International Journal of Maps in Mathematics

*Volume **6**, Issue **1**, 2023, Pages:**22-36***

ISSN: 2636-7467 (Online)

www.journalmim.com

SLANT SUBMANIFOLDS OF ALMOST POLY-NORDEN RIEMANNIAN MANIFOLDS

VİLDAN AYHAN AND SELCEN YÜKSEL PERKTAŞ *

ABSTRACT. In the present paper, we study slant submanifolds of an almost poly-Norden Riemannian manifold and give examples. Also we investigate conditions for the normality of the induced structure provided by the almost poly-Norden structure of the ambient manifold.

Keywords: Almost Poly-Norden Manifold, Slant Submanifold, Induced Structure, Normality.

2010 Mathematics Subject Classification: 53B20, 53B25, 53C15.

1. INTRODUCTION

In Riemannian (as well as semi-Riemannian) manifolds, different geometric structures such as almost complex structures, almost product structures, almost contact structures, almost paracontact structures etc. allow rich differential and geometric features to emerge while investigating geometry of submanifolds.

A solution of the equation $x^2 - x - 1 = 0$, the number $\phi = \frac{1+\sqrt{5}}{2} = 1.618...$, is known as the Golden ratio and it is also considered to be the order relation that gives the best harmony and proportions in art and architecture since ancient times. As a generalization of the Golden ratio, Spinadel introduced metallic means family or metallic proportions in [24].

Received:2022-02-21 *Revised:2022-04-28* *Accepted:2022-07-05*

* Corresponding author

Vildan Ayhan; tayhan@gmail.com; https://orcid.org/0000-0002-4499-8509

Selcen Yüksel Perktaş; sperktas@adiyaman.edu.tr; https://orcid.org/0000-0002-8848-0621

.

Members of metallic means family, namely (p, q) metallic numbers, are the positive solutions of the equation $x^2 - px - q = 0$ and denoted by

$$\sigma_{p,q} = \frac{p + \sqrt{p^2 + 4q}}{2}, \tag{1.1}$$

where p and q are positive integer numbers. The well-known members of the metallic means family are the Golden mean, the Silver mean, the Bronze mean; the Copper mean etc. These means constitute a bridge between mathematics, physics and art.

In recent years, inspired by the Golden mean and the metallic mean, the Golden structure and the metallic structure on Riemannian manifolds were introduced in [10] and [18], respectively. Golden Riemannian manifolds, considered an important subclass of metallic Riemannian manifolds and their submanifolds, have extensively been studied by many geometers (see [13, 11, 15, 16, 17]).

In 2006, by a different approach, Kalia [21] introduced a new Bronze mean and studied Bronze Fibonacci and Lucas numbers. The author revealed the relationship between the convergents of continued fractions of the power of Bronze means and the Bronze Fibonacci and Lucas numbers. Note that, unlike the Bronze mean contained by the metallic means family defined in [24], that new Bronze mean given by Kalia [21] can not be expressed with $\sigma_{p,q}$, for positive integers p and q.

Considering the differentiable structure that may occur on a semi-Riemannian manifold depending on the Bronze mean given by [21] and the study on a Riemannian manifold with the Golden structure [10], a new type of manifold equipped with the Bronze structure was introduced by Şahin [26] and the author named it an almost poly-Norden manifold. After then, Perktaş [28] studied submanifolds of almost poly-Norden Riemannian manifolds and examined fundamental geometric features of such submanifolds with the induced structure provided by the almost poly-Norden structure of the ambient manifold.

Slant submanifolds were first defined by Chen (see, [8], [9]) in complex manifolds. Later, submanifolds of this type have begun to be widely studied on different manifolds. For slant submanifolds in almost contact metric manifolds, in Sasakian manifolds, in para-Hermitian manifolds and in almost product manifolds we refer to [2, 3, 4, 7, 6, 22, 25]. Invariant, anti-invariant, semi-invariant, slant, semi-slant, hemi-slant and bi-slant submanifolds of a metallic Riemannian manifold were studied in [5, 19, 20]. Some types of lightlike submanifolds of a Golden semi-Riemannian manifold and metallic semi-Riemannian manifold were introduced in [1, 12, 14, 23, 27].

In the present paper, we study slant submanifolds of an almost poly-Norden Riemannian manifold and give examples. Also we investigate conditions for the normality of the induced structure provided by the almost poly-Norden structure of the ambient manifold.

2. Preliminaries

The Bronze mean introduced by Kalia [21] is the positive solution of $x^2-mx+1=0$,which is defined by

$$B_m=\frac{m+\sqrt{m^2-4}}{2}. \tag{2.2}$$

For detailed reading on the relations between Bronze Fibonacci numbers, Bronze Lucas numbers and family of sequences given by the recurrences, we refer to [21].

