International Journal of Maps in Mathematics

Volume:1, Issue:1, (2018), Pages:1-14

www.journalmim.com

SOME RESULTS ON COMMON FIXED POINTS FOR RATIONAL TYPE CONTRACTION MAPPINGS IN COMPLEX VALUED METRIC SPACE

DEEPAK KUMAR* AND AMAL CHACKO

ABSTRACT. In this manuscript, we have obtained the sufficient conditions for the existence and uniqueness of a pair of mappings satisfying rational type contractive conditions in the framework of complex valued metric space. Our result generalizes the well known result introduced by Azam et al. [2] in complex valued metric space. Also, various deductions have been provided.

1. Introduction

Azam et al. [2] introduced the concept of more general metric space, which is well known as complex valued metric spaces. He gave sufficient conditions for the existence and uniqueness of common fixed points satisfying contractive conditions. Later, S. Bhatt et al. [4] without using the notion of continuity proved a common fixed point theorem for weakly compatible maps in complex valued metric spaces. F. Rouzkard and M. Imdad [12] considering rational type contractive conditions proved some common fixed point theorems in the framework of complex valued metric space. C. Klin-eam and C. Suanoom [8] proved certain common fixed-point theorems for two single-valued mappings satisfy certain metric inequalities.

Received: 2018-02-16 *Accepted:2018-03-05*

2010 Mathematics Subject Classification: 47H10, 54H25.

Key words: fixed point theorem, contraction mapping, complex valued metric space

* Corresponding author

The notion of complex valued metric space lead to development in non linear analysis. Thereafter, many results have been proved by the researchers in the framework of complex valued metric spaces for references (see [7]-[13]).

2. Preliminaries

To begin with, we recall some basic definitions, notations, and results. The following definitions of Azam et al. [2] are required in the sequel.

Let $\mathbb{C}$ be a set of complex number such that $z_1, z_2 \in \mathbb{C}$. Define a partial order $\precsim$ on $\mathbb{C}$, such that $z_1 \precsim z_2$ if and only if $Re(z_1) \leq Re(z_2)$, $Img(z_1) \leq Img(z_2)$.
It follows that

$$z_1 \precsim z_2$$

if one of the below mentioned conditions is satisfied:

((i)) $Re(z_1) = Re(z_2)$, $Img(z_1) < Img(z_2)$;
((ii)) $Re(z_1) < Re(z_2)$, $Img(z_1) = Img(z_2)$;
((iii)) $Re(z_1) < Re(z_2)$, $Img(z_1) < Img(z_2)$;
((iv)) $Re(z_1) = Re(z_2)$, $Img(z_1) = Img(z_2)$.

In particular, we will write $z_1 \precnsim z_2$, if $z_1 \neq z_2$ and one of (i), (ii) and (iii) is satisfied. We will write $z_1 \prec z_2$ if only (iii) is satisfied.

Remark 2.1. *We obtained that the following statements holds:*

- $a, b \in R$ *and* $a \leq b$ *implies* $az \precsim bz$, *for all* $z \in \mathbb{C}$*;*
- $0 \precsim z_1 \precnsim z_2$ *implies* $|z_1| < |z_2|$*;*
- $z_1 \preceq z_2$ *and* $z_2 \prec z_3$ *imply* $z_1 \prec z_3$.

Definition 2.1. [2] *Let* X *be non-empty set. Suppose that the mapping* $\rho_c : X \times X \to \mathbb{C}$ *satisfies the following conditions:*

((i)) $0 \precsim \rho_c(x, y)$ *for all* $x, y \in X$ *and* $\rho_c(x, y) = 0$ *if* $x = y$*;*
((ii)) $\rho_c(x, y) = \rho_c(y, x)$ *for all* $x, y \in X$*;*
((iii)) $\rho_c(x, y) \precsim \rho_c(x, z) + \rho_c(z, y)$ *for all* $x, y, z \in X$.

Then, ρ_c *is called a complex valued metric on* X*, and* (X, ρ_c) *is called complex valued metric space.*

Definition 2.2. [2] *A point* $x \in X$ *is called an interior of a set* $A \subseteq X$ *whenever their exists* $0 \prec r \in \mathbb{C}$ *such that* $B(x, r) = \{y \in X : \rho_c(x, y) \prec r\} \subseteq A$.

Definition 2.3. [2] *Let $\{x_n\}$ be a sequence in X and $x \in X$. If for every $c \in \mathbb{C}$ with $0 \prec c$, there is $n_0 \in N$ such that for all $n > n_0$, $\rho_c(x_n, x) \prec c$, then $\{x_n\}$ is said to be convergent, $\{x_n\}$ converges to x and x is the limit of $\{x_n\}$. We denote this by $\lim_{n\to\infty} x_n = x$. If for every $c \in \mathbb{C}$ with $0 \prec c$ there is $n_0 \in N$, such that for all $n > n_0$, $\rho_c(x_m, x_{n+m}) \prec c$, then $\{x_n\}$ is called a Cauchy sequence in (X, ρ_c).*

Definition 2.4. [2] *If every Cauchy sequence is convergent in (X, ρ_c) then (X, ρ_c) is called a complete complex valued metric space.*

Lemma 2.1. [2] *Let (X, ρ_c) be a complex valued metric space and let $\{x_n\}$ be a sequence in X. Then, $\{x_n\}$ converges to x if and only if $|\rho_c(x_n, x)| \to 0$ as $n \to \infty$.*

Lemma 2.2. [2] *Let (X, ρ_c) be a complex valued metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a Cauchy sequence if and only if $|\rho_c(x_n, x_{n+m})| \to 0$ as $n \to \infty$.*

3. Some results on Fixed point

Theorem 3.1. *Let (X, ρ_c) be a complete complex valued metric space and $S, T : X \to X$ be self mappings satisfying the following condition:*

$$\rho_c(Sx, Ty) \precsim \alpha\rho_c(x, y) + \beta\frac{\rho_c(x, Sx)\rho_c(y, Ty)}{1 + \rho_c(x, y)} + \gamma\frac{\rho_c(x, Sx)\rho_c(y, Ty)}{1 + \rho_c(x, y) + \rho_c(x, Ty) + \rho_c(y, Sx)}$$

for all $x, y \in X$, where α, β, γ are non-negative reals with $\alpha + \beta + \gamma < 1$. Then S, T have a unique common fixed point.

