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FRENET CURVES IN 3-DIMENSIONAL LORENTZIAN CONCIRCULAR STRUCTURE MANIFOLDS

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Abstract. In this paper, we give some characterizations of Frenet curves in 3-dimensional Lorentzian concircular structure manifolds($(LCS)_3$ manifolds). We define Frenet equations and the Frenet elements of these curves. We also obtain the curvatures of non-geodesic Frenet curves on $(LCS)_3$ manifolds. Finally we give some theorems, corollaries and examples for these curves.

Keywords: Lorentzian manifold, Concircular structure, Frenet curve

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1. Introduction

The differential geometry of curves in manifolds investigated by several authors. Especially the curves in contact and para-contact manifolds drew attention and studied by the authors. B. Olszak[17], derived the conditions for an a.c.m structure on M to be normal and point out some of their consequences. B. Olszak completely characterized the local nature of normal a.c.m. structures on M by giving suitable examples. Moreover B. Olszak gave some restrictions on the scalar curvature in contact metric manifolds which are conformally flat or of constant ϕ-sectional curvature in[16].

J. Welyczko[22], generalized some of the results for Legendre curves in three dimensional normal a.c.m. manifolds, especially, quasi-Sasakian manifolds. J. Welyczko [23], studied

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the curvatures of slant Frenet curves in three-dimensional normal almost paracontact metric manifolds.

B. E. Acet and S. Y. Perktaş [1] obtained the curvatures of Legendre curves in 3-dimensional (ε, δ) trans-Sasakian manifolds. Ji-Eun Lee, defined Lorentzian cross product in a three-dimensional almost contact Lorentzian manifold and proved that $\frac{\kappa}{\tau-1} = cons.$ along a Frenet slant curve in a Sasakian Lorentzian three-manifold. Furthermore, Ji-Eun Lee proved that γ is a slant curve if and only if M is Sasakian for a contact magnetic curve γ in contact Lorentzian 3-manifold M in[12]. Ji-Eun Lee, also gave some characterizations for the generalized Tanaka-Webster connection in a contact Lorentzian manifold in[13].

A. Yıldırım[25] obtained the Frenet apparatus for Frenet curves on three dimensional normal almost contact manifolds and characterized some results for these curves.

U.C.De and K.De[10] studied Lorentzian Trans-Sasakian and conformally flat Lorentzian Trans-Sasakian manifolds.

The LCS manifolds was introduced by [19] with an example. A. A. Shaikh[20] studied various types of $(LCS)_n$-manifolds and proved that in such a manifold the Ricci operator commutes with the structure tensor φ.

In this framework, the paper is organized in the following way. Section 2 with two subsections, we give basic definitions of a $(LCS)_n$-manifolds manifold. In the second subsection we give the Frenet-Serret equations of a curve in $(LCS)_3$ manifold. We give finally the Frenet elements of a Frenet curve in $(LCS)_3$ manifold and give theorems, corollaries and examples for these curves in the third and fourth sections.

2. Preliminaries

2.1. **Lorentzian Concircular Structure Manifolds.** A Lorentzian manifold of dimension n is a doublet $\left(\bar{N}, \bar{g}\right)$, where $\bar{N}$ is a smooth connected para-compact Hausdorff manifold of dimension n and $\bar{g}$ is a Lorentzian metric, that is, $\bar{N}$ admits a smooth symmetric tensor field g of type $(0,2)$such that for each point $p \in \bar{N}$ the tensor $\bar{g}_p : T_p\bar{N} \times T_p\bar{N} \longrightarrow R$ is a non degenerate inner product of signature $(-,+,...,+)$, where $T_p\bar{N}$ denotes the tangent space of $\bar{N}$ at p and R is the real number space. A non zero vector field $V \in T_p\bar{N}$ is called spacelike (resp.non-spacelike, null and timelike) if it satisfies $\bar{g}_p\left(V,V\right) > 0$ (resp., $\leq 0, =, < 0$).[15]