Inspired by the Bronze mean given by (2.2), Şahin [26], defined a structure on a differentiable manifold, precisely the Bronze structure. A differentiable manifold $\hat{M}$ with a $(1,1)$-tensor field $\hat{\Phi}$ satisfying

$$\hat{\Phi}^2=m\hat{\Phi}-I, \tag{2.3}$$

where I is the identity operator on the set of cross sections of tangent bundle $T\hat{M}$ denoted by $\Gamma(T\hat{M})$, is called an almost poly-Norden manifold equipped with a poly-Norden structure $\hat{\Phi}$. Also, an almost poly-Norden manifold $(\hat{M},\hat{\Phi})$ having a semi-Riemannian metric $\hat{g}$ which is $\hat{\Phi}$-compatible, i.e.,

$$g(\hat{\Phi}X,\hat{\Phi}Y)=mg(\hat{\Phi}X,Y)-g(X,Y), \tag{2.4}$$

equivalent to

$$g(\hat{\Phi}X,Y)=g(X,\hat{\Phi}Y), \tag{2.5}$$

for any $X,Y\in\Gamma(T\hat{M})$, is called an almost poly-Norden semi-Riemannian manifold [26]. Every complex structure $\hat{F}$ allows to reduce two poly-Norden structures to a semi-Riemannian manifold given by [26]:

$$\hat{\Phi}_1=\frac{m}{2}I+\frac{\sqrt{4-m^2}}{2}\hat{F},\quad \hat{\Phi}_2=\frac{m}{2}I-\frac{\sqrt{4-m^2}}{2}\hat{F},\quad -2<m<2.$$

Conversely, every poly-Norden structure $\hat{\Phi}$ give rise to define two almost complex structures in the followings [26]:

$$\hat{F}_1=-\frac{m}{\sqrt{4-m^2}}I+\frac{2}{\sqrt{4-m^2}}\hat{\Phi},\quad \hat{F}_2=\frac{m}{\sqrt{4-m^2}}I-\frac{2}{\sqrt{4-m^2}}\hat{\Phi},\quad -2<m<2.$$

A poly-Norden semi-Riemannian manifold is an almost poly-Norden semi-Riemannian manifold with a parallel poly-Norden structure $\hat{\Phi}$ with respect to Levi-Civita connection $\hat{\nabla}$ on the manifold. The integrability of $\hat{\Phi}$ is defined by vanishing of the its Nijenhuis tensor field

$N_{\hat{\Phi}}(X,Y) := [\hat{\Phi}X, \hat{\Phi}Y] - \hat{\Phi}[\hat{\Phi}X, Y] - \hat{\Phi}[X, \hat{\Phi}Y] + \hat{\Phi}^2[X,Y]$, for any $X, Y \in \Gamma(T\hat{M})$. Note that $N_{\hat{\Phi}} = 0$ is equivalent to $\hat{\nabla}\hat{\Phi} = 0$, where $\hat{\nabla}$ is the Levi-Civita connection on $\hat{M}$. It was shown that in case of m is being zero every Norden manifold becomes an almost poly-Norden manifold [26].

Throughout the paper we will consider $m \neq 0$.

3. Submanifolds of almost poly-Norden Riemannian manifolds

Let $(\hat{M}, \hat{\Phi}, g)$ be an $(n+k)$-dimensional almost poly-Norden Riemannian manifold and M be an n-dimensional isometrically immersed submanifold of $\hat{M}$. For any $X \in \Gamma(TM)$ and $U \in \Gamma(TM^{\perp})$, we put

$$\hat{\Phi}X = fX + wX, \tag{3.6}$$

$$\hat{\Phi}U = BU + CU, \tag{3.7}$$

where fX (resp., wX) is the tangential (resp., normal) part of $\hat{\Phi}X$ and BU (resp., CU) is the tangential (resp., normal) part of $\hat{\Phi}U$.

From (2.5) and (3.7) one can easily see that

$$g(fX,Y) = g(X,fY), \quad \forall X,Y \in \Gamma(TM), \tag{3.8}$$

$$g(CU,V) = g(U,CV), \quad \forall U,V \in \Gamma(TM^{\perp}). \tag{3.9}$$

Also, the maps w and B are related by $g(wX,U) = g(X,BU)$, for any $X \in \Gamma(TM)$ and $U \in \Gamma(TM^{\perp})$.

Denoting by $\hat{\nabla}$ and ∇, the Levi-Civita connections on M and $\hat{M}$, respectively, then Gauss and Weingarten formulas are given as follows:

$$\hat{\nabla}_X Y = \nabla_X Y + \sum_{\beta=1}^{k} h_\beta(X,Y)N_\beta, \tag{3.10}$$

$$\hat{\nabla}_X N_\beta = -A_{N_\beta} X + \sum_{\gamma=1}^{k} \sigma_{\beta\gamma}(X)N_\gamma, \tag{3.11}$$

for any $X, Y \in \Gamma(TM)$ and an orthonormal basis $\{N_1, ..., N_k\}$ of $TM^{\perp}$, where $\beta, \gamma \in \{1, ..., k\}$. Here, $h(X,Y) = \sum_{\beta=1}^{k} h_\beta(X,Y)N_\beta$ and A_{N_β} is the shape operator in the direction of N_β defined by $g(A_{N_\beta} X, Y) = h_\beta(X,Y)$. Also, $\sigma_{\beta\gamma}$ $(1 \leq \beta, \gamma \leq k)$ denotes the 1-forms on the submanifold M which satisfy $\hat{\nabla}^{\perp}_X N_\beta = \sum_{\gamma=1}^{k} \sigma_{\beta\gamma}(X)N_\gamma$. Note that by taking the covariant derivative of $g(N_\beta, N_\gamma) = \delta_{\beta\gamma}$ on M, one gets $\sigma_{\beta\gamma} = -\sigma_{\gamma\beta}$.