Proof. Let $x_0 \in X$ be any arbitrary point and define $x_{2k+1} = Sx_{2k}$ and $x_{2k+2} = Tx_{2k+1}$.

Then,

$$\begin{aligned}
\rho_c(x_{2k+1}, x_{2k+2}) &= \rho_c(Sx_{2k}, Tx_{2k+1}) \\
&\precsim \alpha\rho_c(x_{2k}, x_{2k+1}) + \beta\frac{\rho_c(x_{2k}, Sx_{2k})\rho_c(x_{2k+1}, Tx_{2k+1})}{1 + \rho_c(x_{2k}, x_{2k+1})} \\
&\quad + \gamma\frac{\rho_c(x_{2k}, Sx_{2k})\rho_c(x_{2k+1}, Tx_{2k+1})}{1 + \rho_c(x_{2k}, x_{2k+1}) + \rho_c(x_{2k}, Tx_{2k+1}) + \rho_c(x_{2k+1}, Sx_{2k})} \\
&\precsim \alpha\rho_c(x_{2k}, x_{2k+1}) + \beta\frac{\rho_c(x_{2k}, x_{2k+1})\rho_c(x_{2k+1}, x_{2k+2})}{1 + \rho_c(x_{2k}, x_{2k+1})} \\
&\quad + \gamma\frac{\rho_c(x_{2k}, x_{2k+1})\rho_c(x_{2k+1}, x_{2k+2})}{1 + \rho_c(x_{2k}, x_{2k+1}) + \rho_c(x_{2k}, x_{2k+2}) + \rho_c(x_{2k+1}, x_{2k+1})}
\end{aligned}$$

Since,

$$\begin{aligned}\rho_c(x_{2k}, x_{2k+1}) &\leq 1 + \rho_c(x_{2k}, x_{2k+1}) \text{ and}\\ \rho_c(x_{2k}, x_{2k+1}) &\leq 1 + \rho_c(x_{2k}, x_{2k+1}) + \rho_c(x_{2k}, x_{2k+2}).\end{aligned}$$

Therefore,

$$\begin{aligned}\rho_c(x_{2k+1}, x_{2k+2}) &\precsim \alpha\rho_c(x_{2k}, x_{2k+1}) + \beta\rho_c(x_{2k+1}, x_{2k+2}) + \gamma\rho_c(x_{2k+1}, x_{2k+2})\\ \rho_c(x_{2k+1}, x_{2k+2}) &\precsim \frac{\alpha}{1-\beta-\gamma}\rho_c(x_{2k}, x_{2k+1}).\end{aligned}$$

Similarly,

$$\begin{aligned}\rho_c(x_{2k+2}, x_{2k+3}) &= \rho_c(x_{2k+3}, x_{2k+2}) = \rho_c(Sx_{2k+2}, Tx_{2k+1})\\ &\precsim \alpha\rho_c(x_{2k+2}, x_{2k+1}) + \beta\frac{\rho_c(x_{2k+2}, Sx_{2k+2})\rho_c(x_{2k+1}, Tx_{2k+1})}{1+\rho_c(x_{2k+2}, x_{2k+1})}\\ &\quad +\gamma\frac{\rho_c(x_{2k+2}, Sx_{2k+2})\rho_c(x_{2k+1}, Tx_{2k+1})}{1+\rho_c(x_{2k+2}, x_{2k+1})+\rho_c(x_{2k+2}, Tx_{2k+1})+\rho_c(x_{2k+1}, Sx_{2k+2})}\\ &\precsim \alpha\rho_c(x_{2k+2}, x_{2k+1}) + \beta\frac{\rho_c(x_{2k+2}, x_{2k+3})\rho_c(x_{2k+1}, x_{2k+2})}{1+\rho_c(x_{2k+2}, x_{2k+1})}\\ &\quad +\gamma\frac{\rho_c(x_{2k+2}, x_{2k+3})\rho_c(x_{2k+1}, x_{2k+2})}{1+\rho_c(x_{2k+2}, x_{2k+1})+\rho_c(x_{2k+2}, x_{2k+2})+\rho_c(x_{2k+1}, x_{2k+3})}\end{aligned}$$

Since,

$$\begin{aligned}\rho_c(x_{2k+2}, x_{2k+1}) &\leq 1 + \rho_c(x_{2k+2}, x_{2k+1}) \text{ and}\\ \rho_c(x_{2k+2}, x_{2k+1}) &\leq 1 + \rho_c(x_{2k+2}, x_{2k+1}) + \rho_c(x_{2k+1}, x_{2k+3}).\end{aligned}$$

