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FABER POLYNOMIAL COEFFICIENTS ESTIMATES OF BI-UNIVALENT FUNCTIONS

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ABSTRACT. In our present investigation, we use the Faber polynomial expansions to find upper bounds for the $n-th$ $(n \geq 4)$ coefficients of general subclass of analytic bi-univalent functions. In certain cases, our estimates improve some of those existing coefficient bounds.

1. INTRODUCTION

Let A denote the class of all function $f(z)$ which are analytic in the open unit disk $E = \{z : |z| < 1\}$ and has the Taylor-Maclaurin series expansion of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1}$$

By S we mean the subclass A consisting of univalent functions. The every univalent function $f \in S$ has an inverse f^{-1} which is defined as:

$$f^{-1}(f(z)) = z, \quad z \in E,$$

and

$$f(f^{-1}(w)) = w, \ |w| < r_0(f), \ r_0(f) \geq \frac{1}{4},$$

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where

$$g(w) = f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + ...$$
$$= w + \sum_{n=2}^{\infty} A_n w^n. \qquad (2)$$

A function $f \in A$ is said to be bi-univalent in E if both f and f^{-1} are univalent in E. Let Σ denote the class of analytic and bi-univalent functions in E given by the Taylor-Maclaurin series expansion (1). Some examples of functions in the class Σ are given below:

$$h_1(z) = \frac{z}{1-z},\ h_2(z) = -\log(1-z),\ h_3(z) = \frac{1}{2}\log\left(\frac{1+z}{1-z}\right), \qquad z \in E.$$

However, the famous Koebe function $k(z) = \frac{z}{(1-z)^2}$ is not in Σ, for more details we refer [32]. For $f \in \Sigma$, Levin [22] showed that $|a_2| < 1.51$ and Brannan and Clunie [6] proved that $|a_2| \leq \sqrt{2}$. Netanyahu [27] showed that $\max|a_2| = \frac{4}{3}$. Brannan and Taha [7] introduced certain subclass of the bi-univalent functions. For a brief history and interesting examples of bi-univalent functions we refer, [5, 12, 13, 18, 21, 22, 23, 24, 25, 26, 28, 32].

Not much is known about the bounds on the general coefficient $|a_n|$ for $n \geq 4$. Here, in this paper, we use the Faber polynomial expansions for a subclass of analytic bi-univalent functions to determine estimates for the general coefficient bounds $|a_n|$ for $n \geq 4$.

The Faber polynomials introduced by Faber [11] play an important role in various areas of mathematical sciences, especially in geometric function theory. In the literature, there are only a few works determining the general coefficient bounds $|a_n|$ for the analytic bi-univalent functions given by (1) using Faber polynomial expansions see [16, 15, 19]. A very little is known about the bounds of Maclaurin's series coefficient $|a_n|$ for $n \geq 4$ by using a Faber polynomials we refer [4, 2, 8, 9, 14, 17, 31, 30, 34].

Firstly, we consider class of analytic bi-univalent functions defined by Bulut [8] and class of analytic bi-univalent functions defined by Jahangiri and Hamidi [20]. The purpose of this article is to extend the work of [8, 20] by using well known Faber polynomials. In this paper, we use the Faber polynomial expansions to obtain bounds for the general coefficients $|a_n|$ of bi-univalent functions in $N_{\Sigma}^{\mu}(\delta, \lambda, \alpha, \beta)$ as well as providing estimates for the initial coefficients of these functions.

2. The class $\mathbf{N}_{\Sigma}^{\mu}(\delta, \lambda, \alpha, \beta)$

Definition 1.1. *A function $f \in \Sigma$, $0 \leq \delta \leq 1$, $\lambda \geq 1$, $\mu \geq 0$, and $0 \leq \beta \leq 1$ we introduce a new class of bi-univalent functions $N_{\Sigma}^{\mu}(\delta, \lambda, \alpha, \beta)$ as $f \in N_{\Sigma}^{\mu}(\delta, \lambda, \alpha, \beta)$ if and only if*