For any $X \in \Gamma(TM)$, $\hat{\Phi}X$ and $\hat{\Phi}N_\beta$ $(1 \leq \beta \leq k)$ can be written respectively in the following forms:

$$\hat{\Phi}X = fX + \sum_{\beta=1}^{k} \upsilon_\beta(X)\, N_\beta, \tag{3.12}$$

$$\hat{\Phi}N_\beta = \zeta_\beta + \sum_{\gamma=1}^{k} \theta_{\beta\gamma}\, N_\gamma, \tag{3.13}$$

where f is a tensor field of type $(1,1)$ on M which transforms tangent vector field X on M to the tangential component of $\hat{\Phi}X$, υ_β are 1-forms and $\theta_{\beta\gamma}$ are differentiable real valued functions on M providing a $k \times k$ matrix denoted by $(\theta_{\beta\gamma})_{k\times k}$.

Since $g(\hat{\Phi}X, N_\beta) = g(X, \hat{\Phi}N_\beta)$ and $g(\hat{\Phi}N_\beta, N_\gamma) = g(N_\beta, \hat{\Phi}N_\gamma)$, by using (2.4) and (3.8) we have

Lemma 3.1. [28] *In a submanifold M of an almost poly-Norden Riemannian manifold $(\hat{M}, \hat{\Phi}, g)$, we have*

$$\upsilon_\beta(X) = g(\hat{\Phi}X, N_\beta) = g(X, \zeta_\beta), \tag{3.14}$$

$$g(fX, fY) = mg(X, fY) - g(X,Y) + \sum_{\beta,\gamma=1}^{k} \upsilon_\beta(X)\upsilon_\gamma(Y), \tag{3.15}$$

$$\theta_{\beta\gamma} = \theta_{\gamma\beta}, \tag{3.16}$$

for any $X, Y \in \Gamma(TM)$ and $1 \leq \beta, \gamma \leq k$.

Proposition 3.1. [28] *Let M be an n-dimensional isometrically immersed submanifold of an $(n+k)$-dimensional almost poly-Norden Riemannian manifold $(\hat{M}, \hat{\Phi}, g)$. Then the structure $(f, g, \upsilon_\beta, \zeta_\beta, (\theta_{\beta\gamma})_{k\times k})$ on M induced by the almost-poly Norden structure of $\hat{M}$ satisfies*

$$(\nabla_X f)\, Y = \sum_{\beta=1}^{k} \left\{ g(wY, N_\beta) A_{N_\beta} X + h_\beta(X,Y) BN_\beta \right\}, \tag{3.17}$$

$$\begin{aligned} f^2X &= mfX - X - \sum_{\beta=1}^{k} \upsilon_\beta(X)\zeta_\beta, & (3.18)\\ \upsilon_\beta(fX) &= m\upsilon_\beta(X) - \sum_{\gamma=1}^{k} \theta_{\beta\gamma}\upsilon_\gamma(X), & (3.19)\\ \upsilon_\gamma(\zeta_\beta) &= m\theta_{\beta\gamma} - \delta_{\beta\gamma} - \sum_{\lambda=1}^{k} \theta_{\beta\lambda}\theta_{\lambda\gamma}, & (3.20)\\ f\,\zeta_\beta &= m\zeta_\beta - \sum_{\gamma=1}^{k} \theta_{\beta\gamma}\zeta_\gamma, & (3.21) \end{aligned}$$

for any $X \in \Gamma(TM)$. Moreover, in case of $\hat{M}$ is being a poly-Norden semi-Riemannian manifold, we have

$$f\, A_{N_\beta} X + \nabla_X \zeta_\beta - \sum_{\gamma=1}^{k} \theta_{\beta\gamma} A_{N_\gamma} X - \sum_{\gamma=1}^{k} \sigma_{\beta\gamma}(X)\zeta_\gamma = 0. \tag{3.22}$$

4. Slant Submanifolds

Let M be a submanifold of an almost poly-Norden Riemannian manifold $(\hat{M}, \hat{\Phi}, g)$. By using the Cauchy-Schwartz inequality, namely,

$$g(\hat{\Phi}X, fX) \leq \left\|\hat{\Phi}X\right\| \|fX\|, \quad \forall X \in \Gamma(TM),$$

we can state that there exists a function $\theta : T_xM \to \left[0, \frac{\pi}{2}\right]$ satisfying

$$\left|g(\hat{\Phi}X, fX)\right| = \cos\theta(X) \left\|\hat{\Phi}X\right\| \|fX\|,$$

for any $X \in \Gamma(TM)$. Here $\theta(X)$ is called the Wirtinger angle of X.

Now we define the slant submanifolds of an almost poly-Norden Riemannian manifold similar to the definition given in [8]:

Definition 4.1. *Let M be a submanifold of an almost poly-Norden Riemannian manifold $(\hat{M}, \hat{\Phi}, g)$. If for any $X \in \Gamma(TM)$ the angle $\theta(X)$ between $\hat{\Phi}X$ and T_xM does not depend on $X_x \in T_xM$, then M is called a slant submanifold of $(\hat{M}, \hat{\Phi}, g)$.*

In this case, θ is called the slant angle of M. Furthermore, we have

$$\cos\theta = \frac{g(\hat{\Phi}X, fX)}{\left\|\hat{\Phi}X\right\| \|fX\|} = \frac{\|fX\|}{\left\|\hat{\Phi}X\right\|}, \tag{4.23}$$

for any $X \in \Gamma(TM)$ and $\hat{\Phi}X \neq 0$. The invariant and anti-invariant submanifolds of an almost poly-Norden Riemannian manifold are slant submanifolds with the slant angle $\theta = 0$ and $\theta = \frac{\pi}{2}$, respectively.