Therefore,

$$\begin{aligned}\rho_c(x_{2k+2}, x_{2k+3}) &\precsim \alpha\rho_c(x_{2k+2}, x_{2k+1}) + \beta\rho_c(x_{2k+2}, x_{2k+3}) + \gamma\rho_c(x_{2k+2}, x_{2k+3})\\ \rho_c(x_{2k+2}, x_{2k+3}) &\precsim \frac{\alpha}{1-\beta-\gamma}\rho_c(x_{2k+2}, x_{2k+1})\\ &\text{or}\\ \rho_c(x_{2k+2}, x_{2k+3}) &\precsim \frac{\alpha}{1-\beta-\gamma}\rho_c(x_{2k+1}, x_{2k+2}).\end{aligned}$$

Assume, $h = \dfrac{\alpha}{1-\beta-\gamma} < 1$, we have

$$\rho_c(x_{n+1}, x_{n+2}) \precsim hd(x_n, x_{n+1}) \precsim ... \precsim h^{n+1}\rho_c(x_0, x_1).$$

For some $m > n$, we have

$$\begin{aligned}\rho_c(x_n, x_m) &\precsim \rho_c(x_n, x_{n+1}) + \rho_c(x_{n+1}, x_{n+2}) + ... + \rho_c(x_{m-1}, x_m) \\ &\precsim [h^n + h^{n+1} + ... + h^{m-1}]\rho_c(x_0, x_1) \\ &\precsim \left[\frac{h^n}{1-h}\right]\rho_c(x_0, x_1).\end{aligned}$$

This implies,

$$|\rho_c(x_m, x_n)| \leq \left[\frac{h^n}{1-h}\right]|\rho_c(x_0, x_1)| \to 0, \ as \ m, n \to \infty.$$

Hence, $\{x_n\}$ is a Cauchy sequence. Since X is complete complex valued metric space, therefore there exists $u \in X$ such that $x_n \to u$, we shall show that $u = Su$. To prove that $\rho_c(u, Su) = z > 0$. Therefore, by using triangle inequality, we have

$$\begin{aligned}\rho_c(u, Su) = z &\precsim \rho_c(u, x_{2k+2}) + \rho_c(x_{2k+2}, Su) \\ &\precsim \rho_c(u, x_{2k+2}) + \rho_c(Tx_{2k+1}, Su) \\ &\precsim \rho_c(u, x_{2k+2}) + \alpha\rho_c(x_{2k+1}, u) + \beta\frac{\rho_c(u, Su)\rho_c(x_{2k+1}, Tx_{2k+1})}{1 + \rho_c(x_{2k+1}, u)} \\ &\quad +\gamma\frac{\rho_c(u, Su)\rho_c(x_{2k+1}, Tx_{2k+1})}{1 + \rho_c(x_{2k+1}, u) + \rho_c(x_{2k+1}, Su) + \rho_c(u, Tx_{2k+1})} \\ &\precsim \rho_c(u, x_{2k+2}) + \alpha\rho_c(x_{2k+1}, u) + \beta\frac{z\rho_c(x_{2k+1}, x_{2k+2})}{1 + \rho_c(x_{2k+1}, u)} \\ &\quad +\gamma\frac{zd(x_{2k+1}, x_{2k+2})}{1 + \rho_c(x_{2k+1}, u) + \rho_c(x_{2k+1}, u) + \rho_c(u, x_{2k+2})}.\end{aligned}$$

This implies,

$$\begin{aligned}|\rho_c(u, Su)| &\leq |\rho_c(u, x_{2k+2})| + \alpha|\rho_c(x_{2k+1}, u)| + \beta\frac{|z||\rho_c(x_{2k+1}, x_{2k+2})|}{|1 + \rho_c(x_{2k+1}, u)|} \\ &\quad +\gamma\frac{|z||\rho_c(x_{2k+1}, x_{2k+2})|}{|1 + \rho_c(x_{2k+1}, u) + \rho_c(u, x_{2k+2}) + \rho_c(x_{2k+1}, u)|}.\end{aligned}$$

Letting $k \to \infty$, we have $|\rho_c(u, Su)| \leq 0$, hence $\rho_c(u, Su) = 0$. That is $z = 0$, a contradiction. Hence our supposition is wrong. Therefore, $z = 0$, ie $Su = u$. On the same lines, we can show that $u = Tu$. Therefore, u is a common fixed point of S and T.

Now, we shall show that u is a unique common fixed point of S and T. For this, Consider $u^* = u$ be another common fixed point of S and T.

Therefore,

$$\begin{aligned}\rho_c(u,u^*) &= \rho_c(Su,Tu^*)\\ &\precsim \alpha\rho_c(u,u^*)+\beta\frac{\rho_c(u,Su)\rho_c(u,Tu^*)}{1+\rho_c(u,u^*)}\\ &\quad+\gamma\frac{\rho_c(u,Su)\rho_c(u,Tu^*)}{1+\rho_c(u,u^*)+\rho_c(u,Tu^*)+\rho_c(u^*,Su)}\\ &\precsim \alpha\rho_c(u,u^*).\end{aligned}$$

This implies $(1-\alpha)\rho_c(u,u^*)\precsim 0$ and hence, $(1-\alpha)|\rho_c(u,u^*)|\leq 0$.

Therefore, $\rho_c(u,u^*)=0$ and hence, $u=u^*$, which implies uniqueness. Thus u is a unique common fixed point of S and T.