Definition 2.1. *In a Lorentzian manifold* $\left(\bar{N}, \bar{g}\right)$ *a vector field w is defined by*

$$\bar{g}(U, \rho) = A(U) \tag{2.1}$$

for any $U \in \chi(\bar{N})$ is said to be a concircular vector field if

$$(\nabla_U A)(V) = \alpha \left\{ \bar{g}(U,V) + w(U)w(V) \right\}, \tag{2.2}$$

where α is a non-zero scalar and w is a closed 1-form.[24]

If a Lorentzian manifold $\bar{N}$ admits a unit timelike concircular vector field ξ, called generator of the manifold, then we have

$$\bar{g}(\xi,\xi) = -1. \tag{2.3}$$

Since ξ is the unit concircular vector field on $\bar{N}$, there exists a non-zero 1-form η such that

$$\bar{g}(U,\xi) = \eta(U), \tag{2.4}$$

which satisfies the following equation

$$(\nabla_U \eta)(V) = \alpha \left\{ \bar{g}(U,V) + \eta(U)\eta(V) \right\}, \quad (\alpha \neq 0) \tag{2.5}$$

for all vector fields U and V, where ∇ gives the covariant differentiation with respect to the Lorentzian metric $\bar{g}$ and α is a non-zero scalar function satisfies

$$(\nabla_U \alpha) = U\alpha = d\alpha(U) = \rho\eta(U), \tag{2.6}$$

where ρ is a certain scalar function defined by $\rho = -(\xi\alpha)$. If we take

$$\varphi U = \frac{1}{\alpha}\nabla_U \xi, \tag{2.7}$$

then with the help of (2.3), (2.4) and (2.6), we can find

$$\varphi U = U + \eta(U)\xi, \tag{2.8}$$

which shows that φ is a tensor field of type (1,1), called the structure tensor of the manifold $\bar{N}$. Hence the Lorentzian manifold $\bar{N}$ of class C^∞ equipped with a unit timelike concircular vector field ξ, its associated 1-form η and (1,1) tensor field φ is said to be a Lorentzian concircular structure manifold (i.e. $(LCS)_n$ manifold)[19]. Moreover, if $\alpha = 1$, then we have the LP-Sasakian structure of Matsumoto[14]. So we can say the generalization of LP-Sasakian manifold gives us the $(LCS)_n$ manifold. It is noteworthy to mention that LCS-manifold is invariant under a conformal change whereas LP-Sasakian structure is not so[18]. In $(LCS)_3$ manifolds, the following relations hold[19]

$$\varphi^2 U = U + \eta(U)\xi, \quad \eta(\xi) = -1, \tag{2.9}$$

$$\varphi(\xi) = 0, \quad \eta(\varphi U) = 0,$$

and

$$\bar{g}(\varphi U, \varphi V) = \bar{g}(U, V) + \eta(U)\eta(V). \tag{2.10}$$

2.2. **Frenet Curves.** Let $\zeta : I \to \bar{N}$ be a unit speed curve in $(LCS)_3$ manifold $\bar{N}$ such that $\zeta^{'}$ satisfies $\bar{g}(\zeta^{'}, \zeta^{'}) = \varepsilon_1 = \mp 1$. The constant ε_1 is called the casual character of ζ. The constants ε_2 and ε_3 defined by $\bar{g}(n, n) = \varepsilon_2$ and $\bar{g}(b, b) = \varepsilon_3$ and called the second casual character and third casual character of ζ, respectively. Thus we $\varepsilon_1\varepsilon_2 = -\varepsilon_3$.

A unit speed curve ζ is said to be a spacelike or timelike if its casual character is 1 or -1, respectively. A unit speed curve ζ is said to be a Frenet curve if $\bar{g}(\zeta^{'}, \zeta^{'}) \neq 0$. A Frenet curve ζ admits an orthonormal frame field $\{t = \zeta^{'}, n, b\}$ along ζ. Then the Frenet-Serret equations given as follows:

$$\begin{aligned} \nabla_{\zeta'} t &= \varepsilon_2 \kappa n \\ \nabla_{\zeta'} n &= -\varepsilon_1 \kappa t - \varepsilon_3 \tau b \\ \nabla_{\zeta'} b &= \varepsilon_2 \tau n \end{aligned} \tag{2.11}$$

where $\kappa = |\nabla_{\zeta'} \zeta^{'}|$ is the geodesic curvature of ζ and τ is geodesic torsion [12]. The vector fields t, n and b are called the tangent vector field, the principal normal vector field and the binormal vector field of ζ, respectively.