Proposition 4.1. *Let M be an n-dimensional submanifold of an $(n+k)$-dimensional almost poly-Norden Riemannian manifold $(\hat{M}, \hat{\Phi}, g)$. If M is a slant submanifold with the slant angle θ, then we have*

$$\begin{aligned} g(fX, fY) &= \cos^2\theta\{mg(\hat{\Phi}X, Y) - g(X, Y)\}, && (4.24)\\ g(wX, wY) &= \sin^2\theta\{mg(\hat{\Phi}X, Y) - g(X, Y)\}, && (4.25)\end{aligned}$$

for any $X, Y \in \Gamma(TM)$.

Proof. Since M is a slant submanifold with the slant angle θ, then by putting $X+Y$ instead of X in (4.23) we get

$$\cos^2\theta g(\hat{\Phi}X,\hat{\Phi}Y)=g(fX,fY). \tag{4.26}$$

From (2.4) and the last equation we obtain (4.24).

On the other hand, using (3.6) we write

$$g(\hat{\Phi}X,\hat{\Phi}Y)=g(fX,fY)+g(wX,wY),$$

which implies

$$g(wX,wY)=(1-\cos^2\theta)\{mg(\hat{\Phi}X,Y)-g(X,Y)\}$$

via (4.24) and (2.4). Hence we obtain (4.25).

Theorem 4.1. *A submanifold M of an almost poly-Norden Riemannian manifold $(\hat{M},\hat{\Phi},g)$ is slant if and only if there exists a constant $\lambda\in[0,1]$ such that*

$$f^2=\lambda\,(mf-I)\,. \tag{4.27}$$

Proof. Since M is a slant submanifold, from (3.8) and (4.24) we write

$$\begin{aligned} g(f^2X,Y) &= g(fX,fY)=\cos^2\theta\{mg(\hat{\Phi}X,Y)-g(X,Y)\} \\ &= \cos^2\theta g(mfX-X,Y), \end{aligned}$$

for any $X,Y\in\Gamma(TM)$, which implies

$$f^2X=\cos^2\theta\,(mf-I)\,(X).$$

For $\lambda=\cos^2\theta$ gives (4.27).

Conversely, assume that there exists a constant $\lambda\in[0,1]$ which satisfies (4.27). Then, for any $X\in\Gamma(TM)$ with $fX\neq 0$, we have

$$\begin{aligned} \cos\theta &= \frac{g(\hat{\Phi}X,fX)}{\left\|\hat{\Phi}X\right\|\|fX\|}=\frac{g(X,f^2X)}{\left\|\hat{\Phi}X\right\|\|fX\|} \\ &= \lambda\frac{mg(\hat{\Phi}X,X)-g(X,X)}{\left\|\hat{\Phi}X\right\|\|fX\|}. \end{aligned}$$

By using (2.4) in the last equation we get $\cos\theta=\lambda\frac{\|\hat{\Phi}X\|}{\|fX\|}$, which shows that $\cos^2\theta=\lambda=constant$ and hence, M is a slant submanifold. This completes the proof.

Proposition 4.2. *If M is a slant submanifold with the slant angle θ of an $(n+k)$-dimensional almost poly-Norden Riemannian manifold $(\hat{M}, \hat{\Phi}, g)$, then we have*

$$(\nabla_X f^2)Y = m\cos^2\theta(\nabla_X f)Y, \tag{4.28}$$

for any $X, Y \in \Gamma(TM)$.

Proof. From (4.27), for all $X, Y \in \Gamma(TM)$, we write

$$\nabla_X f^2 Y = \cos^2\theta(m\nabla_X fY - \nabla_X Y)$$

and

$$f^2(\nabla_X Y) = \cos^2\theta(mf\,\nabla_X Y - \nabla_X Y),$$

which completes the proof.

Hence, from (3.17) and (4.28) we give

Proposition 4.3. *Let M be an n-dimensional slant submanifold of an $(n+k)$-dimensional poly-Norden Riemannian manifold $(\hat{M}, \hat{\Phi}, g)$. Then, for any $X, Y \in \Gamma(TM)$, we have*

$$\left(\nabla_X f^2\right) Y = m\cos^2\theta \sum_{\beta=1}^{k} \left\{\upsilon_\beta(Y) A_{N_\beta} X + h_\beta(X,Y)\zeta_\beta\right\}.$$

Proposition 4.4. *If M is a slant submanifold with the slant angle θ ($\theta \neq \frac{\pi}{2}$) of an $(n+k)$-dimensional poly-Norden Riemannian manifold $(\hat{M}, \hat{\Phi}, g)$, then we have*

$$f^2 = \cot^2\theta \sum_{\beta=1}^{k} \nu_\beta \otimes \zeta_\beta.$$

Proof. It follows from (3.18) and (4.27).

