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CERTAIN RESULTS OF RICCI SOLITON ON 3-DIMENSIONAL LORENTZIAN PARA α-SASAKIAN MANIFOLDS

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Abstract. The paper deals with the study of almost Ricci (AR) soliton and gradient almost Ricci (GAR) soliton on 3-dimensional Lorentzian para α-Sasakian manifolds (α- LPS manifolds). Finally, we also provide an example of AR soliton.

Keywords: Lorentzian Para α-Sasakian manifold, Ricci soliton , Gradient Ricci soliton, Almost Ricci soliton, Gradient almost Ricci soliton.

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1. Introduction

As a generalization of an Einstein metric [6], Ricci soliton first defined in 1982 by Hamilton [19]. A pseudo-Riemannian manifold (M, g_*) defines a Ricci soliton with a smooth vector field V on M such that

$$\pounds_V g_* + 2S - 2\tau_1 g_* = 0, \tag{1.1}$$

where $\pounds_V$ is the Lie derivative along the vector field V and S is the Ricci tensor on M and τ_1 is a real scalar. Ricci soliton is said to be shrinking $\tau_1 < 0$, steady $\tau_1 = 0$ or expanding $\tau_1 > 0$, [8]. A Ricci soliton is changed into Einstein equation with V zero or killing vector field.

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The study of almost Ricci soliton was presented by Pigola et al. [23], in this manner they gave new version of the definition of Ricci soliton by adding new condition on the parameter τ_1 to be a variable function, we say that a Riemannian manifold (M, g_*) admits an almost Ricci soliton, if there exists a complete vector field V, called potential vector field and a smooth soliton function $\tau_1 : M \to \mathbb{R}$ satisfying

$$S + \frac{1}{2}\pounds_V g_* = \tau_1 g_*, \tag{1.2}$$

where S and $\pounds$ represent Ricci tensor and Lie derivative along the direction of soliton vector field V. We shall now refer to this equation as the fundamental equation of an almost Ricci soliton (M, g_*, V, τ_1). Ricci soliton will be called shrinking, steady or expanding, respectively, if $\tau_1 < 0, \tau_1 = 0$ or $\tau_1 > 0$. For remaining it will be called indefinite. When the vector field V is gradient of a smooth function $f : M \to \mathbb{R}$ the metric will be called gradient almost Ricci soliton. So, we obtain

$$S + \bar{\nabla}^2 f = \tau_1 g_*, \tag{1.3}$$

where $\bar{\nabla}^2 f$ means for the Hessian of f.

Additionally, if the vector field X_1 is trivial, or the potential f is constant, the almost Ricci soliton is said to be trivial, otherwise it is said to be non-trivial almost Ricci soliton. We observe that when $n \geq 3$ and X_1 is a killing vector field almost Ricci solitons will be Ricci solitons. So in this situtation we have an Einstein manifold. The soliton function τ_1 is not necessarily constant, certainly comparison with soliton theory will be modified. In particular the rigidity result contained in Theorem 1.3 of [23] inform that almost Ricci solitons should reveal a reasonably broad generalization of the important concept of classical soliton.

The presence of Ricci almost soliton has been affirmed by Pigola et al. [23] on some specific class of warped product manifolds. Some characterization of Ricci almost soliton on Riemannian manifolds can be found in [1, 4, 5, 7, 18, 26]. It is important to note that if the potential vector field V of the Ricci almost soliton (M, g_*, V, τ_1) is Killing then the soliton becomes trivial, provided the dimension of $M > 2$. Additionally, if V is conformal then M is isometric to Euclidean sphere S^n. Thus the Ricci almost soliton is a generalization of Einstein metric as well as Ricci soliton.

In [15], authors studied Ricci solitons and gradient Ricci solitons geometric properties on 3-dimensional normal almost contact metric manifolds. In [16] authors studied compact Ricci soliton. In [17] author studied K-contact and Sasakian manifolds whose metric is gradient almost Ricci solitons. Conditions of K-contact and Sasakian manifolds are more stronger than almost normal contact metric manifolds in the sense of the 1-form of almost normal contact metric manifolds are not contact form. Ricci soliton as well as gradient Ricci soliton have been studied by many authors such as [2, 13, 14].