If the geodesic curvature of the curve ζ vanishes, then the curve is called a geodesic curve. If $\kappa = cons.$ and $\tau = 0$, then the curve is called a pseudo-circle and pseudo-helix if the geodesic curvature and torsion are constant.

A curve in a three dimensional Lorentzian manifold is a slant curve if the tangent vector field of the curve has constant angle with the Reeb vector field,i.e. $\eta(\zeta') = -\bar{g}(\zeta', \xi) = cos\theta = constant$. If $\eta(\zeta') = -\bar{g}(\zeta', \xi) = 0$, then the curve ζ is called a Legendre curve[12].

3. Main Results

In this section we consider a $(LCS)_3$ manifold $\bar{N}$. Let $\zeta : I \to \bar{N}$ be a Frenet curve with the geodesic curvature $\kappa \neq 0$, given with the arc-parameter s and $\bar{\nabla}$ be the Levi-Civita connection on $\bar{N}$. From the basis $(\zeta^{'}, \varphi\zeta^{'}, \xi)$ we obtain an orthonormal basis $\{e_1, e_2, e_3\}$ which satisfy the equations

$$\begin{aligned} e_1 &= \zeta^{'}, \\ e_2 &= \frac{\varepsilon_2\varphi\zeta^{'}}{\sqrt{\varepsilon_1+\rho^2}}, \\ e_3 &= \varepsilon_2\frac{\varepsilon_1\xi-\rho\zeta^{'}}{\sqrt{\varepsilon_1+\rho^2}} \end{aligned} \tag{3.12}$$

where

$$\eta(\zeta^{'}) = \bar{g}(\zeta^{'},\xi) = \rho. \tag{3.13}$$

Then if we write the covariant differentiation of $\zeta^{'}$ as

$$\bar{\nabla}_{\zeta'} e_1 = \nu e_2 + \mu e_3 \tag{3.14}$$

such that

$$\nu = \bar{g}(\bar{\nabla}_{\zeta'} e_1, e_2) \tag{3.15}$$

is a certain function. Moreover we obtain ν by

$$\mu = \bar{g}(\bar{\nabla}_{\zeta'} e_1, e_3) = \varepsilon_2\left(\frac{\rho'}{\sqrt{\varepsilon_1+\rho^2}} - \varepsilon_1\alpha\sqrt{\varepsilon_1+\rho^2}\right), \tag{3.16}$$

where $\rho'(s) = \frac{d\rho(\zeta(s))}{ds}$. Then we find

$$\bar{\nabla}_{\zeta'} e_2 = -\nu e_1 + \left(\varepsilon_3\alpha + \frac{\varepsilon_1\rho\nu}{\sqrt{\varepsilon_1+\rho^2}}\right)e_3 \tag{3.17}$$

and

$$\bar{\nabla}_{\zeta'} e_3 = -\mu e_1 - \left(\varepsilon_3\alpha + \frac{\varepsilon_1\rho\nu}{\sqrt{\varepsilon_1+\rho^2}}\right)e_2. \tag{3.18}$$

The fundamental forms of the tangent vector ζ' on the basis of the equation (3.12) is

$$[\omega_{ij}(\zeta')] = \begin{pmatrix} 0 & \nu & \mu \\ -\nu & 0 & \varepsilon_3\alpha + \frac{\varepsilon_1\rho\nu}{\sqrt{\varepsilon_1+\rho^2}} \\ -\mu & -\varepsilon_3\alpha - \frac{\varepsilon_1\rho\nu}{\sqrt{\varepsilon_1+\rho^2}} & 0 \end{pmatrix} \tag{3.19}$$

and the Darboux vector connected to the vector ζ' is

$$\omega(\zeta') = \left(\varepsilon_3\alpha + \frac{\varepsilon_1\rho\nu}{\sqrt{\varepsilon_1+\rho^2}}\right)e_1 - \mu e_2 + \nu e_3. \tag{3.20}$$