$$\operatorname{Re}\left[(1-\delta)\left\{(1-\lambda)\left(\frac{f(z)}{z}\right)^{\mu}+\lambda f^{'}(z)\left(\frac{f(z)}{z}\right)^{\mu-1}\right\}+\delta\left(\frac{zf^{'}(z)}{f(z)}\right)\left(\frac{f(z)}{z}\right)^{\beta}\right]>\alpha, \quad (3)$$

and

$$\operatorname{Re}\left[(1-\delta)\left\{(1-\lambda)\left(\frac{g(w)}{w}\right)^{\mu}+\lambda g^{'}(w)\left(\frac{g(w)}{w}\right)^{\mu-1}\right\}+\delta\left(\frac{wf^{'}(w)}{f(w)}\right)\left(\frac{f(w)}{w}\right)^{\beta}\right]>\alpha, \quad (4)$$

where $0 \leq \alpha < 1$, $z, w \in E$, $g(w) = f^{-1}(w)$ is defined by

Remark 1.1. *In the following special cases of Definition 1 we show how the class of analytic bi-univalent functions $N_{\Sigma}^{\mu}(\delta, \lambda, \alpha, \beta)$ for suitable choices of λ, δ, β and μ lead to certain new as well as known classes of analytic bi-univalent functions studied earlier in the literature.*

(i) For $\delta = 0$, we obtain the class of bi-univalent functions introduced by Bulut [8].

(ii) For $\delta = 1$, we obtain the class of bi-univalent functions introduced by Jahangiri and Hamidi [20].

(iii) For $\delta = 0$ and $\mu = 1$ we obtain the class of bi-univalent function introduced by Frasin and Aouf [13].

(iv) For $\delta = 0$, $\lambda = 1$ and $\mu = 1$ we obtain class of bi-univalent function introduced by Srivastava et al [33].

(v) For $\delta = 0$, and $\lambda = 1$ we have the bi-Bazilevic function class introduced by Prema and Keerthi [29].

(vi) For $\delta = 1$, and $\beta = 1$ we get the class which is consists of functions $f \in \Sigma$, satisfying $Re\Big((f^{'}(z)\Big) > \alpha$ and $Re\Big((g^{'}(w)\Big) > \alpha$, where $0 \leq \alpha < 1$, and $z, w \in E$ and $g = f^{-1}$.

2. Main Results

Using the Faber polynomial expansion of functions $f \in A$ of the form (1), the coefficients of its inverse map $g = f^{-1}$ are given by,

$$g(w) = f^{-1}(w) = w + \sum_{n=2}^{\infty} \frac{1}{n} K_{n-1}^{-n}(a_2, a_3, ...)w^n,$$

where

$$\begin{aligned} K_{n-1}^{-n} &= \frac{(-n)!}{(-2n+1)!(n-5)!}a_2^{n-1} + \frac{(-n)!}{[2(-n+1)]!(n-3)!}a_2^{n-3}a_3 \\ &+ \frac{(-n)!}{(-2n+3)!(n-4)!}a_2^{n-4}a_4 \\ &+ \frac{(-n)!}{[2(-n+2)]!(n-5)!}a_2^{n-5}\left[a_5 + (-n+2)a_3^2\right] \\ &+ \frac{(-n)!}{(-2n+5)!(n-6)!}a_2^{n-6}\left[a_6 + (-2n+5)a_3a_4\right] \\ &+ \sum_{j\geq 7} a_2^{n-j}V_j, \end{aligned} \tag{4}$$

such that V_j with $7 \leq j \leq n$ is a homogeneous polynomial in the variables $\mid a_2 \mid, \mid a_3 \mid, ... |a_n|$, [1]. In particular, the first three terms of K_{n-1}^{-n} are

$$\frac{1}{2}K_1^{-2} = -a_2,$$

$$\frac{1}{3}K_2^{-3} = 2a_2^2 - a_3,$$

$$\frac{1}{4}K_3^{-4} = -(5a_2^3 - 5a_2a_3 + a_4). \tag{5}$$

In general, for any $p \in N$ and $n \geq 2$, an expansion of K_{n-1}^p is as, [2],

$$K_{n-1}^p = pa_n + \frac{p(p-1)}{2}E_{n-1}^2 + \frac{p!}{(p-3)!3!}E_{n-1}^3 + ... + \frac{p!}{(p-n+1)!(n-1)!}E_{n-1}^{n-1}, \tag{6}$$

where $E_{n-1}^p = E_{n-1}^p(a_2, a_3....)$ and by [3],

$$E_{n-1}^m(a_2, ..., a_n) = \sum_{n=2}^{\infty} \frac{m!(a_2)^{\mu_1}...(a_n)^{\mu_{n-1}}}{\mu_{1!}, ..., \mu_{n-1}!}, \qquad \text{for} \ \ m \leq n.$$