Corollary 3.1. *Let (X,ρ_c) be a complete complex valued metric space and $T:X\to X$ be a self mapping satisfying the following condition:*

$$\rho_c(Tx,Ty)\precsim\alpha\rho_c(x,y)+\beta\frac{\rho_c(x,Tx)\rho_c(y,Ty)}{1+\rho_c(x,y)}+\gamma\frac{\rho_c(x,Tx)\rho_c(y,Ty)}{1+\rho_c(x,y)+\rho_c(x,Ty)+\rho_c(y,Tx)}$$

for all $x,y\in X$, where α,β,γ are non-negative reals with $\alpha+\beta+\gamma<1$. Then T has a unique fixed point.

Corollary 3.2. *Let (X,ρ_c) be a complete complex valued metric space and $T:X\to X$ be a self mapping satisfying the following condition:*

$$\rho_c(T^nx,T^ny)\precsim\alpha\rho_c(x,y)+\beta\frac{\rho_c(x,T^nx)\rho_c(y,T^ny)}{1+\rho_c(x,y)}+\gamma\frac{\rho_c(x,T^nx)\rho_c(y,T^ny)}{1+\rho_c(x,y)+\rho_c(x,T^ny)+\rho_c(y,T^nx)}$$

for all $x,y\in X$, where α,β,γ are non-negative reals with $\alpha+\beta+\gamma<1$. Then T has a unique fixed point.

Proof. By Corollary 3.1, we obtain $\eta\in X$ such that $T^n\eta=\eta$.

The result then follows from the fact that,

$$\begin{aligned}\rho_c(T^n\eta,\eta) &= \rho_c(TT^n\eta,T^n\eta)=\rho_c(T^nT\eta,T^n\eta)\\ &\precsim \alpha\rho_c(T\eta,\eta)+\beta\frac{\rho_c(T\eta,T^nT\eta)d(\eta,T^n\eta)}{1+\rho_c(T\eta,\eta)}\\ &\quad+\gamma\frac{\rho_c(T\eta,T^nT\eta)d(\eta,T^n\eta)}{1+\rho_c(T\eta,\eta)+\rho_c(T\eta,T^n\eta)+d(\eta,T^nT\eta)}\\ &\precsim \alpha\rho_c(T\eta,\eta)+\beta\frac{\rho_c(T\eta,T^nT\eta)d(\eta,\eta)}{1+\rho_c(T\eta,\eta)}\\ &\quad+\gamma\frac{\rho_c(T\eta,T^nT\eta)d(\eta,\eta)}{1+\rho_c(T\eta,\eta)+\rho_c(T\eta,T^n\eta)+d(\eta,T^nT\eta)}\\ &= \alpha\rho_c(T\eta,\eta)\end{aligned}$$

Therefore, $(1-\alpha)\rho_c(T\eta,\eta) \precsim 0$, this implies, $(1-\alpha)|\rho_c(T\eta,\eta)| \leq 0$, hence $\rho_c(T^n\eta,\eta)=0$. Thus, η is a fixed point of T. On the same lines of Theorem 3.1, we can prove the uniqueness.

Theorem 3.2. *Let (X,ρ_c) be a complete complex valued metric space and $S,T : X \to X$ be self mappings satisfying the following condition:*

$$\rho_c(Sx,Ty) \precsim \alpha\rho_c(x,y) + \beta\frac{\rho_c(x,Sx)\rho_c(y,Ty)}{1+\rho_c(x,y)} + \gamma\frac{\rho_c(x,Sx)\rho_c(y,Ty)}{1+\rho_c(x,Sx)+\rho_c(y,Ty)}$$

for all $x,y \in X$, where α,β,γ are non-negative reals with $\alpha+\beta+\gamma<1$. Then S,T have a unique common fixed point.

Proof. Let $x_0 \in X$ be any arbitrary point and define $x_{2k+1}=Sx_{2k}$ and $x_{2k+2}=Tx_{2k+1}$. Then,

$$\begin{aligned}
\rho_c(x_{2k+1},x_{2k+2}) &= \rho_c(Sx_{2k},Tx_{2k+1})\\
&\precsim \alpha\rho_c(x_{2k},x_{2k+1}) + \beta\frac{\rho_c(x_{2k},Sx_{2k})\rho_c(x_{2k+1},Tx_{2k+1})}{1+\rho_c(x_{2k},x_{2k+1})}\\
&\quad +\gamma\frac{\rho_c(x_{2k},Sx_{2k})\rho_c(x_{2k+1},Tx_{2k+1})}{1+\rho_c(x_{2k},Sx_{2k})+\rho_c(x_{2k+1},Tx_{2k+1})}\\
&\precsim \alpha\rho_c(x_{2k},x_{2k+1}) + \beta\frac{\rho_c(x_{2k},x_{2k+1})\rho_c(x_{2k+1},x_{2k+2})}{1+\rho_c(x_{2k},x_{2k+1})}\\
&\quad +\gamma\frac{\rho_c(x_{2k},x_{2k+1})\rho_c(x_{2k+1},x_{2k+2})}{1+\rho_c(x_{2k},x_{2k+1})+\rho_c(x_{2k+1},x_{2k+2})}.
\end{aligned}$$

Following cases arises,

Case 1. If,

$$\begin{aligned}
\rho_c(x_{2k},x_{2k+1}) &\leq 1+\rho_c(x_{2k},x_{2k+1}) \text{ and}\\
\rho_c(x_{2k+1},x_{2k+2}) &\leq 1+\rho_c(x_{2k},x_{2k+1})+\rho_c(x_{2k+1},x_{2k+2}).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\rho_c(x_{2k+1},x_{2k+2}) &\precsim \alpha\rho_c(x_{2k},x_{2k+1}) + \beta\rho_c(x_{2k+1},x_{2k+2}) + \gamma\rho_c(x_{2k},x_{2k+1})\\
\rho_c(x_{2k+1},x_{2k+2}) &\precsim \frac{\alpha+\gamma}{1-\beta}\rho_c(x_{2k},x_{2k+1})
\end{aligned}$$