Sharma [24] obtained results on Ricci almost solitons in K-contact geometry, also in author [17] studied Ricci almost solitons and gradient Ricci almost solitons in (k, μ)-contact geometry and Majhi [22] on 3-dimensional f-Kenmotsu manifolds also De and Mandal [12] studied for structure (k, μ)-Paracontact geometry. Motivated by above studies in this paper, we are interested to study almost Ricci solitons and gradient Ricci almost solitons with Lorentzian para α-Sasakian manifolds.

We are studying the following sections: Section 2 contains important definitions and some preliminary results of Lorentzian para α-Sasakian (α- LPS) manifolds needed for the study. In section 3, we deal second order parallel symmetric tensors α- LPS manifolds. In section 4, we obtain result for almost Ricci (AR) soliton in 3-dimensional α-LPS manifolds. In the Section 5, we deduce theorem for such manifolds with gradient almost Ricci (GAR) solitons. Finally, we give an example of 3-dimensional (α- LPS)manifolds with almost Ricci soliton.

2. α- LPS manifolds

A differentiable manifold M of $(2n+1)$ dimensional is said to be an α- LPS manifolds, if it cosist a tensor field J of type $(1,1)$, a characteristic vector field ζ_1, a 1-form η_* and g_* as Lorentzian metric satisfy (see [10, 21]) :

$$J^2X_1 = X_1 + \eta_*(X_1)\zeta_1, \tag{2.4}$$

$$\eta_*(\zeta_1) = -1, \ \eta_*(X_1) = g_*(X_1, \zeta_1), \tag{2.5}$$

$$J\zeta_1 = 0, \ \eta_* \circ J = 0, \tag{2.6}$$

$$g_*(JX_1, JY_1) = g_*(X_1, Y_1) + \eta_*(X_1)\eta_*(Y_1). \tag{2.7}$$

Definition 2.1. *A differentiable manifold M with an almost contact Lorentzian metric structure $(J, \zeta_1, \eta_*, g_*)$ is said to be an α-LS manifold if*

$$(\bar{\nabla}_{X_1} J)Y_1 = \alpha\{g_*(X_1, Y_1)\zeta_1 + \eta_*(Y_1)X_1\}, \tag{2.8}$$

where α is a constant function on M.

An almost contact metric structure is called a LPS manifold (or simply Lorentzian para-Sasakian manifold) if, (for details see [27, 11, 9])

$$(\bar{\nabla}_{X_1} J)Y_1 = g_*(X_1, Y_1)\zeta_1 + \eta_*(Y_1)X_1 + 2\eta_*(X_1)\eta_*(Y_1)\zeta_1, \tag{2.9}$$

where $\bar{\nabla}$ is the Levi-Civita connection with respect to g_*. Using above equation, one can obtain

$$\bar{\nabla}_{X_1}\zeta_1 = JX_1, \quad (\bar{\nabla}_{X_1}\eta_*)Y_1 = g_*(X_1, JY_1). \tag{2.10}$$

Definition 2.2. *A differentiable manifold M with an almost contact Lorentzian metric structure $(J, \zeta_1, \eta_*, g_*)$ is called an α-LPS manifold if*

$$(\bar{\nabla}_{X_1} J)Y_1 = \alpha\{g_*(X_1, Y_1)\zeta_1 + \eta_*(Y_1)X_1 + 2\eta_*(X_1)\eta_*(Y_1)\zeta_1\}, \tag{2.11}$$

where α is a smooth function on M.

Remark- Note that if $\alpha = 1$, then LPS manifold is the special case of α-LPS manifold. For an α-LPS manifold following relations are holds [3]:

$$\bar{\nabla}_{X_1}\zeta_1 = \alpha JX_1, \tag{2.12}$$

$$(\bar{\nabla}_{X_1}\eta_*)Y_1 = \alpha g_*(JX_1, Y_1), \tag{2.13}$$

$$\begin{aligned} R(X_1, Y_1)\zeta_1 &= \alpha^2\{\eta_*(Y_1)X_1 - \eta_*(X_1)Y_1\} \\ &\quad +\{(X_1\alpha)JY_1 - (Y_1\alpha)JX_1\}, \end{aligned} \tag{2.14}$$