So we can write

$$\bar{\nabla}_{\zeta'} e_i = \omega(\zeta') \wedge \varepsilon_i e_i \quad (1 \leq i \leq 3). \tag{3.21}$$

Thus, for any vector field $Z = \sum_{i=1}^{3} \theta^i e_i \in \chi(\bar{N})$ strictly dependent on the curve ζ on $\bar{N}$ and we have the following equation

$$\bar{\nabla}_{\zeta'} Z = \omega(\zeta') \wedge Z + \sum_{i=1}^{3} \varepsilon_i e_i [\theta^i] e_i. \tag{3.22}$$

3.1. **Frenet Elements of ζ.** Let a curve $\zeta : I \to \bar{N}$ be a Frenet curve with the geodesic curvature $\kappa \neq 0$, given with the arc parameter s and the elements $\{t, n, b, \kappa, \tau\}$. The Frenet elements of the curve ζ can be calculated as above:

If we consider the equation (3.14), then we get

$$\varepsilon_2 \kappa n = \bar{\nabla}_{\zeta'} e_1 = \nu e_2 + \mu e_3. \tag{3.23}$$

If we consider (3.16) and (3.23) we find

$$\kappa = \sqrt{\nu^2 + \left(\frac{\rho'}{\sqrt{\varepsilon_1 + \rho^2}} - \varepsilon_1 \alpha \sqrt{\varepsilon_1 + \rho^2} \right)^2}. \tag{3.24}$$

On the other hand

$$\begin{aligned} \bar{\nabla}_{\zeta'} n &= \left(\frac{\nu}{\varepsilon_2 \kappa} \right)' e_2 + \frac{\nu}{\varepsilon_2 \kappa} \nabla_{\zeta'} e_2 + \left(\frac{\mu}{\varepsilon_2 \kappa} \right)' e_3 + \frac{\mu}{\varepsilon_2 \kappa} \nabla_{\zeta'} e_3 \\ &= -\varepsilon_1 \kappa t - \varepsilon_3 \tau B. \end{aligned} \tag{3.25}$$

By means of the equation (3.17) and (3.18) we find

$$\begin{aligned} -\varepsilon_3 \tau B &= \left[\left(\frac{\nu}{\varepsilon_2 \kappa} \right)' - \frac{\mu}{\varepsilon_2 \kappa} \left(\varepsilon_3 \alpha + \frac{\varepsilon_1 \rho \nu}{\sqrt{\varepsilon_1 + \rho^2}} \right) \right] e_2 \\ &+ \left[\left(\frac{\mu}{\varepsilon_2 \kappa} \right)' + \frac{\nu}{\varepsilon_2 \kappa} \left(\varepsilon_3 \alpha + \frac{\varepsilon_1 \rho \nu}{\sqrt{\varepsilon_1 + \rho^2}} \right) \right] e_3. \end{aligned} \tag{3.26}$$

By a direct computation we find following

$$\left[\left(\frac{\nu}{\varepsilon_2 \kappa} \right)' \right]^2 + \left[\left(\frac{\mu}{\varepsilon_2 \kappa} \right)' \right]^2 = \left[- \left(\frac{\nu}{\varepsilon_2 \kappa} \right)' \frac{\mu}{\varepsilon_2 \kappa} + \frac{\nu}{\varepsilon_2 \kappa} \left(\frac{\mu}{\varepsilon_2 \kappa} \right)' \right]^2. \tag{3.27}$$

Taking the norm of the last equation by using (3.26) and if we consider the equations (3.16) and (3.27) in (3.26) we obtain

$$\tau = \left| \varepsilon_3 \alpha + \frac{\varepsilon_1 \rho \nu}{\sqrt{\varepsilon_1 + \rho^2}} - \sqrt{ \left[\left(\frac{\nu}{\varepsilon_2 \kappa} \right)' \right]^2 + \left[\left(\frac{\varepsilon_2 \left(\frac{\rho'}{\sqrt{\varepsilon_1 + \rho^2}} - \varepsilon_1 \alpha \sqrt{\varepsilon_1 + \rho^2} \right)}{\kappa} \right)' \right]^2 } \right|. \tag{3.28}$$