Example 4.1. *Let R^4 be the 4-dimensional real number space with a coordinate system (x, y, z, t). We define*

$$\begin{array}{cccc} \hat{\Phi}: & R^4 & \rightarrow & R^4 \\ & (x,y,z,t) & \rightarrow & \hat{\Phi}(x,y,z,t) = (B_m x, B_m y, (m - B_m)\,z, (m - B_m)\,t)\,, \end{array}$$

where $B_m = \frac{m+\sqrt{m^2-4}}{2}$. Then $(R^4, \hat{\Phi})$ is an almost poly-Norden manifold [26]*. If we consider usual scalar product $\langle .,. \rangle$ on R^4, then we see that it is $\hat{\Phi}$-compatible and $(R^4, \hat{\Phi}, \langle .,. \rangle)$ is an almost poly-Norden Riemannian manifold. Now assume that M is a submanifold of $(R^4, \hat{\Phi}, \langle .,. \rangle)$ defined by the immersion*

$$\Omega(u_1, u_2) = (u_1 + u_2, u_1 - u_2, \sqrt{2}u_2, \sqrt{2}u_1).$$

In this case, TM is generated by

$$X = (1, 1, 0, \sqrt{2}), \quad Y = (1, -1, \sqrt{2}, 0).$$

One can see that

$$\hat{\Phi}X = \left(B_m, B_m, 0, \sqrt{2}(m - B_m)\right), \hat{\Phi}Y = \left(B_m, -B_m, \sqrt{2}(m - B_m), 0\right),$$

and

$$\begin{aligned}\left\langle \hat{\Phi}X, X \right\rangle &= 2(B_m + (m - B_m)) = 2m = \left\langle \hat{\Phi}Y, Y \right\rangle, \\ \|X\| &= \|Y\| = 2, \quad \left\|\hat{\Phi}X\right\| = \left\|\hat{\Phi}Y\right\| = \sqrt{2\left(m^2 - 2\right)},\end{aligned}$$

which imply that M is a slant submanifold of $\left(R^4, \hat{\Phi}, \langle ., . \rangle\right)$ with the slant angle

$$\theta = \cos^{-1}\left(\frac{m}{\sqrt{2\left(m^2 - 2\right)}}\right), \quad -\sqrt{2} < m < \sqrt{2}.$$

Example 4.2. *Consider the almost poly-Norden structure given by*

$$\hat{\Phi}(x_i, y_j, t) = \left(B_m x_i, \bar{B}_m y_j, B_m t\right), \quad 1 \leq i, j \leq 4,$$

and the scalar product $\langle ., . \rangle$ on R^9. Then $(R^9, \hat{\Phi}, \langle ., . \rangle)$ is an almost poly-Norden Riemannian manifold. Now let M be a submanifold of $(R^9, \hat{\Phi}, \langle ., . \rangle)$ by

$$\begin{aligned}\Psi(u, v, w, z) &= (\bar{B}_m u \cos\theta, \bar{B}_m v \cos\theta, \bar{B}_m w \cos\theta, \bar{B}_m z \cos\theta, \\ &\quad B_m u \sin\theta, B_m v \sin\theta, B_m w \sin\theta, B_m z \sin\theta, 0).\end{aligned}$$

In this case the tangent bundle of the submanifold is generated by

$$\begin{aligned}E_1 &= (\bar{B}_m \cos\theta, 0, 0, 0, B_m \sin\theta, 0, 0, 0, 0), \\ E_2 &= (0, \bar{B}_m \cos\theta, 0, 0, 0, B_m \sin\theta, 0, 0, 0), \\ E_3 &= (0, 0, \bar{B}_m \cos\theta, 0, 0, 0, B_m \sin\theta, 0, 0), \\ E_4 &= (0, 0, 0, \bar{B}_m \cos\theta, 0, 0, 0, B_m \sin\theta, 0).\end{aligned}$$

Then we calculate

$$\begin{aligned}\hat{\Phi}E_1 &= (\cos\theta, 0, 0, 0, \sin\theta, 0, 0, 0, 0), \\ \hat{\Phi}E_2 &= (0, \cos\theta, 0, 0, 0, \sin\theta, 0, 0, 0), \\ \hat{\Phi}E_3 &= (0, 0, \cos\theta, 0, 0, 0, \sin\theta, 0, 0), \\ \hat{\Phi}E_4 &= (0, 0, 0, \cos\theta, 0, 0, 0, \sin\theta, 0).\end{aligned}$$

and

$$\begin{aligned}\left\langle \hat{\Phi} E_k, E_k \right\rangle &= \bar{B}_m \cos^2\theta + B_m \sin^2\theta, \\ \|E_k\| &= \sqrt{\bar{B}_m^2 \cos^2\theta + B_m^2 \sin^2\theta}, \\ \left\|\hat{\Phi} E_k\right\| &= 1,\end{aligned}$$

where $1 \leq k \leq 4$, *which imply that*

$$\frac{\left\langle \hat{\Phi} E_k, E_k \right\rangle}{\left\|\hat{\Phi} E_k\right\| \|E_k\|} = \frac{\bar{B}_m \cos^2\theta + B_m \sin^2\theta}{\sqrt{\bar{B}_m^2 \cos^2\theta + B_m^2 \sin^2\theta}}.$$

Hence M *is a* 4*-dimensional slant submanifold of* $(R^9, \hat{\Phi}, \langle ., . \rangle)$ *with the slant angle* t *given by*

$$\cos t = \frac{\bar{B}_m \cos^2\theta + B_m \sin^2\theta}{\sqrt{\bar{B}_m^2 \cos^2\theta + B_m^2 \sin^2\theta}}.$$

5. Nijenhuis Tensor Field and Normality of the Structure

Let M be an n-dimensional isometrically immersed submanifold of an $(n+k)$-dimensional almost poly-Norden Riemannian manifold $(\hat{M}, \hat{\Phi}, \hat{g})$. We consider the structure

$$\Pi = (f, g, \upsilon_\beta, \zeta_\beta, (\theta_{\beta\gamma})_{k\times k})$$

on M induced by the almost poly-Norden structure of $\hat{M}$ which satisfies the properties given by Proposition 3.1.