While $a_1 = 1$, and the sum is taken over all nonnegative integer $\mu_1, ..., \mu_n$ satisfying

$$\mu_1 + \mu_2 + ... + \mu_n = m,$$

$$\mu_1 + 2\mu_2 + ... + (n-1)\mu_{n-1} = n - 1.$$

Evidently, $E_{n-1}^{n-1}(a_2, ..., a_n) = a_2^{n-1}$, [4]; or equivalently,

$$E_n^m(a_1, a_2, ..., a_n) = \sum_{n=1}^{\infty} \frac{m!(a_1)^{\mu_1}...(a_n)^{\mu_n}}{\mu_{1!}, ..., \mu_n!}, \qquad \text{for } m \leq n,$$

while $a_1 = 1$, and the sum is taken over all nonnegative integer $\mu_1, ..., \mu_n$ satisfying:

$$\mu_1 + \mu_2 + ... + \mu_n = m,$$

$$\mu_1 + 2\mu_2 + ... + (n)\mu_n = n.$$

It is clear that $E_n^n(a_1, ..., a_n) = E_1^n$ the first and last polynomials are:

$$E_n^n = a_1^n, \qquad E_n^1 = a_n.$$

Theorem 2.1. *For* $1 \leq \delta \leq 0$, $\lambda \geq 1$, $\mu \geq 0$, $0 \leq \beta \leq 1$ *and* $0 \leq \alpha < 1$. *Let* $f \in N_\Sigma^\mu(\delta, \lambda, \alpha, \beta)$, *if* $a_m = 0$; $2 \leq m \leq n-1$, *then*

$$|a_n| \leq \frac{2(1-\alpha)}{(1-\delta)\{\mu + (n-1)\lambda\} + \delta\{\beta + (n-1)\}}; \qquad n \geq 4. \tag{7}$$

Proof. For the function $f \in N_\Sigma^\mu(\delta, \lambda, \alpha, \beta)$ of the form (1), we have

$$(1-\delta)\left\{(1-\lambda)\left(\frac{f(z)}{z}\right)^\mu + \lambda f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1}\right\} + \delta\left(\frac{zf'(z)}{f(z)}\right)\left(\frac{f(z)}{z}\right)^\beta$$

$$= 1 + \sum_{n=2}^{\infty} F_{n-1}(a_2, a_3...., a_n)z^{n-1}, \tag{8}$$

and for its inverse map $g = f^{-1}$, we have

$$(1-\delta)\left\{(1-\lambda)\left(\frac{g(w)}{w}\right)^\mu + \lambda f'(w)\left(\frac{g(w)}{w}\right)^{\mu-1}\right\} + \delta\left(\frac{wg'(w)}{g(w)}\right)\left(\frac{g(w)}{w}\right)^\beta$$

$$= 1 + \sum_{n=2}^{\infty} F_{n-1}(A_2, A_3...., A_n)w^{n-1}, \tag{9}$$

where, $A_n = \frac{1}{n}K_{n-1}^{-n}(a_2, a_3, ...)$.

$$F_1 = \{(1-\delta)(\mu+\lambda) + \delta(\beta+1)\}\, a_2,$$

$$F_2 = \{(1-\delta)(\mu+2\lambda) + \delta(\beta+2)\}\left[\frac{(\mu-1)+(\beta-1)}{2}a_2^2 + a_3\right],$$

$$F_3 = \{(1-\delta)(\mu+3\lambda) + \delta(\beta+3)\}\begin{bmatrix} \frac{(\mu-1)(\mu-2)+(\beta-1)(\beta-2)}{3!}a_2^3 \\ -\{(\mu-1)+(\beta-1)\}\, a_2a_3 + a_4 \end{bmatrix}.$$