Moreover we can write the Frenet vector fields of ζ as in the following theorem

Theorem 3.1. *Let $\bar{N}$ be a $(LCS)_3$ manifold and ζ be a Frenet curve on $\bar{N}$. The Frenet vector fields t, n and b are in the form of*

$$
\begin{aligned}
t &= \zeta' = e_1, \\
n &= \frac{\nu}{\varepsilon_2\kappa} e_2 + \frac{\mu}{\varepsilon_2\kappa} e_3, \\
b &= -\frac{1}{\varepsilon_3\tau}\left[\left(\frac{\nu}{\varepsilon_2\kappa}\right)' - \frac{\mu}{\varepsilon_2\kappa}\left(\varepsilon_3\alpha + \frac{\varepsilon_1\rho\nu}{\sqrt{\varepsilon_1+\rho^2}}\right)\right] e_2 \\
&\quad - \frac{1}{\varepsilon_3\tau}\left[\left(\frac{\mu}{\varepsilon_2\kappa}\right)' + \frac{\nu}{\varepsilon_2\kappa}\left(\varepsilon_3\alpha + \frac{\varepsilon_1\rho\nu}{\sqrt{\varepsilon_1+\rho^2}}\right)\right] e_3.
\end{aligned}
\tag{3.29}
$$

Note that

$$
\begin{aligned}
\xi &= \varepsilon_1\rho t - \frac{\mu\sqrt{\varepsilon_1+\rho^2}}{\kappa} n \\
&\quad - \frac{\sqrt{\varepsilon_1+\rho^2}}{\varepsilon_3\tau}\left[\left(\frac{\mu}{\varepsilon_2\kappa}\right)' + \frac{\nu}{\varepsilon_2\kappa}\left(\varepsilon_3\alpha + \frac{\varepsilon_1\rho\nu}{\sqrt{\varepsilon_1+\rho^2}}\right)\right] b.
\end{aligned}
\tag{3.30}
$$

Let ζ be a non-geodesic Frenet curve given with the arc-parameter s in $(LCS)_3$ manifold $\bar{N}$. So one can state the above theorems.

Theorem 3.2. *Let $\bar{N}$ be a $(LCS)_3$ manifold and ζ be a Frenet curve on $\bar{N}$. ζ is a slant curve ($\rho = \eta(\zeta') = cos\theta = cons.$) on $\bar{N}$ if and only if the Frenet elements $\{t, n, b, \kappa, \tau\}$ of ζ are as follows*

$$
\begin{aligned}
t &= e_1 = \zeta', \\
n &= e_2 = \frac{\varepsilon_2\varphi\zeta'}{\sqrt{\varepsilon_1 + cos^2\theta}}, \\
b &= e_3 = \varepsilon_2\frac{\varepsilon_1\xi - cos\theta\zeta'}{\sqrt{\varepsilon_1 + cos^2\theta}}, \\
\kappa &= \sqrt{\nu^2 + \alpha^2\left(\varepsilon_1 + cos^2\theta\right)}, \\
\tau &= \left|\varepsilon_3\alpha + \frac{\varepsilon_1 cos\theta\nu}{\sqrt{\varepsilon_1 + cos^2\theta}} - \sqrt{\left[\left(\frac{\nu}{\varepsilon_2\kappa}\right)'\right]^2 + \left[\left(\frac{\alpha\sqrt{\varepsilon_1 + \cos^2\theta}}{\kappa}\right)'\right]^2}\right|.
\end{aligned}
\tag{3.31}
$$

Proof. Let the curve ζ be a slant curve in $(LCS)_3$ manifold $\bar{N}$. If we take account the condition $\rho = \eta(\zeta') = cos\theta = constant$ in the equations (3.12), (3.24) and (3.28) we find (3.31). If the equations in (3.31) hold, from the definition of slant curves it is obvious that the curve ζ is a slant curve.