Definition 5.1. *Let* M *be an* n*-dimensional submanifold of an* $(n+k)$*-dimensional almost poly-Norden Riemannian manifold* $(\hat{M}, \hat{\Phi}, g)$*. The structure* Π *is called normal if the Nijenhuis torsion tensor field of* f *satisfies*

$$N_f = 2\sum_{\beta=1}^{k} d\upsilon_\beta \otimes \zeta_\beta.$$

Lemma 5.1. *If* M *is an* n*-dimensional submanifold of an* $(n+k)$*-dimensional poly-Norden Riemannian manifold* $(\hat{M}, \hat{\Phi}, g)$ *and* $\Pi = (f, g, \upsilon_\beta, \zeta_\beta, (\theta_{\beta\gamma})_{k\times k})$ *is the induced structure on* M*, then we have*

$$N_f(X,Y) = \sum_{\beta=1}^{k} \left\{ g(X,\zeta_\beta) B_\beta Y - g(Y,\zeta_\beta) B_\beta X - g(B_\beta X, Y)\zeta_\beta \right\}, \tag{5.29}$$

$$2d\upsilon_\beta(X,Y) = -g(B_\beta X,Y) + \sum_{\gamma=1}^{k}\left\{\sigma_{\beta\gamma}(X)g(Y,\zeta_\gamma) - \sigma_{\beta\gamma}(Y)g(X,\zeta_\gamma)\right\}, \tag{5.30}$$

where $A_\beta = A_{N_\beta}$ *and* $B_\beta = fA_\beta - A_\beta f$, $1 \le \beta \le k$.

Proof. Since the Nijenhuis torsion tensor field of f is given by

$$N_f(X,Y) = (\nabla_{fX} f)\,Y - (\nabla_{fY} f)\,X - f\left[(\nabla_X f)\,Y - (\nabla_Y f)\,X\right],$$

then by using (3.17) we have

$$N_f(X,Y) = \sum_{\beta=1}^{k}\left\{\begin{array}{c} g(wY,N_\beta)A_{N_\beta}fX + g(A_{N_\beta}fX,Y)\zeta_\beta - g(wX,N_\beta)A_{N_\beta}fY \\ -g(X,A_{N_\beta}fY)\zeta_\beta - fg(wY,N_\beta)A_{N_\beta}X + fg(wX,N_\beta)A_{N_\beta}Y \end{array}\right\},$$

which implies

$$N_f(X,Y) = \sum_{\beta=1}^{k}\left\{\begin{array}{c} g(A_{N_\beta}fX - fA_{N_\beta}X,Y)\zeta_\beta \\ -g(X,\zeta_\beta)(A_{N_\beta}f - fA_{N_\beta})Y + g(Y,\zeta_\beta)(A_{N_\beta}f - fA_{N_\beta})X \end{array}\right\},$$

and we obtain (5.29).

From the definition of $d\upsilon_\beta$, it is well-known that

$$2d\upsilon_\beta(X,Y) = g(\nabla_X\zeta_\beta,Y) - g(X,\nabla_Y\zeta_\beta),$$

for any $X,Y \in \Gamma(TM)$. By using (3.22) we get

$$\begin{aligned} 2d\upsilon_\beta(X,Y) &= -g(B_\beta X,Y) \\ &\quad + \sum_{\gamma=1}^{k}\left\{g(A_{N_\gamma}X,Y) - g(X,A_{N_\gamma}Y)\right\}\theta_{\beta\gamma} \\ &\quad + \sum_{\gamma=1}^{k}\left\{g(Y,\zeta_\gamma)\sigma_{\beta\gamma}(X) - g(X,\zeta_\gamma)\sigma_{\beta\gamma}(Y)\right\}, \end{aligned}$$

which gives (5.30).

From (5.29) and (5.30), we obtain

Theorem 5.1. *Let M be an n-dimensional submanifold of an $(n+k)$-dimensional poly-Norden Riemannian manifold $(\hat{M},\hat{\Phi},g)$ with the induced structure $\Pi = (f,g,\upsilon_\beta,\zeta_\beta,(\theta_{\beta\gamma})_{k\times k})$. Then we have*

$$\begin{aligned} N_f(X,Y) - 2\sum_{\beta=1}^{k} d\upsilon_\beta(X,Y)\zeta_\beta &= \sum_{\beta=1}^{k}\{g(X,\zeta_\beta)B_\beta Y - g(Y,\zeta_\beta)B_\beta X\} \\ &\quad - \sum_{\beta=1}^{k}\sum_{\gamma=1}^{k}\{g(Y,\zeta_\gamma)\sigma_{\beta\gamma}(X) \\ &\qquad - g(X,\zeta_\gamma)\sigma_{\beta\gamma}(Y)\zeta_\beta\}, \end{aligned} \tag{5.31}$$

for any $X, Y \in \Gamma(TM)$.