$$\begin{aligned} R(\zeta_1, Y_1)\zeta_1 &= \alpha^2\{Y_1 + \eta_*(Y_1)\zeta_1\} \\ &\quad +(\zeta_1\alpha)JY_1, \end{aligned} \tag{2.15}$$

$$R(\zeta_1, \zeta_1)\zeta_1 = 0, \tag{2.16}$$

$$\begin{aligned} R(\zeta_1, Y_1)X_1 &= \alpha^2\{g_*(X_1, Y_1)\zeta_1 - \eta_*(X_1)Y_1\} \\ &\quad -(X_1\alpha)JY_1 + g_*(JX_1, Y_1)(grad\alpha), \end{aligned} \tag{2.17}$$

$$S(Y_1,\zeta_1) = 2n\alpha^2\eta_*(Y_1) - \{(Y_1\alpha)w + (JY_1)\alpha\}, \tag{2.18}$$

for any vector field Y_1 on M, $w = g_*(J(e_i), e_i)$ and S defines the Ricci curvature on M.

$$S(\zeta_1,\zeta_1) = -2n\alpha^2 - (\zeta_1\alpha)w, \tag{2.19}$$

and

$$\begin{aligned} \eta_*(R(X_1,Y_1)Z_1) &= \alpha^2\{g_*(Y_1,Z_1)\eta_*(X_1) - g_*(X_1,Z_1)\eta_*(Y_1)\} \\ &\quad -\{(X_1\alpha)g_*(JY_1,Z_1) - (Y_1\alpha)g_*(X_1J,Z_1)\}. \end{aligned} \tag{2.20}$$

In a 3-dimensional Riemannian manifold, we always have

$$\begin{aligned} R(X_1,Y_1)Z_1 &= g_*(Y_1,Z_1)QX_1 - g_*(X_1,Z_1)QY_1 \\ &\quad +S(Y_1,Z_1)X_1 - S(X_1,Z_1)Y_1 \\ &\quad -\frac{r}{2}[g_*(Y_1,Z_1)X_1 - g_*(X_1,Z_1)Y_1]. \end{aligned} \tag{2.21}$$

In a 3-dimensional α-LPS manifold, we have

$$\begin{aligned} R(X_1,Y_1)Z_1 &= [\frac{r}{2} - \alpha^2][g_*(Y_1,Z_1)X_1 - g_*(X_1,Z_1)Y_1] \\ &\quad +[\frac{r}{2} - 3\alpha^2][g_*(Y_1,Z_1)\eta_*(X_1)\zeta_1 \\ &\quad -g_*(X_1,Z_1)\eta_*(Y_1)\zeta_1 + \eta_*(Y_1)\eta_*(Z_1)X_1 \\ &\quad -\eta_*(X_1)\eta_*(Z_1)Y_1], \end{aligned} \tag{2.22}$$

and

$$\begin{aligned} S(X_1,Z_1) &= [\frac{r}{2} - \alpha^2]g_*(X_1,Z_1) \\ &\quad +[\frac{r}{2} - 3\alpha^2]\eta_*(X_1)\eta_*(Y_1). \end{aligned} \tag{2.23}$$

Putting $Z_1 = \zeta_1$ in (2.17), we have

$$R(X_1, Y_1)\zeta_1 = \eta_*(Y_1)QX_1 - \eta_*(X_1)QY_1 + S(Y_1, \zeta_1)X_1 - S(X_1, \zeta_1)Y_1 - \frac{r}{2}[\eta_*(Y_1)X_1 - \eta_*(X_1)Y_1], \quad (2.24)$$

and

$$S(X_1, \zeta_1) = 2\alpha^2\eta_*(X_1). \quad (2.25)$$

where Q is the Ricci operator define by $S(X_1, Y_1) = g_*(QX_1, Y_1)$.

