5, 5]$.

Figure 7 is obtained by $n = 4$, $k = -8$, $a = 2$, and $y \in \{0, 1, 2, 3, 4, 5\}$ using (1.31) for $x \in [-5, 5]$.

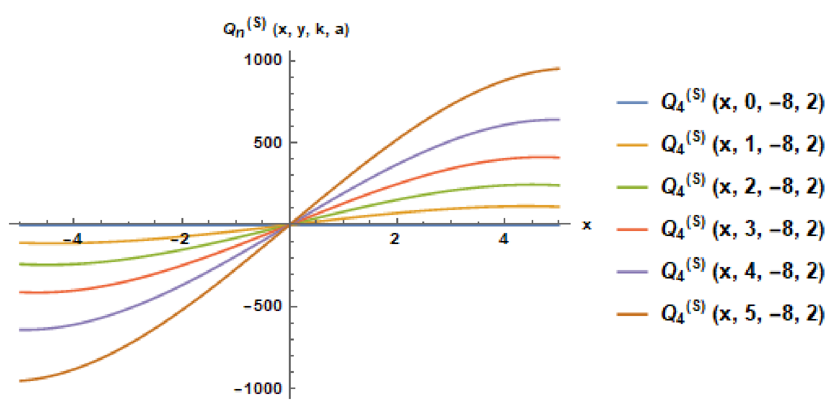

FIGURE 7. Plots of the polynomials $Q_n^{(S)}(x, y, -8, 2)$ for randomly selected special cases when $y \in \{0, 1, 2, 3, 4, 5\}$ with $n = 4$ and $x \in [-5, 5]$.

Figure 8 is obtained by $n = 15$, $k = -8$, $a = 2$, and $y \in \{0, 1, 2, 3, 4, 5\}$ using (1.31) for $x \in [-6, 6]$.

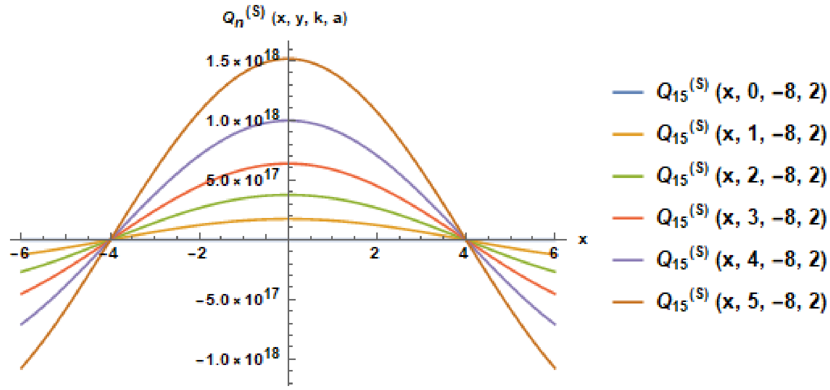

FIGURE 8. Plots of the polynomials $Q_n^{(S)}(x, y, -8, 2)$ for randomly selected special cases when $y \in \{0, 1, 2, 3, 4, 5\}$ with $n = 15$ and $x \in [-6, 6]$.

5. CONCLUSION

Special numbers, special polynomials and trigonometric functions are among remarkably wide used in applied mathematics, combinatorial analysis, mathematical analysis, analytic number theory, mathematical physics, and engineering. Recently using different techniques and methods, many properties of parametrically polynomials involving trigonometric functions have been studied by many researchers. Using both the generating functions and their functional equations techniques and some known results, we obtained many interesting identities, combinatorial sums and inequalities including the Euler numbers and polynomials of higher order, the Bernoulli numbers of higher order, the Changhee numbers, the Daehee numbers, the parametrically generalized polynomials, the Stirling numbers and also well-known special polynomials. By using Mathematica with the help of the Wolfram programming language, we gave some plots of the parametrically generalized polynomials under the special cases. Consequently, the results of this article have the potential to be used both pure and applied mathematics, physics, engineering and other related areas, and to attract the attention of researchers working in this areas.

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