Since $\sigma_{\beta\gamma}$ are the components of the normal connection $\hat{\nabla}^{\perp}$ and $B_\beta = fA_\beta - A_\beta f$, from (5.31) we have

Corollary 5.1. *Let M be an n-dimensional submanifold of an $(n+k)$-dimensional poly-Norden Riemannian manifold $(\hat{M}, \hat{\Phi}, g)$. Then the induced structure $\Pi = (f, g, \upsilon_\beta, \zeta_\beta, (\theta_{\beta\gamma})_{k\times k})$ on M is normal provided that the tensor field f commutes with the Weingarten operator A_β, for all $\beta \in \{1, ..., k\}$ and the normal connection $\hat{\nabla}^{\perp}$ identically vanishes on the normal bundle.*

Lemma 5.2. *Let M be a non-invariant submanifold of codimension $k \geq 1$ in a poly-Norden Riemannian manifold $(\hat{M}, \hat{\Phi}, g)$. If the normal connection $\hat{\nabla}^{\perp}$ vanishes on the normal bundle, then the vector fields $\zeta_1, ..., \zeta_k$ are linearly independent.*

Proof. From (3.14) and (3.20) we write

$$\upsilon_\gamma(\zeta_\beta) = m\theta_{\beta\gamma} - \delta_{\beta\gamma} - \sum_{\lambda=1}^{k} \theta_{\beta\lambda}\theta_{\lambda\gamma} = g(\zeta_\gamma, \zeta_\beta).$$

Assume that $\sum_{i=1}^{k} c_i\zeta_i = 0$, for some real numbers $c_1, ..., c_k$. Then we have

$$0 = \sum_{i=1}^{k} c_i g(\zeta_i, \zeta_\gamma), \quad \gamma \in \{1, ..., k\},$$

which implies a linear equation system defined by

$$\sum_{i=1}^{k} c_i \Upsilon_{ij} = 0, \tag{5.32}$$

for any index $j \in \{1, ..., k\}$. Here, $\Upsilon_{ii} = m\theta_{ii} - 1 - \sum_{\lambda=1}^{k} \theta_{i\lambda}^2$ and $\Upsilon_{ij} = m\theta_{ij} - \sum_{\lambda=1}^{k} \theta_{i\lambda}\theta_{\lambda j}$, for $i, j \in \{1, ..., k\}$ and $i \neq j$. The determinant of the coefficient matrix of the linear system (5.32) is the determinant of the matrix given by

$$P = m\Theta - I_k - \Theta^2, \quad \Theta = \left(\theta_{\beta\gamma}\right)_{k\times k}.$$

In case of M is being a non-invariant submanifold with respect to $\hat{\Phi}$, the determinant of P cannot be zero which implies that the linear equation system (5.32) has only the trivial solution. This completes the proof.

Theorem 5.2. *Let M be a non-invariant submanifold of codimension $k \geq 1$ in a poly-Norden Riemannian manifold $(\hat{M}, \hat{\Phi}, g)$ with vanishing normal connection $\hat{\nabla}^{\perp}$ on the normal bundle.*

Then the induced structure $\Pi = (f, g, \upsilon_\beta, \zeta_\beta, (\theta_{\beta\gamma})_{k\times k})$ *on* M *is normal if and only if the induced* $(1,1)$*-tensor field* f *commutes with the Weingarten operator* A_β*, for all* $\beta \in \{1, ..., k\}$.

Proof. Assume that the induced structure Π is normal. Since $\hat{\nabla}^\perp = 0$ (equivalently, $\sigma_{\beta\gamma} = 0$) on the normal bundle, from (5.31) we have, for any $X, Y \in \Gamma(TM)$:

$$\sum_{\beta=1}^{k} g(X, \zeta_\beta) B_\beta Y = \sum_{\beta=1}^{k} g(Y, \zeta_\beta) B_\beta X,$$

which implies

$$\sum_{\beta=1}^{k} g(X, \zeta_\beta) g(B_\beta Y, Z) = \sum_{\beta=1}^{k} g(Y, \zeta_\beta) g(B_\beta X, Z), \tag{5.33}$$

for any $X, Y, Z \in \Gamma(TM)$. Replacing Y by Z in the last equation we write

$$\sum_{\beta=1}^{k} g(X, \zeta_\beta) g(B_\beta Z, Y) = \sum_{\beta=1}^{k} g(Z, \zeta_\beta) g(B_\beta X, Y). \tag{5.34}$$

By summing the last two equations side by side and using the skew-symmetry property of B_β, we obtain

$$\sum_{\beta=1}^{k} \{g(B_\beta X, Z)\zeta_\beta + g(Z, \zeta_\beta) B_\beta X\} = 0.$$

Interchanging X with Z in the last equation and summing these equations we get

$$\sum_{\beta=1}^{k} \{g(Z, \zeta_\beta) B_\beta X + g(X, \zeta_\beta) B_\beta Z\} = 0,$$

which gives

$$\sum_{\beta=1}^{k} \{g(Z, \zeta_\beta) g(B_\beta X, Y) + g(X, \zeta_\beta) g(B_\beta Z, Y)\} = 0. \tag{5.35}$$

From (5.33) and (5.35), we obtain

$$\sum_{\beta=1}^{k} g(Z, \zeta_\beta) g(B_\beta X, Y) = 0,$$

for any $X, Y, Z \in \Gamma(TM)$. By considering the hypothesis and using Lemma 5.2, we can observe that there exists a vector field $W \in \Gamma(TM)$ such that it is orthogonal on $Span\,\{\{\zeta_1, ..., \zeta_r\} \backslash \zeta_\beta\}$ and $g(W, \zeta_\beta) \neq 0$. So from the last equation we obtain that $B_\beta = 0$, for all $\beta \in \{1, ..., k\}$.

The proof of the converse part is obvious from (5.31).