Similarly,

$$\begin{aligned}\rho_c(x_{2k+2}, x_{2k+3}) &= \rho_c(Sx_{2k+2}, Tx_{2k+1}) \\ &\precsim \alpha\rho_c(x_{2k+2}, x_{2k+1}) + \beta\frac{\rho_c(x_{2k+2}, Sx_{2k+2})\rho_c(x_{2k+1}, Tx_{2k+1})}{1+\rho_c(x_{2k+2}, x_{2k+1})} \\ &\quad +\gamma\frac{\rho_c(x_{2k+2}, Sx_{2k+2})\rho_c(x_{2k+1}, Tx_{2k+1})}{1+\rho_c(x_{2k+2}, Sx_{2k+2})+\rho_c(x_{2k+1}, Tx_{2k+1})} \\ &\precsim \alpha\rho_c(x_{2k+2}, x_{2k+1}) + \beta\frac{\rho_c(x_{2k+2}, x_{2k+3})\rho_c(x_{2k+1}, x_{2k+2})}{1+\rho_c(x_{2k+2}, x_{2k+1})} \\ &\quad +\gamma\frac{\rho_c(x_{2k+2}, x_{2k+3})\rho_c(x_{2k+1}, x_{2k+2})}{1+\rho_c(x_{2k+2}, x_{2k+3})+\rho_c(x_{2k+1}, x_{2k+2})}\end{aligned}$$

Since,

$$\begin{aligned}\rho_c(x_{2k+2}, x_{2k+1}) &\leq 1+\rho_c(x_{2k+2}, x_{2k+1}) \text{ and} \\ \rho_c(x_{2k+2}, x_{2k+3}) &\leq 1+\rho_c(x_{2k+2}, x_{2k+3})+\rho_c(x_{2k+1}, x_{2k+2}).\end{aligned}$$

Therefore,

$$\begin{aligned}\rho_c(x_{2k+2}, x_{2k+3}) &\precsim \alpha\rho_c(x_{2k+1}, x_{2k+2}) + \beta\rho_c(x_{2k+2}, x_{2k+3}) + \gamma\rho_c(x_{2k+1}, x_{2k+2}) \\ \rho_c(x_{2k+2}, x_{2k+3}) &\precsim \frac{\alpha+\gamma}{1-\beta}\rho_c(x_{2k+1}, x_{2k+2}).\end{aligned}$$

Assume, $h = \dfrac{\alpha+\gamma}{1-\beta} < 1$, we have

$$\rho_c(x_{n+1}, x_{n+2}) \precsim hd(x_n, x_{n+1}) \precsim ... \precsim h^{n+1}\rho_c(x_0, x_1).$$

For some $m > n$, we have

$$\begin{aligned}\rho_c(x_n, x_m) &\precsim \rho_c(x_n, x_{n+1}) + \rho_c(x_{n+1}, x_{n+2}) + ... + \rho_c(x_{m-1}, x_m) \\ &\precsim [h^n + h^{n+1} + ... + h^{m-1}]\rho_c(x_0, x_1) \\ &\precsim \left[\frac{h^n}{1-h}\right]\rho_c(x_0, x_1).\end{aligned}$$

This implies,

$$|\rho_c(x_m, x_n)| \leq \left[\frac{h^n}{1-h}\right]|\rho_c(x_0, x_1)| \to 0, \ as\ m, n \to \infty.$$

Hence, $\{x_n\}$ is a Cauchy sequence. Since X is complete complex valued metric space, therefore there exists $u \in X$ such that $x_n \to u$, we shall show that $u = Su$. To prove, consider

$\rho_c(u, Su) = z > 0$. Therefore, by using triangle inequality, we have

$$\begin{aligned}
\rho_c(u, Su) = z &\precsim \rho_c(u, x_{2k+2}) + \rho_c(x_{2k+2}, Su) \\
&\precsim \rho_c(u, x_{2k+2}) + \rho_c(Tx_{2k+1}, Su) \\
&\precsim \rho_c(u, x_{2k+2}) + \alpha\rho_c(x_{2k+1}, u) + \beta\frac{\rho_c(u, Su)\rho_c(x_{2k+1}, Tx_{2k+1})}{1 + \rho_c(x_{2k+1}, u)} \\
&\quad +\gamma\frac{\rho_c(u, Su)\rho_c(x_{2k+1}, Tx_{2k+1})}{1 + \rho_c(u, Su) + \rho_c(x_{2k+1}, Tx_{2k+1})} \\
&\precsim \rho_c(u, x_{2k+2}) + \alpha\rho_c(x_{2k+1}, u) + \beta\frac{z\rho_c(x_{2k+1}, x_{2k+2})}{1 + \rho_c(x_{2k+1}, u)} \\
&\quad +\gamma\frac{z\rho_c(x_{2k+1}, x_{2k+2})}{1 + z + \rho_c(x_{2k+1}, x_{2k+2})}.
\end{aligned}$$

This implies,

$$\begin{aligned}
|\rho_c(u, Su)| &\precsim |\rho_c(u, x_{2k+2})| + \alpha|\rho_c(x_{2k+1}, u)| + \beta\frac{|z||\rho_c(x_{2k+1}, x_{2k+2})|}{|1 + \rho_c(x_{2k+1}, u)|} \\
&\quad +\gamma\frac{|z||\rho_c(x_{2k+1}, x_{2k+2})|}{|1 + z + \rho_c(x_{2k+1}, x_{2k+2})|}.
\end{aligned}$$

Letting $k \to \infty$, we have $|\rho_c(u, Su)| \leq 0$, hence $\rho_c(u, Su) = 0$. That is $z = 0$, a contradiction. Hence our supposition is wrong. Therefore, $z = 0$, ie $Su = u$. On the same lines, we can show that $u = Tu$. Therefore u is a common fixed point of S and T.