Corollary 3.1. *Let $\bar{N}$ be a $(LCS)_3$ manifold and ζ be a slant curve on $\bar{N}$. If the geodesic curvature κ of the curve ζ is non-zero constant, then the geodesic torsion of ζ is $\tau = \left|\left(\varepsilon_3\alpha + \varepsilon_1 \frac{cos\theta\nu}{\sqrt{\varepsilon_1+\cos^2\theta}}\right)\right|$ and ζ is a pseudo-helix on $\bar{N}$.*

Corollary 3.2. *Let $\bar{N}$ be a $(LCS)_3$ manifold and ζ be a slant curve on $\bar{N}$. If the geodesic curvature κ of the curve ζ is not constant and the geodesic torsion of ζ is $\tau = 0$ then ζ is a plane curve on $\bar{N}$ and function ν satisfies the equation*

$$\nu = \int (c_1 + c_2\nu)\, \kappa^2 ds, \tag{3.32}$$

where $c_1 = \frac{\varepsilon_3}{\sqrt{\varepsilon_1+\cos^2\theta}}$ and $c_2 = \frac{\varepsilon_1 cos\theta}{\alpha(\varepsilon_1+\cos^2\theta)}$.

Theorem 3.3. *Let $\bar{N}$ be a $(LCS)_3$ manifold and ζ is a Frenet curve on $\bar{N}$. ζ is a spacelike Legendre curve($\rho = \eta(\zeta') = 0$) in this manifold if and only if the Frenet elements $\{t, n, b, \kappa, \tau\}$ of ζ are as follows*

$$\begin{aligned} t &= e_1 = \zeta', \\ n &= e_2 = \varepsilon_2\varphi\zeta', \\ b &= e_3 = -\varepsilon_3\xi, \\ \kappa &= \sqrt{\nu^2+\alpha^2}, \\ \tau &= \left|\varepsilon_3\alpha - \sqrt{\left[\left(\frac{\nu}{\varepsilon_2\kappa}\right)'\right]^2 + \alpha^2\left[\frac{\kappa'}{\kappa^2}\right]^2}\right|. \end{aligned} \tag{3.33}$$

Proof. Let the curve ζ be a Legendre curve in $(LCS)_3$ manifold $\bar{N}$. If we take account the condition $\rho = \eta(\zeta') = 0$ in the equations (3.12), (3.24) and (3.28) we find (3.33). If the equations in (3.33) hold, from the definition of Legendre curves it is obvious that the curve ζ is a Legendre curve on $\bar{N}$.

Corollary 3.3. *Let the curve ζ is a Legendre curve in $(LCS)_3$ manifold $\bar{N}$. If the geodesic curvature κ of the curve ζ is non-zero constant, then the geodesic torsion of ζ is $\tau = 0$ and ζ is a plane curve on $\bar{N}$.*

4. Examples

Let $\bar{N}$ be the 3-dimensional manifold given

$$\bar{N} = \left\{(x, y, z) \in \Re^3, z \neq 0\right\}, \tag{4.34}$$

where (x,y,z) denote the standart co-ordinates in $\Re^3$. Then

$$E_1 = e^z\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right), \qquad E_2 = e^z\frac{\partial}{\partial y}, E_3 = \frac{\partial}{\partial z} \tag{4.35}$$

are linearly independent of each point of $\bar{N}$. Let g be the Lorentzian metric tensor defined by

$$\bar{g}(E_1, E_1) = \bar{g}(E_2, E_2) = 1, \qquad \bar{g}(E_3, E_3) = -1, \tag{4.36}$$
$$\bar{g}(E_i, E_j) = 0, \quad i \neq j,$$

for $i, j = 1, 2, 3$[2]. Let η be the 1-form defined by $\eta(Z) = \bar{g}(Z, E_3)$ for any $Z \in \Gamma(T\bar{N})$. Let φ be the (1,1)-tensor field defined by

$$\varphi E_1 = E_1, \quad \varphi E_2 = E_2, \quad \varphi E_3 = 0. \tag{4.37}$$

Then using the condition of the linearity of φ and $\bar{g}$, we obtain $\eta(E_3) = -1$,

$$\varphi^2 Z = Z + \eta(Z)E_3, \tag{4.38}$$
$$\bar{g}(\varphi Z, \varphi W) = \bar{g}(Z, W) + \eta(Z)\eta(W),$$

for all $Z, W \in \Gamma(T\bar{N})$. Thus for $\xi = E_3$, $(\varphi, \xi, \eta, \bar{g})$ defines a Lorentzian paracontact structure on $\bar{N}$.