Definition 2.3. *An α-LPS manifold M is called an Einstein like if its Ricci tensor S satisfies*

$$S(X_1, Y_1) = ag_*(X_1, Y_1) + bg_*(JX_1, Y_1) + c\eta_*(X_1)\eta_*(Y_1), \quad (2.26)$$

$X_1, Y_1 \in (M)$ *for some real constants a, b and c.*

3. Second order parallel symmetric tensors in an α-LPS manifold

Fix h a symmetric tensor field of $(0, 2)$-type which we suppose to be parallel with respect to $\bar{\nabla}$ that is $\bar{\nabla}h = 0$. Applying the Ricci identity [25]

$$\bar{\nabla}^2 h(X_1, Y_1; Z_1, W_1) - \bar{\nabla}^2 h(X_1, Y_1; W_1, Z_1) = 0, \quad (3.27)$$

we obtain the relation

$$h(R(X_1, Y_1)Z_1, W_1) + h(Z_1, R(X_1, Y_1)W_1) = 0. \quad (3.28)$$

Replacing $Z_1 = W_1 = \zeta_1$ in (3.2) and by using (2.11) and by the symmetry of h, we have

$$\alpha^2[\eta_*(Y_1)h(X_1, \zeta_1) - \eta_*(X_1)h(Y_1, \zeta_1)] + (X_1\alpha)h(JY_1, \zeta_1) - (Y_1\alpha)h(JX_1, \zeta_1) = 0. \quad (3.29)$$

Putting $X_1 = \zeta_1$ in (3.3) and by virtue of (2.2) and (2.3), we obtain

$$\alpha^2[\eta_*(Y_1)h(\zeta_1, \zeta_1) + h(Y_1, \zeta_1)] + (\zeta_1\alpha)h(JY_1, \zeta_1) = 0. \quad (3.30)$$

Replacing $Y_1 = JY_1$ in (3.4), we have

$$(\zeta_1\alpha)[\eta_*(Y_1)h(\zeta_1,\zeta_1)+h(Y_1,\zeta_1)]+\alpha^2 h(JY_1,\zeta_1)=0. \tag{3.31}$$

Solving (3.4) and (3.5), we have

$$(\alpha^4-(\zeta_1\alpha)^2)[\eta_*(Y_1)h(\zeta_1,\zeta_1)+h(Y_1,\zeta_1)]=0. \tag{3.32}$$

Since $\alpha^4-(\zeta_1\alpha)^2\neq 0$, it results

$$h(Y_1,\zeta_1)=-\eta_*(Y_1)h(\zeta_1,\zeta_1), \tag{3.33}$$

from (3.7), we obtain

$$h(Y_1,\zeta_1)+g_*(Y_1,\zeta_1)h(\zeta_1,\zeta_1)=0. \tag{3.34}$$

Putting $Y_1=\bar{\nabla}_{X_1}Y_1$ in (3.7), we have

$$h(\bar{\nabla}_{X_1}Y_1,\zeta_1)+g_*(\bar{\nabla}_{X_1}Y_1,\zeta_1)h(\zeta_1,\zeta_1)=0. \tag{3.35}$$

Covariantly differentiating (3.7) with respect to X_1, we obtain

$$\begin{aligned} & (\bar{\nabla}_{X_1}h)(Y_1,\zeta_1)+h(\bar{\nabla}_{X_1}Y_1,\zeta_1)+h(Y_1,\bar{\nabla}_{X_1}\zeta_1) \qquad (3.36)\\ = & -[g_*(\bar{\nabla}_{X_1}Y_1,\zeta_1)+g_*(Y_1,\bar{\nabla}_{X_1}\zeta_1)]h(\zeta_1,\zeta_1)\\ & -\eta_*(Y_1)[(\bar{\nabla}_{X_1}h)(\zeta_1,\zeta_1)+2h(\bar{\nabla}_{X_1}\zeta_1,\zeta_1)]\\ = & \ 0. \end{aligned}$$

Applying the parallel condition $\bar{\nabla}h=0$, $\eta_*(\bar{\nabla}_{X_1}\zeta_1)=0$ and using (2.9) and (3.6) in (3.9), we infer

$$\alpha[h(Y_1,JX_1)+g_*(Y_1,JX_1)h(\zeta_1,\zeta_1)]=0. \tag{3.37}$$

Replacing $X_1=JX_1$ in (3.11) and on simplification, we get

$$\alpha[h(X_1,Y_1)+g_*(X_1,Y_1)h(\zeta_1,\zeta_1)]=0, \tag{3.38}$$

since α is non-zero smooth function in an α-LPS manifold and this implies that

$$h(X_1,Y_1)=-g_*(X_1,Y_1)h(\zeta_1,\zeta_1), \tag{3.39}$$

which is together with the standard fact that the parallelism of h implies that $h(\zeta_1, \zeta_1)$ is a constant, via (3.6). Now using the above conditions, we can write the following:

Theorem 3.1. *A second order covariant symmetric parallel tensor in an α-LPS manifold is a constant multiple of the metric tensor.*

4. AR solitons on 3-dimensional α-LPS manifolds

This section deal with the characterization of AR solitons on 3-dimensional α-LPS manifolds. Consider the potential vector field V be pointwise collinear, $V = b\zeta_1$, where b is a function on M. Then from (1.1) we have

$$g_*(\bar{\nabla}_{X_1} b\zeta_1, Y_1) + g_*(\bar{\nabla}_{Y_1} b\zeta_1, X_1) + 2S(X_1, Y_1) = 2\tau_1 g_*(X_1, Y_1). \tag{4.40}$$

By virtue of (2.9) and (4.1), we have

$$\begin{aligned} & 2b\alpha g_*(JX_1, Y_1) + (X_1 b)\eta_*(Y_1) \\ & +(Y_1 b)\eta_*(X_1) + 2S(X_1, Y_1) \\ = \;& 2\tau_1 g_*(X_1, Y_1). \end{aligned} \tag{4.41}$$

Substituting $Y_1 = \zeta_1$ in (4.2) and using (2.21), we get

$$-(X_1 b) + (\zeta_1 b)\eta_*(X_1) + 4\alpha^2\eta_*(X_1) = 2\tau_1\eta_*(X_1). \tag{4.42}$$

Taking $X_1 = \zeta_1$ in (4.3), we infer

$$\zeta_1 b = \tau_1 - 2\alpha^2. \tag{4.43}$$

Substituting the value of $\zeta_1 b$ in (4.3), we have

$$db = (2\alpha^2 - \tau_1)\eta_*. \tag{4.44}$$

Operating d on (4.5) and using $d^2 = 0$, we obtain

$$0 = d^2 b = (2\alpha^2 - \tau_1)d\eta_*. \tag{4.45}$$

It follows from the above equation

$$\tau_1 = 2\alpha^2,$$

which implies $db = 0$, i.e., $b =$ constant, by virtue of $db = (2\alpha^2 - \tau_1)\eta_*$. Thus, using constancy of b in (4.2), we infer

$$\begin{aligned} S(X_1, Y_1) &= \tau_1 g_*(X_1, Y_1) - \alpha b g_*(JX_1, Y_1) \\ &\quad -2(2\alpha^2 - \tau_1)\eta_*(X_1)\eta_*(Y_1), \end{aligned} \tag{4.46}$$

which is of the form $S(X_1, Y_1) = a g_*(X_1, Y_1) + b g_*(JX_1, Y_1) + c\eta_*(X_1)\eta_*(Y_1)$. Hence, we can state the following result:

Theorem 4.1. *A 3-dimensional α-LPS manifold $(M, \zeta_1, \eta_*, g_*)$ with constant α admitting an AR soliton with pointwise collinear vector field V with the structure vector field ζ_1, is an Einstein like manifold provided $\tau_1 = 2\alpha^2 > 0$ i.e., expanding.*

Now let $V = \zeta_1$. Then (4.1) reduces to

$$(\pounds_{\zeta_1} g_*)(X_1, Y_1) + 2S(X_1, Y_1) = 2\tau_1 g_*(X_1, Y_1). \tag{4.47}$$

Now, by using (2.9) we have

$$\begin{aligned} (\pounds_{\zeta_1} g_*)(X_1, Y_1) &= g_*(\bar{\nabla}_{X_1}\zeta_1, Y_1) + g_*(\bar{\nabla}_{Y_1}\zeta_1, X_1) \\ &= 2\alpha g_*(JX_1, Y_1). \end{aligned} \tag{4.48}$$

Using (2.19), we get

$$\begin{aligned} (\pounds_{\zeta_1} g_*)(X_1, Y_1) &= -2[\left(\frac{r}{2} - \alpha^2\right) g_*(X_1, Y_1) \\ &\quad + \left(\frac{r}{2} - 3\alpha^2\right) \eta_*(X_1)\eta_*(Y_1)] \\ &\quad +2\tau_1 g_*(X_1, Y_1). \end{aligned} \tag{4.49}$$