Now, we shall show that u is a unique common fixed point of S and T. For this, Consider $u^* = u$ be another common fixed point of S and T.

Therefore,

$$\begin{aligned}
\rho_c(u, u^*) &= \rho_c(Su, Tu^*) \\
&\precsim \alpha\rho_c(u, u^*) + \beta\frac{\rho_c(u, Su)\rho_c(u, Tu^*)}{1 + \rho_c(u, u^*)} + \gamma\frac{\rho_c(u, Su)\rho_c(u, Tu^*)}{1 + \rho_c(u, Su) + \rho_c(u, Tu^*)} \\
&\precsim \alpha\rho_c(u, u^*).
\end{aligned}$$

This implies $(1 - \alpha)\rho_c(u, u^*) \precsim 0$ and hence, $(1 - \alpha)|\rho_c(u, u^*)| \leq 0$.

Therefore, $\rho_c(u, u^*) = 0$ and hence, $u = u^*$, which implies uniqueness. Thus, u is a unique common fixed point of S and T.

Case 2. If,

$$\begin{aligned}
\rho_c(x_{2k}, x_{2k+1}) &\leq 1 + \rho_c(x_{2k}, x_{2k+1}) \text{ and} \\
\rho_c(x_{2k}, x_{2k+1}) &\leq 1 + \rho_c(x_{2k}, x_{2k+1}) + \rho_c(x_{2k+1}, x_{2k+2}).
\end{aligned}$$

Therefore,

$$\begin{aligned}\rho_c(x_{2k+1}, x_{2k+2}) &\precsim \alpha\rho_c(x_{2k}, x_{2k+1}) + \beta\rho_c(x_{2k+1}, x_{2k+2}) + \gamma\rho_c(x_{2k+1}, x_{2k+2})\\ \rho_c(x_{2k+1}, x_{2k+2}) &\precsim \frac{\alpha}{1-\beta-\gamma}\rho_c(x_{2k}, x_{2k+1}).\end{aligned}$$

Similarly,

$$\begin{aligned}\rho_c(x_{2k+2}, x_{2k+3}) &= \rho_c(Sx_{2k+2}, Tx_{2k+1})\\ &\precsim \alpha\rho_c(x_{2k+2}, x_{2k+1}) + \beta\frac{\rho_c(x_{2k+2}, Sx_{2k+2})\rho_c(x_{2k+1}, Tx_{2k+1})}{1+\rho_c(x_{2k+2}, x_{2k+1})}\\ &\quad+\gamma\frac{\rho_c(x_{2k+2}, Sx_{2k+2})\rho_c(x_{2k+1}, Tx_{2k+1})}{1+\rho_c(x_{2k+2}, Sx_{2k+2})+\rho_c(x_{2k+1}, Tx_{2k+1})}\\ &\precsim \alpha\rho_c(x_{2k+2}, x_{2k+1}) + \beta\frac{\rho_c(x_{2k+2}, x_{2k+3})\rho_c(x_{2k+1}, x_{2k+2})}{1+\rho_c(x_{2k+2}, x_{2k+1})}\\ &\quad+\gamma\frac{\rho_c(x_{2k+2}, x_{2k+3})\rho_c(x_{2k+1}, x_{2k+2})}{1+\rho_c(x_{2k+2}, x_{2k+3})+\rho_c(x_{2k+1}, x_{2k+2})}\end{aligned}$$

Since,

$$\begin{aligned}\rho_c(x_{2k+2}, x_{2k+1}) &\leq 1+\rho_c(x_{2k+2}, x_{2k+1}) \text{ and}\\ \rho_c(x_{2k+2}, x_{2k+1}) &\leq 1+\rho_c(x_{2k+2}, x_{2k+3})+\rho_c(x_{2k+1}, x_{2k+2}).\end{aligned}$$

Therefore,

$$\begin{aligned}\rho_c(x_{2k+2}, x_{2k+3}) &\precsim \alpha\rho_c(x_{2k+1}, x_{2k+2}) + \beta\rho_c(x_{2k+2}, x_{2k+3}) + \gamma\rho_c(x_{2k+2}, x_{2k+3})\\ \rho_c(x_{2k+2}, x_{2k+3}) &\precsim \frac{\alpha}{1-\beta-\gamma}\rho_c(x_{2k}, x_{2k+1})\rho_c(x_{2k+1}, x_{2k+2}).\end{aligned}$$

Assume, $h = \dfrac{\alpha}{1-\beta-\gamma} < 1$, we have

$$\rho_c(x_{n+1}, x_{n+2}) \precsim hd(x_n, x_{n+1}) \precsim ... \precsim h^{n+1}\rho_c(x_0, x_1).$$

For some $m > n$, we have

$$\begin{aligned}\rho_c(x_n, x_m) &\precsim \rho_c(x_n, x_{n+1}) + \rho_c(x_{n+1}, x_{n+2}) + ... + \rho_c(x_{m-1}, x_m)\\ &\precsim [h^n + h^{n+1} + ... + h^{m-1}]\rho_c(x_0, x_1)\\ &\precsim \left[\frac{h^n}{1-h}\right]\rho_c(x_0, x_1).\end{aligned}$$