Now, let ∇ be the Levi-Civita connection with respect to the Lorentzian metric $\bar{g}$. Then we obtain

$$[E_1, E_2] = -e^z E_2, \quad [E_1, E_3] = -E_1, \quad [E_2, E_3] = -E_2. \tag{4.39}$$

If we use the Koszul formulae for the Lorentzian metric tensor $\bar{g}$, we can easily calculate the covariant derivations as follows:

$$\begin{gathered} \nabla_{E_1}E_1 = -E_3, \quad \nabla_{E_2}E_1 = e^z E_2, \quad \nabla_{E_1}E_3 = -E_1, \\ \nabla_{E_2}E_3 = -E_2, \qquad \nabla_{E_2}E_2 = -e^z E_1 - E_3, \\ \nabla_{E_1}E_2 = \nabla_{E_3}E_1 = \nabla_{E_3}E_2 = \nabla_{E_3}E_3 = 0. \end{gathered} \tag{4.40}$$

From the about represantations, one can easily see that $(\varphi, \xi, \eta, \bar{g})$ is a $(LCS)_3$ structure on $\bar{N}$, that is, $\bar{N}$ is an $(LCS)_3$-manifold with $\alpha = -1$ and $\rho = 0$.

Example 4.1. *Let β be a spacelike Legendre curve in the $(LCS)_3$ manifold $\bar{N}$ and defined as*

$$\begin{aligned} \beta: \; I \; &\to \bar{N} \\ s \; &\to \beta(s) = \left(s^2, s^2, ln2\right), \end{aligned}$$

where the curve β parametrized by the arc length parameter t. If we differentiate $\beta(t)$ and consider (3.12) we find

$$e_1 = \beta'(t), \tag{4.41}$$

$$e_2 = \frac{1}{\sqrt{2}}E_1 + \frac{1}{\sqrt{2}}E_2, \tag{4.42}$$

$$e_3 = \varepsilon_2 E_3. \tag{4.43}$$

If we consider the equations (3.13), (3.14), (3.16), (3.24) and (3.28) we can write

$$\rho = 0, \quad \mu = -\varepsilon_2\alpha, \quad \nu = -\frac{1}{\sqrt{2}}, \tag{4.44}$$

$$\kappa = \sqrt{\alpha^2 + \frac{1}{2}} = \sqrt{\frac{3}{2}}, \quad \tau = |\alpha| = 1.$$

From the above equations we see that the curve β is a Legendre helix curve in $\bar{N}$.

Example 4.2. *Let υ be a spacelike Legendre curve in the $(LCS)_3$ manifold $\bar{N}$ and defined as*

$$\begin{aligned} \upsilon: \; I \; &\to \bar{N} \\ s \; &\to \upsilon(s) = (coss, sins, 1), \end{aligned}$$

where the curve υ parametrized by the arc length parameter t. If we differentiate $\upsilon(t)$ and consider (3.12) we find

$$e_1 = \upsilon'(t), \tag{4.45}$$

$$e_2 = \varepsilon_2 \left(-sin(\frac{t}{e})E_1 + cos(\frac{t}{e})E_2\right), \tag{4.46}$$

$$e_3 = \varepsilon_2 \partial_3. \tag{4.47}$$

If we consider the equations (3.13), (3.14), (3.16), (3.24) and (3.28) we can write

$$\rho = 0, \quad \mu = -\varepsilon_2\alpha, \quad \nu = 0, \tag{4.48}$$

$$\kappa = \tau = |\alpha|.$$

So, the curve $\upsilon(s)$ is a Legendre helix curve in $\bar{N}$.