In general

$$F_{n-1}(a_2, a_3...., a_n) = \left[\left\{\begin{array}{c} (1-\delta)\{\mu + (n-1)\lambda\} + \delta\{\beta + (n-1)\} \\ \times\{(\mu-1)! + (\beta-1)!\} \end{array}\right\} \times G\right],$$

where

$$G = \sum_{i_1+2i_2+...(n-1)i_{n-1}=n-1} \frac{(a_2)^{i_1}a_3^{i_2}...(a_n)^{i_{n-1}}}{i_1!i_2!...,i_n!\left[\{\mu-(i_1+i_2+...i_{n-1})\}! + \{\beta-(i_1+i_2+...i_{n-1})\}!\right]}.$$

On the other hand, since $f \in N_{\Sigma}^{\mu}(\delta,\lambda,\alpha,\beta)$ and $g = f^{-1} \in N_{\Sigma}^{\mu}(\delta,\lambda,\alpha,\beta)$ by definition, there exist two positive real-part functions $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ and $q(w) = 1 + \sum_{n=1}^{\infty} c_n w^n \in A$ where $\operatorname{Re}(p(z)) > 0$ and $\operatorname{Re}(q(w)) > 0$ in E, such that

$$\begin{aligned}
&(1-\delta)\left\{(1-\lambda)\left(\frac{f(z)}{z}\right)^{\mu} + \lambda f^{'}(z)\left(\frac{f(z)}{z}\right)^{\mu-1}\right\} + \delta\left(\frac{zf^{'}(z)}{f(z)}\right)\left(\frac{f(z)}{z}\right)^{\beta}\\
&= \alpha + (1-\alpha)p(z)\\
&= 1 + (1-\alpha)\sum_{n=1}^{\infty} K_n^1(c_1,c_2,...,c_n)z^n \qquad (10)
\end{aligned}$$

and

$$\begin{aligned}
&(1-\delta)\left\{(1-\lambda)\left(\frac{g(w)}{w}\right)^{\mu} + \lambda f^{'}(w)\left(\frac{g(w)}{w}\right)^{\mu-1}\right\} + \delta\left(\frac{wg^{'}(w)}{g(w)}\right)\left(\frac{g(w)}{w}\right)^{\beta}\\
&= \alpha + (1-\alpha)q(w)\\
&= 1 + (1-\alpha)\sum_{n=1}^{\infty} K_n^1(d_1,d_2,...,d_n)w^n. \qquad (11)
\end{aligned}$$

Note that, by the Caratheodory lemma [10], $|c_n| \leq 2$ and $|d_n| \leq 2$, $(n \in N)$. Comparing the corresponding coefficients of (8) and (10) for any $n \geq 2$, we have

$$F_{n-1}(a_2,a_3....,a_n) = (1-\alpha)K_{n-1}^1(c_1,c_2,...,c_{n-1}), \quad n \geq 2. \qquad (12)$$

Which under the assumption $a_m = 0$; $2 \leq m \leq n-1$, we have

$$(1-\delta)\{\mu + (n-1)\lambda\} + \delta\{\beta + (n-1)\}a_n = (1-\alpha)c_{n-1}, \quad n \geq 2.$$

Similarly corresponding coefficients of (9) and (11) we have

$$F_{n-1}(A_2,A_3....,A_n) = (1-\alpha)K_{n-1}^1(d_1,d_2,...,d_{n-1}), \quad n \geq 2. \qquad (13)$$

Which by the hypothesis, we obtain

$$(1-\delta)\{\mu + (n-1)\lambda\} + \delta\{\beta + (n-1)\}A_n = (1-\alpha)d_{n-1}. \qquad (14)$$

Note that for $a_m = 0$; $2 \leq m \leq n-1$ we have $A_n = -a_n$, and so

$$\begin{aligned}
(1-\delta)\{\mu + (n-1)\lambda\} + \delta\{\beta + (n-1)\}a_n &= (1-\alpha)c_{n-1},\\
-(1-\delta)\{\mu + (n-1)\lambda\} + \delta\{\beta + (n-1)\}a_n &= (1-\alpha)d_{n-1}. \qquad (15)
\end{aligned}$$