This implies,

$$|\rho_c(x_m, x_n)| \leq \left[\frac{h^n}{1-h}\right]|\rho_c(x_0, x_1)| \to 0,\ as\ m, n \to \infty.$$

Hence, $\{x_n\}$ is a Cauchy sequence. Since X is complete complex valued metric space, therefore there exists $u \in X$ such that $x_n \to u$, we shall show that $u = Su$. To prove, consider $\rho_c(u, Su) = z > 0$. Therefore, by using triangle inequality, we have

$$\begin{aligned}
\rho_c(u, Su) = z &\precsim \rho_c(u, x_{2k+2}) + \rho_c(x_{2k+2}, Su) \\
&\precsim \rho_c(u, x_{2k+2}) + \rho_c(Tx_{2k+1}, Su) \\
&\precsim \rho_c(u, x_{2k+2}) + \alpha\rho_c(x_{2k+1}, u) + \beta\frac{\rho_c(u, Su)\rho_c(x_{2k+1}, Tx_{2k+1})}{1 + \rho_c(x_{2k+1}, u)} \\
&\quad +\gamma\frac{\rho_c(u, Su)\rho_c(x_{2k+1}, Tx_{2k+1})}{1 + \rho_c(u, Su) + \rho_c(x_{2k+1}, Tx_{2k+1})} \\
&\precsim \rho_c(u, x_{2k+2}) + \alpha\rho_c(x_{2k+1}, u) + \beta\frac{z\rho_c(x_{2k+1}, x_{2k+2})}{1 + \rho_c(x_{2k+1}, u)} \\
&\quad +\gamma\frac{z\rho_c(x_{2k+1}, x_{2k+2})}{1 + z + \rho_c(x_{2k+1}, x_{2k+2})}.
\end{aligned}$$

This implies,

$$\begin{aligned}
|\rho_c(u, Su)| &\precsim |\rho_c(u, x_{2k+2})| + \alpha|\rho_c(x_{2k+1}, u)| + \beta\frac{|z||\rho_c(x_{2k+1}, x_{2k+2})|}{|1 + \rho_c(x_{2k+1}, u)|} \\
&\quad +\gamma\frac{|z||\rho_c(x_{2k+1}, x_{2k+2})|}{|1 + z + \rho_c(x_{2k+1}, x_{2k+2})|}.
\end{aligned}$$

Letting $k \to \infty$, we have $|\rho_c(u, Su)| \leq 0$, hence $\rho_c(u, Su) = 0$. That is $z = 0$, a contradiction. Hence our supposition is wrong. Therefore, $z = 0$, ie $Su = u$. On the same lines, we can show that $u = Tu$. Therefore u is a common fixed point of S and T.

Now, we shall show that u is a unique common fixed point of S and T. For this, Consider $u^* = u$ be another common fixed point of S and T. Therefore,

$$\begin{aligned}
\rho_c(u, u^*) &= \rho_c(Su, Tu^*) \\
&\precsim \alpha\rho_c(u, u^*) + \beta\frac{\rho_c(u, Su)\rho_c(u, Tu^*)}{1 + \rho_c(u, u^*)} + \gamma\frac{\rho_c(u, Su)\rho_c(u, Tu^*)}{1 + \rho_c(u, Su) + \rho_c(u, Tu^*)} \\
&\precsim \alpha\rho_c(u, u^*).
\end{aligned}$$

This implies $(1 - \alpha)\rho_c(u, u^*) \precsim 0$ and hence, $(1 - \alpha)|\rho_c(u, u^*)| \leq 0$. Therefore, $\rho_c(u, u^*) = 0$ and hence, $u = u^*$, which implies uniqueness. Thus, u is a unique common fixed point of S and T.

Corollary 3.3. *Let (X, ρ_c) be a complete complex valued metric space and $T : X \to X$ be a self mapping satisfying the following condition:*

$$\rho_c(Tx, Ty) \precsim \alpha\rho_c(x, y) + \beta\frac{\rho_c(x, Tx)\rho_c(y, Ty)}{1 + \rho_c(x, y)} + \gamma\frac{\rho_c(x, Tx)\rho_c(y, Ty)}{1 + \rho_c(x, Tx) + \rho_c(y, Ty)}$$

for all $x, y \in X$, where α, β, γ are non-negative reals with $\alpha+\beta+\gamma<1$. Then T has a unique fixed point.

Corollary 3.4. *Let (X, ρ_c) be a complete complex valued metric space and $T : X \to X$ be a self mapping satisfying the following condition:*

$$\rho_c(T^n x, T^n y) \precsim \alpha\rho_c(x, y) + \beta\frac{\rho_c(x, T^n x)\rho_c(y, T^n y)}{1+\rho_c(x, y)} + \gamma\frac{\rho_c(x, T^n x)\rho_c(y, T^n y)}{1+\rho_c(x, T^n x)+\rho_c(y, T^n y)}$$

for all $x, y \in X$, where α, β, γ are non-negative reals with $\alpha+\beta+\gamma<1$. Then T has a unique fixed point.