In view of (4.9) and (4.10), we obtain

$$\begin{aligned} \alpha g_*(JX_1, Y_1) &= -[\left(\frac{r}{2} - \alpha^2\right) g_*(X_1, Y_1) \\ &\quad + \left(\frac{r}{2} - 3\alpha^2\right) \eta_*(X_1)\eta_*(Y_1)] \\ &\quad +\tau_1 g_*(X_1, Y_1). \end{aligned} \tag{4.50}$$

Taking $X_1 = Y_1 = \zeta_1$ in (4.11), we obtain

$$\tau_1 = 2\alpha^2. \tag{4.51}$$

Since α is constant. This implies $\tau_1 = 2\alpha^2$ =constant. Hence, we can establish the following result.

Theorem 4.2. *A 3-dimensional α-LPS manifold $(M, \zeta_1, \eta_*, g_*)$ admits AR soliton then it reduces to a Ricci soliton for α =constant.*

5. Gradient Almost Ricci (GAR) Solitons

In this part, we study 3-dimensional α-LPS manifolds admitting GAR soliton. For a GAR soliton, we have

$$\bar{\nabla}_{Y_1} Df = \tau_1 Y_1 - QY_1, \tag{5.52}$$

where D symbolize the gradient operator of g_*.

Now taking covariant differentiation of (5.1) along arbitrary vector field X_1, we have

$$\bar{\nabla}_{X_1}\bar{\nabla}_{Y_1} Df = d\tau_1(X_1)Y_1 + \tau_1\bar{\nabla}_{X_1}Y_1 - (\bar{\nabla}_{X_1}Q)Y_1. \tag{5.53}$$

In above equation d is exterior derivative, using this similarly we obtain

$$\bar{\nabla}_{Y_1}\bar{\nabla}_{X_1} Df = d\tau_1(Y_1)X_1 + \tau_1\bar{\nabla}_{Y_1}X_1 - (\bar{\nabla}_{Y_1}Q)X_1, \tag{5.54}$$

and

$$\bar{\nabla}_{[X_1,Y_1]} Df = \tau_1[X_1, Y_1] - Q[X_1, Y_1]. \tag{5.55}$$

In view of $(5.2), (5.3)$ and (5.4), we get

$$\begin{aligned} R(X_1, Y_1)Df &= \bar{\nabla}_{X_1}\bar{\nabla}_{Y_1} Df - \bar{\nabla}_{Y_1}\bar{\nabla}_{X_1} Df - \bar{\nabla}_{[X_1,Y_1]} Df \qquad (5.56)\\ &= (\bar{\nabla}_{Y_1}Q)X_1 - (\bar{\nabla}_{X_1}Q)Y_1 - (Y_1\tau_1)X_1 + (X_1\tau_1)Y_1. \end{aligned}$$

From (2.19), we have

$$QX_1 = [\frac{r}{2} - \alpha^2]X_1 + [\frac{r}{2} - 3\alpha^2]\eta_*(X_1)\zeta_1. \tag{5.57}$$

Taking covariant differentiation of (5.6) along arbitrary vector field X_1 and using (2.9), we have

$$\begin{aligned}(\bar{\nabla}_{X_1}Q)Y_1 &= \left(\frac{X_1 r}{2}\right)[Y_1+\eta_*(Y_1)\zeta_1] \\ &\quad +\alpha\left(\frac{r}{2}-3\alpha^2\right)[g_*(JX_1,Y_1)+\eta_*(Y_1)JX_1]. \end{aligned} \tag{5.58}$$

Similarly, we have

$$\begin{aligned}(\bar{\nabla}_{Y_1}Q)X_1 &= \left(\frac{Y_1 r}{2}\right)[X_1+\eta_*(X_1)\zeta_1] \\ &\quad +\alpha\left(\frac{r}{2}-3\alpha^2\right)[g_*(JY_1,X_1)+\eta_*(X_1)JY_1]. \end{aligned} \tag{5.59}$$

Using (5.7) and (5.8) in (5.5), we have

$$\begin{aligned}R(X_1,Y_1)Df &= \left(\frac{Y_1 r}{2}\right)[X_1+\eta_*(X_1)\zeta_1]+\alpha\left(\frac{r}{2}-3\alpha^2\right)\eta_*(X_1)JY_1 \\ &\quad -\left(\frac{X_1 r}{2}\right)[Y_1+\eta_*(Y_1)\zeta_1]-\alpha\left(\frac{r}{2}-3\alpha^2\right)\eta_*(Y_1)JX_1 \\ &\quad -(Y_1\tau_1)X_1+(X_1\tau_1)Y_1. \end{aligned} \tag{5.60}$$