Now taking the absolute values of equation (14) and (15) and using the fact that $|c_{n-1}| \leq 2$ and $|d_{n-1}| \leq 2$, we obtain

$$\begin{aligned} |a_n| &= \frac{|(1-\alpha)c_{n-1}|}{(1-\delta)\{\mu+(n-1)\lambda\}+\delta\{\beta+(n-1)\}} \\ &= \frac{|(1-\alpha)d_{n-1}|}{(1-\delta)\{\mu+(n-1)\lambda\}+\delta\{\beta+(n-1)\}} \\ &\leq \frac{2(1-\alpha)}{(1-\delta)\{\mu+(n-1)\lambda\}+\delta\{\beta+(n-1)\}} \end{aligned}$$

which completes the proof of Theorem 2.1.

Remark 2.1. *(i) For $\delta = 1$ in Theorem 2.1 we obtain the estimates $|a_n|$, proved by Jahangiri and Hamidi in* [20].
(ii) For $\delta = 0$ in Theorem 2.1 we obtain the estimates $|a_n|$, proved by Bulut in [8].
(iii) For $\delta = 0$, $\mu = 1$ in Theorem 1 we obtain the Corollary 1, proved by Bulut in [8].

Theorem 2.2. *For $1 \leq \delta \leq 0$, $\lambda \geq 1$, $\mu \geq 0$, $0 \leq \beta \leq 1$ and $0 \leq \alpha < 1$. Let $f \in N_{\Sigma}^{\mu}(\delta, \lambda, \alpha, \beta)$. Then*

$$|a_2| \leq \frac{2(1-\alpha)}{\{(1-\delta)(\mu+\lambda)+\delta(\beta+1)\}}, \tag{1a}$$

$$|a_3| \leq \frac{4(1-\alpha)^2}{\{(1-\delta)(\mu+\lambda)+\delta(\beta+1)\}^2} + \frac{2(1-\alpha)}{\{(1-\delta)(\mu+2\lambda)+\delta(\beta+2)\}}, \tag{1b}$$

$$\left|a_3 - a_2^2\right| \leq \frac{2(1-\alpha)}{\{(1-\delta)(\mu+2\lambda)+\delta(\beta+2)\}}. \tag{1c}$$

Proof. Replacing n by 2 and 3 in (12) and (13), respectively, we find that

$$\{(1-\delta)(\mu+\lambda)+\delta(\beta+1)\}\, a_2 = (1-\alpha)c_1, \tag{16}$$

$$\{(1-\delta)(\mu+2\lambda)+\delta(\beta+2)\}\left[\frac{(\mu-1)+(\beta-1)}{2}a_2^2 + a_3\right] = (1-\alpha)c_2, \tag{17}$$

$$-\{(1-\delta)(\mu+2\lambda)+\delta(\beta+2)\}\, a_2 = (1-\alpha)d_1, \tag{18}$$

$$\{(1-\delta)(\mu+2\lambda)+\delta(\beta+2)\}\left[\frac{(\mu+1)+(\beta+1)}{2}a_2^2 - a_3\right] = (1-\alpha)d_2. \tag{19}$$

From (16) and (18) we obtain

$$|a_2| = \left|\frac{(1-\alpha)c_1}{\{(1-\delta)(\mu+\lambda)+\delta(\beta+1)\}}\right| = \left|\frac{(1-\alpha)d_1}{-\{(1-\delta)(\mu+\lambda)+\delta(\beta+1)\}}\right|$$

$$\leq \frac{2(1-\alpha)}{\{(1-\delta)(\mu+\lambda)+\delta(\beta+1)\}}. \tag{20}$$

Adding (17) and (19) we have

$$[\{(1-\delta)(\mu+2\lambda)+\delta(\beta+2)\}(\mu+\beta)]\, a_2^2 = (1-\alpha)(c_2+d_2). \tag{21}$$

Using the Caratheodory lemma, we have

$$|a_2| \leq \sqrt{\frac{4(1-\alpha)}{\{(1-\delta)(\mu+2\lambda)+\delta(\beta+2)\}(\mu+\beta)}}. \tag{22}$$