Proof. By Corollary 3.3, we obtain $\eta \in X$ such that $T^n\eta = \eta$.
The result then follows from the fact that,

$$\begin{aligned}
\rho_c(T^n\eta, \eta) &= \rho_c(TT^n\eta, T^n\eta) = \rho_c(T^nT\eta, T^n\eta) \\
&\precsim \alpha\rho_c(T\eta, \eta) + \beta\frac{\rho_c(T\eta, T^nT\eta)d(\eta, T^n\eta)}{1+\rho_c(T\eta, \eta)} + \gamma\frac{\rho_c(T\eta, T^nT\eta)d(\eta, T^n\eta)}{1+\rho_c(T\eta, T^nT\eta)+d(\eta, T^n\eta)} \\
&\precsim \alpha\rho_c(T\eta, \eta) + \beta\frac{\rho_c(T\eta, T^nT\eta)d(\eta, \eta)}{1+\rho_c(T\eta, \eta)} + \gamma\frac{\rho_c(T\eta, T^nT\eta)d(\eta, \eta)}{1+\rho_c(T\eta, T^nT\eta)+d(\eta, T^n\eta)} \\
&= \alpha\rho_c(T\eta, \eta).
\end{aligned}$$

Therefore, $(1-\alpha)\rho_c(T\eta, \eta) \precsim 0$, this implies, $(1-\alpha)|\rho_c(T\eta, \eta)| \le 0$, hence $\rho_c(T^n\eta, \eta) = 0$. Thus, η is a fixed point of T. On the same lines of Theorem 3.2, we can prove the uniqueness.

4. Deduction

Theorem 4.1. [2] *[Azam et al.] Let (X, ρ_c) be a complete complex valued metric space and let the mappings $S, T : X \to X$ satisfy:*

$$\rho_c(Sx, Ty) \preceq \lambda\rho_c(x, y) + \frac{\mu\rho_c(x, Sx)\rho_c(y, Ty)}{1+\rho_c(x, y)}$$

for all $x, y \in X$, where λ, μ are non-negative reals with $\lambda+\mu<1$. Then S, T have a unique common fixed point.

Proof. The required result can be obtained by assuming $\gamma = 0$ in Theorem 3.1 and 3.2.

References

[1] Ahmad, J., Azam, A., and Saejung, S. , Common fixed point results for contractive mappings in complex valued metric spaces, Fixed Point Theory and Applications 2014; 2014: 67.

[2] Azam, A., Fisher, B. and Khan, M., Common fixed point theorems in complex valued metric spaces, Numerical Functional Analysis and Optimization 2011; 32(3): 243-253.

[3] Azam, A., Ahmad, J., and Kumam, P., Common fixed point theorems for multi-valued mappings in complex-valued metric spaces, Journal of Inequalities and Applications 2013: 578.

[4] Bhatt, S., Chaukiyal, S. and Dimri, R.C., A common fixed point theorem for weakly compatible maps in complex valued metric spaces, International Journal of Mathematical Sciences and Applications 2011; 1(3).

[5] Chandok, S. and Kumar, D., Some common fixed points for rational type contraction mappings in complex valued metric spaces, Journal of Operators 2013: Article ID 813707, 6 pages.

[6] Kang, S.M. Coupled fixed point theorems in complex valued metric spaces, International Journal of Mathematical Analysis 7(46), (2013), 2269-2277.

[7] Khan, S.U., Arshad, M., Nashine, H.K. and Nazam, M., Some common fixed points of generalized contractive mappings on complex valued metric spaces, Journal of Analysis and Number Theory 2017; 5(1): 73-80.

[8] Klin-eam, C. and Suanoom, C., Some common fixed point theorems for generalized contractive type mappings on complex valued metric spaces, Abstract and Applied Analysis 2013; Article ID 604215, 6 pages.

[9] Kumar, M., Kumar, P., and Kumar, S., Common fixed point theorems in complex valued metric spaces, Journal of Analysis and Number Theory 2014; 2: 103-109.

[10] Kumar, T.S. and Hussain, R.J., Common coupled fixed point theorem for contractive type mappings in closed ball of complex valued metric spaces, Advances in Inequalities and Applications 2014; 2014: 34.

[11] Nashine, H.K. , Imdad, M., and Hasan,M., Common fixed point theorems under rational contractions in complex valued metric spaces, Journal of Non-linear Sciences and Applications 2014; 7: 42-50.

[12] Rouzkard, F., and Imdad, M., Some common fixed point theorems on complex valued metric spaces, Computers and Mathematics with Applications 2012; 64: 1866-1874.

[13] Sastry,K.P.R, Naidu, G.A. and Bekeshie, T., Metrizability of complex valued metric spaces and some remarks on fixed point theorems in complex valued metric spaces, International Journal of Mathematical Archive 3(7),(2012), 2686-2690.

[14] Sitthikul, K., and Saejung, S., Some fixed point theorems in complex valued metric spaces, Fixed Point Theory and Applications 189, 2012.

[15] Sintunavarat, W., and Kumam, W., Generalized common fixed point theorems in complex valued metric spaces and applications, Journal of Inequalities and Applications,84, 2012.

[16] Sintunavarat, W., Cho, Y.J. and Kumam, P., Urysohn integral equation approach by common fixed points in complex valued metric spaces, Advances in Difference Equations 2013: 49.

[17] Savitri and Hooda,N., Common fixed point theorems for mappings satisfying (CLRg) property, Mathematical Journal of Interdisciplinary Sciences 2015; 4(1): 65-75.

[18] Singh, N., Singh, D., Badal, A. and Joshi,V., Fixed point theorems in complex valued metric spaces, Journal of the Egyptian Mathematical Society 2016; 24: 402-409.

Department of Mathematics, Lovely Professional University, Phagwara, Punjab-144411, India.

E-mail address: deepakanand@live.in

Department of Mathematics, Lovely Professional University, Phagwara, Punjab-144411, India.

E-mail address: amaljpc@outlook.com