Taking an inner product with ζ_1 in above equation, then we obtain

$$g_*(R(X_1,Y_1)Df,\zeta_1) = -(Y_1\tau_1)\eta_*(X_1)+(X_1\tau_1)\eta_*(Y_1). \tag{5.61}$$

Taking $Y_1=\zeta_1$, then we infer

$$g_*(R(X_1,\zeta_1)Df,\zeta_1) = -(\zeta_1\tau_1)\eta_*(X_1)-(X_1\tau_1). \tag{5.62}$$

Also from (2.18), it follows that

$$g_*(R(X_1,\zeta_1)Df,\zeta_1) = \alpha^2[(\zeta_1 f)\eta_*(X_1)-(X_1 f)]. \tag{5.63}$$

Using (5.9) in (5.10), we get

$$\alpha^2[(\zeta_1 f)\eta_*(X_1)-(X_1 f)] = -(\zeta_1\tau_1)\eta_*(X_1)-(X_1\tau_1). \tag{5.64}$$

Assuming that f is constant. Then it follows from (5.11) that

$$d\tau_1 + (\zeta_1\tau_1)\eta_* = 0. \tag{5.65}$$

Applying d both sides of (5.14), we obtain

$$\zeta_1\tau_1 = 0. \tag{5.66}$$

By virtue of (5.14) and (5.15), we get

$$d\tau_1 = 0. \tag{5.67}$$

This implies τ_1 is constant. Hence, we can establish the following result:

Theorem 5.1. *A 3-dimensional α-LPS manifold $(M, \zeta_1, \eta_*, g_*)$ admits a GAR soliton then it reduces to a Ricci soliton provided f is constant.*

6. Example

We consider the 3-dimensional manifold $M = \{(x, y, t) \in R^3 : t \neq 0\}$, where (x, y, t) are the standard coordinates in R^3. We choose the vector fields

$$\tilde{E}_1 = e_*^t \frac{\partial}{\partial y}, \tilde{E}_2 = e_*^t (\frac{\partial}{\partial x} + \frac{\partial}{\partial y}) \text{ and } \tilde{E}_3 = e_*^t \frac{\partial}{\partial t},$$

which are linearly independent at each point of M. Let g_* be the Lorentzian metric defined by

$$g_*(\tilde{E}_1, \tilde{E}_2) = g_*(\tilde{E}_2, \tilde{E}_3) = g_*(\tilde{E}_3, \tilde{E}_1) = 0,$$

$$g_*(\tilde{E}_1, \tilde{E}_1) = g_*(\tilde{E}_2, \tilde{E}_2) = 1, \; g_*(\tilde{E}_3, \tilde{E}_3) = -1.$$

Let η_* be the 1- form defined by $\eta_*(Z_1) = g_*(Z_1, \tilde{E}_3)$ for any vector field Z_1 on M. We define the $(1,1)$ tensor field J as $J(\tilde{E}_1) = -\tilde{E}_1, J(\tilde{E}_2) = -\tilde{E}_2$ and $J(\tilde{E}_3) = 0$. Then using the linearity of J and g_*, we have

$$\eta_*(\tilde{E}_3) = -1, J^2 Z_1 = Z_1 + \eta_*(Z_1)\tilde{E}_3,$$

$$g_*(JZ_1, JW_1) = g_*(Z_1, W_1) + \eta_*(Z_1)\eta_*(W_1),$$

for any vector fields Z_1, W_1 on M. Thus for $\tilde{E}_3 = \zeta_1$, the structure $(J, \zeta_1, \eta_*, g_*)$ defines an almost contact metric structure on M.