Combining inequality (20) and (22) we obtain required result (i). Next in order to find the bound on the coefficient $|a_3|$, we subtract (19) from (17) we thus obtain,

$$2\{(1-\delta)(\mu+2\lambda)+\delta(\beta+2)\}(a_3-a_2^2) = (1-\alpha)(c_2-d_2), \tag{23}$$

or

$$a_3 = a_2^2 + \frac{|(1-\alpha)(c_2-d_2)|}{2\{(1-\delta)(\mu+2\lambda)+\delta(\beta+2)\}}. \tag{24}$$

Substituting the value of a_2^2 from (20) into (24), we obtain

$$a_3 = \frac{(1-\alpha)^2 c_1^2}{\{(1-\delta)(\mu+\lambda)+\delta(\beta+1)\}^2} + \frac{(1-\alpha)(c_2-d_2)}{2\{(1-\delta)(\mu+2\lambda)+\delta(\beta+2)\}}. \tag{25}$$

Taking the absolute of (25) and using the Caratheodory lemma we have

$$|a_3| \leq \frac{4(1-\alpha)^2}{\{(1-\delta)(\mu+\lambda)+\delta(\beta+1)\}^2} + \frac{2(1-\alpha)}{\{(1-\delta)(\mu+2\lambda)+\delta(\beta+2)\}}. \tag{26}$$

Again substituting the value of a_2^2 from (21) into (24), we obtain

$$a_3 = \frac{(1-\alpha)(c_2+d_2)}{\{(1-\delta)(\mu+2\lambda)+\delta(\beta+2)\}(\mu+\beta)} + \frac{(1-\alpha)(c_2-d_2)}{2\{(1-\delta)(\mu+2\lambda)+\delta(\beta+2)\}}. \tag{27}$$

Again taking the absolute of (27) and using the Caratheodory lemma we have

$$|a_3| \leq \frac{4(1-\alpha)}{\{(1-\delta)(\mu+2\lambda)+\delta(\beta+2)\}(\mu+\beta)}. \tag{28}$$

From (26) and (28) we obtain required result ($1b$). Taking the absolute values of both sides of the equation (23), we obtain

$$|a_3-a_2^2| = \left|\frac{2(1-\alpha)}{\{(1-\delta)(\mu+2\lambda)+\delta(\beta+2)\}}\right| \leq \frac{2(1-\alpha)}{\{(1-\delta)(\mu+2\lambda)+\delta(\beta+2)\}}. \tag{29}$$

Which is the desired inequality$(1c)$.

Remark 2.2. *(i) For $\delta = 1$, $\mu = 1$ in Theorem 2.2 we obtained the estimates $|a_2|$, $|a_3 - a_2^2|$ proved by Jahangiri and Hamidi in [20].*

(ii) For $\delta = 0$ and $\beta = 1$ in Theorem 2.2 we obtain the estimates $|a_2|$ and $|a_3|$, proved by Bulut in [8].

(iii) For $\delta = 0$, $\beta = 1$ and $\mu = 1$ in Theorem 2.2 we obtain the estimates $|a_2|$ and $|a_3|$ of Corollary 2 proved by Bulut in [8].

(iv) For $\delta = 0$, $\lambda = 1$, and $\beta = 1$ in Theorem 2.2 we obtain the Corollary 3, proved by Bulut in [8].

(v) For $\delta = 1$, $\mu = 1$ and $\beta = 1$ in Theorem 2.2 we obtain the Corollary 2.2, proved by Jahangiri and Hamidi in [20].

Letting $\delta = 1$, $\lambda = 1$, $\mu = 1$ and $\beta = 0$ in Theorem 2.2 we obtain the following corollary for analytic bi-Starlike functions of order α, $0 \leq \alpha < 1$.

Corollary 2.1. *Let $f \in N_{\Sigma}^{1}(1, 1, \alpha, 0)$ be bi-Starlike of order α in E. Then*

$$|a_2| \leq 2(1-\alpha),$$

$$|a_3| \leq 3(1-\alpha),$$

$$\left|a_3 - a_2^2\right| \leq 1-\alpha.$$

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