References

[1] Acet B. E., Perktaş S. Y. (2018). Curvature and torsion of a legendre curve in (ε, δ) Trans-Sasakian manifolds. Malaya Journal of Matematik, Vol. 6, No. 1, 140-144.

[2] Atçeken M. (2013). Some Curvature Properties of $(LCS)_n$-Manifolds. Hindawi Publishing Corporation Abstract and Applied Analysis, Volume 2013, http://dx.doi.org/10.1155/2013/380657.

[3] Blair D.E. (2002). Riemannian Geometry of Contact and Symplectic Manifolds. Progr. Math., Birkhäuser, Boston.

[4] Calvaruso G. (2007). Homogeneous structures on three-dimensional Lorentzian manifolds. J. Geom. Phys. 57, 1279–1291. G. Calvaruso, J. Geom. Phys. 58 (2008) 291–292, Addendum.

[5] Calvaruso G. (2007). Einstein-like metrics on three-dimensional homogeneous Lorentzian manifolds. Geometriae Dedicata. 127, 99–119.

[6] Calvaruso G., Perrone D. (2010). Contact pseudo-metric manifolds. Differ. Geom. App. 28, 615–634.

[7] Calvaruso G. (2011). Contact Lorentzian manifolds. Differ. Geom. App. 29, S41–S51.

[8] Camci, C. (2011). Extended cross product in a 3-dimensional almost contact metric manifold with applications to curve theory. Turk. J. Math. 35, 1–14.

[9] Cho, J. T., Inoguchi, J. I., and Lee, J. E. (2006). On slant curves in Sasakian 3-manifolds. Bulletin of the Australian Mathematical Society, 74(3), 359-367.

[10] De, U. C. and De, K. (2013). On Lorentzian Trans-Sasakian Manifolds. Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat., 62(2), 37-51.

[11] Duggal, K.L. and Jin, D.H. (2007). Null Curves and Hypersurfaces of Semi-Riemannian Manifolds. World Scientific Publishing: Singapore.

[12] Lee, J. E. (2019). Slant Curves and Contact Magnetic Curves in Sasakian Lorentzian 3-Manifolds. Symmetry. 11, 784; doi:10.3390/sym11060784.

[13] Lee, J. E. (2020). Slant Curves in Contact Lorentzian Manifolds with CR Structures. Mathematics. 8, 46; doi:math8010046.

[14] Matsumoto K. (1989). On Lorentzian paracontact manifolds. Bull. Yamagata Univ. Natur. Sci. 12, no.2, 151-156.

[15] O'Neill B. (1983). Semi-Riemannian Geometry. Academic Press, New York.

[16] Olszak B. (1979). On contact metric manifolds. Tôhoku Math. J. 31, 247–253.

[17] Olszak B. (1986). Normal Almost Contact Metric Manifolds of Dimension Three. Ann. Pol. Math., XLVII.

[18] Shaikh, A. A., and De, U. C. (2000). On 3-dimensional LP-Sasakian manifolds. Soochow Journal of Mathematics, 26(4), 359-368.

[19] Shaikh, A. A. (2003). On Lorentzian almost paracontact manifolds with a structure of the concircular type. Kyungpook Math. J, 43(2), 305-314.

[20] Shaikh A.A. (2009). Some Results on $(LCS)_n$-manifolds. J. Korean Math. Soc. 46(3), 449-461.

[21] Şahin, B. (2012). Manifoldların diferensiyel geometrisi. Nobel Yayın, 310.

[22] Welyczko J. (2007). On Legendre curves in 3-dimensional normal almost contact metric manifolds. Soochow Journal of Mathematics, 33(4), 929-937.

[23] Welyczko J. (2014). Slant Curves in 3-Dimensional Normal Almost Paracontact Metric Manifolds. Mediterr. J. Math. 11, 965–978, DOI 10.1007/s00009-013-0361-2 0378-620X/14/030965-14.

[24] Yano K. (1940). Concircular Geometry.I. Concircular transformations. proc. Imp. Acad. Tokyo. 16, 195-200.

[25] Yıldırım A. (2020). On curves in 3-dimensional normal almost contact metric manifolds. Int. J. Geom. Methods M. Physics, 1-18.

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