Let $\bar{\nabla}$ be the Levi-Civita connection with respect to the Lorentzian metric g_*. Then, we have

$$[\tilde{E}_1, \tilde{E}_2] = 0, \quad [\tilde{E}_1, \tilde{E}_3] = -e_*^t \tilde{E}_1 \text{ and } \quad [\tilde{E}_2, \tilde{E}_3] = -e_*^t \tilde{E}_2.$$

Koszul's formula is defined by

$$\begin{aligned} 2g_*(\bar{\nabla}_{X_1} Y_1, Z_1) &= X_1 g_*(Y_1, Z_1) + Y_1 g_*(Z_1, X_1) - Z_1 g_*(X_1, Y_1) \\ &\quad - g_*(X_1, [Y_1, Z_1]) - g_*(Y_1, [X_1, Z_1]) + g_*(Z_1, [X_1, Y_1]). \end{aligned}$$

Using Koszul's formula, we can easily calculate

$$\bar{\nabla}_{\tilde{E}_1} \tilde{E}_3 = -e_*^t \tilde{E}_1, \quad \bar{\nabla}_{\tilde{E}_1} \tilde{E}_2 = 0, \quad \bar{\nabla}_{\tilde{E}_1} \tilde{E}_1 = -e_*^t \tilde{E}_3,$$

$$\bar{\nabla}_{\tilde{E}_2} \tilde{E}_3 = -e_*^t \tilde{E}_2, \quad \bar{\nabla}_{\tilde{E}_2} \tilde{E}_2 = -e_*^t \tilde{E}_3, \quad \bar{\nabla}_{\tilde{E}_2} \tilde{E}_1 = 0,$$

$$\bar{\nabla}_{\tilde{E}_3} \tilde{E}_3 = 0, \quad \bar{\nabla}_{\tilde{E}_3} \tilde{E}_2 = 0, \quad \bar{\nabla}_{\tilde{E}_3 \tilde{E}_1 = 0}.$$

From the above, it follows that the manifold satisfies

$$(\bar{\nabla}_{X_1} J) Y_1 = \alpha \{ g_*(X_1, Y_1) \zeta_1 + \eta_*(Y_1) X_1 + 2\eta_*(X_1) \eta_*(Y_1) \zeta_1 \},$$

for $\tilde{E}_3 = \zeta_1$. and $\alpha = e_*^t$, $(J, \zeta_1, \eta_*, g_*)$ is a 3-dimensional α-LPS structure on M. Consequently $M^3(J, \zeta_1, \eta_*, g_*)$ is a 3-dimensional α-LPS manifold. Also, the Riemannian curvature tensor R is given by

$$R(X_1, Y_1) Z_1 = \bar{\nabla}_{X_1} \bar{\nabla}_{Y_1} Z_1 - \bar{\nabla}_{Y_1} \bar{\nabla}_{X_1} Z_1 - \bar{\nabla}_{[X_1, Y_1]} Z_1.$$

With the help of above results, we obtain

$$R(\tilde{E}_1, \tilde{E}_2) \tilde{E}_1 = -e_*^{2t} \tilde{E}_2, \; R(\tilde{E}_1, \tilde{E}_2) \tilde{E}_3 = 0, \; R(\tilde{E}_1, \tilde{E}_2) \tilde{E}_2 = -e_*^{2t} \tilde{E}_1,$$

$$R(\tilde{E}_1, \tilde{E}_3) \tilde{E}_1 = -e_*^{2t} \tilde{E}_3, \mathbf{R}(\tilde{E}_1, \tilde{E}_3) \tilde{E}_2 = 0, \mathbf{R}(\tilde{E}_1, \tilde{E}_3) \tilde{E}_3 = -e_*^{2t} \tilde{E}_3.$$

$$R(\tilde{E}_2, \tilde{E}_3) \tilde{E}_1 = 0, \mathbf{R}(\tilde{E}_2, \tilde{E}_3) \tilde{E}_2 = -e_*^{2t} \tilde{E}_3, \mathbf{R}(\tilde{E}_2, \tilde{E}_3) \tilde{E}_3 = -e_*^{2t} \tilde{E}_2.$$

Then, the Ricci tensor S is given by

$$S(\tilde{E}_1, \tilde{E}_1) = 0, S(\tilde{E}_2, \tilde{E}_2) = 0 \text{ and } \quad S(\tilde{E}_3, \tilde{E}_3) = -2e_*^{2t}.$$

from equation (1.2) and above calculation, we find $\tau_1 = 2e_*^t(1 - e_*^t)$.

Thus 3-dimensional α-LPS manifold admitting an AR soliton.

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