References

[1] Acet, B. E. (2018). Lightlike hypersurfaces of metallic semi-Riemannian manifolds. Int. J. Geom. Methods Mod. Phys., doi:10.1142/S0219887818502018.

[2] Adati, T. (1981). Submanifolds of an almost product Riemannian manifold. Kodai Math. J. 4(2) (1981) 327-343.

[3] Alegre, P., & Carriazo, A. (2017). Slant submanifolds of para-Hermitian manifolds. Mediterranean Journal of Mathematics, Vol. 14, No. 5, 1–14.

[4] Atçeken, M. (2010). Slant submanifolds of a Riemannian product manifold. Acta Mathematica Scientia B, Vol. 30, No. 1, 215–224.

[5] Blaga, A. M., & Hretcanu, C. E. (2018). Invariant, anti-invariant and slant submanifolds of a metallic Riemannian manifold. Novi Sad J. Math., doi:10.30755/NSJOM.06365.

[6] Cabrerizo, J. L., Carriazo, A., & Fernandez, L. M. (1999). Semi-slant submanifolds of a Sasakian manifold,"Geometriae Dedicata, Vol. 78, No. 2, 183–199.

[7] Cabrerizo, J. L., Carriazo, A., Fernandez, L. M., & Fernandez, M. (2000). Slant submanifolds in Sasakian manifolds. Glasgow Mathematical Journal, Vol. 42, No. 1, 125–138.

[8] Chen, B. Y. (1990). Geometry of Slant Submanifolds. Katholieke Universiteit Leuven, Leuven, Belgium.

[9] Chen, B. Y. (1990). Slant immersions. Bulletin of the Australian Mathematical Society, Vol. 41, No. 1, 135–147.

[10] Crasmareanu, M., & Hretcanu, C. E. (2008). Golden differential geometry. Chaos, Solitons & Fractals 38, 1229–1238.

[11] Erdoğan, F. E., & Yıldırım, C. (2017). Semi-invariant submanifolds of golden Riemannian manifolds. AIP Conference Proceedings, 1833, 020044, doi:10.1063/1.4981692.

[12] Erdoğan, F. E. (2018). Transversal lightlike submanifolds of metallic semi-Riemannian manifolds. Turk J Math, doi: 10.3906/mat-1804-88.

[13] Erdoğan, F. E., & Yıldırım, C. (2018). On a study of the totally umbilical semi-invariant submanifolds of golden Riemannian manifolds. Journal of Polytechnic, 21(4): 967-970.

[14] Erdoğan, F. E., Yüksel Perktaş, S., Acet, B. E., & Blaga, A. M. (2019). Screen Transversal Lightlike Submanifolds of Metallic Semi-Riemannian Manifolds, Journal of Geometry and Physics, Volume 142, 111-120.

[15] Gezer, A., & Cengiz, N. (2013). On integrability of golden Riemannian structures. Turk J Math, 37, 693–703.

[16] Hretcanu, C. E., & Crasmareanu, M. (2007). On some invariant submanifolds in a Riemannian manifold with golden structure. An. Stiint. Univ. Al. I. Cuza Iasi. Mat. (N.S.), 53, 199–211.

[17] Hretcanu, C. E., & Crasmareanu, M. (2009). Applications of the golden ratio on Riemannian manifolds. Turk J Math, 33 (2), 179–191.

[18] Hretcanu, C. E., & Crasmareanu, M. (2013). Metallic structures on Riemannian manifolds. Rev. Un. Mat. Argentina 54, 15–27.

[19] Hretcanu, C. E., & Blaga, A. M. (2018). Submanifolds in metallic Riemannian manifolds. Diff. Geom. Dynm. Syst., 20, 83–97.

[20] Hretcanu, C. E., & Blaga, A. M. (2018). Slant and semi-slant submanifolds in metallic Riemannian manifolds. J. Func. Spac., Vol.2018 Article ID 2864263, 13 pages.

[21] Kalia, S. (2006). The generalizations of the Golden ratio: their powers, continued fractions, and convergents, http://math.mit.edu/research/highschool/primes/papers.php.

[22] Lotta, A. (1996). Slant submanifolds in contact geometry. Bulletin math´ematique de la Soci´et´e des Sciences Math´ematiques de Roumanie, Vol. 39, 183–198.

[23] Poyraz, N. Ö., & Yaşar, E. (2017). Lightlike hypersurfaces of a golden semi-Riemannian manifold. Mediterr. J. Math., 14 (204).

[24] Spinadel, V. W. (1999). The metallic means family and multifractal spectra. Nonlinear Anl. Ser. B: Real World Appl., 36(6), 721-745.

[25] Şahin, B. (2006). Slant submanifolds of an almost product Riemannian manifold. Journal of the Korean Mathematical Society, Vol. 43, No. 4, 717–732.

[26] Şahin, B. (2018). Almost poly-Norden manifolds,International Journal of Maps in Mathematics, Vol.1, Issue 1, 68-79.

[27] Yüksel Perktaş, S., Erdoğan, F. E., & Acet, B. E. (2020). Lightlike Submanifolds of Metallic Semi-Riemannian Manifolds, Filomat, 34(6), 1781–1794.

[28] Yüksel Perktaş, S. (2020). Submanifolds of almost poly-Norden Riemannian manifolds, Turk J Math, 44, 31 – 49.

Faculty of Arts and Sciences, Department of Mathematics, Adiyaman University, Adiyaman, TURKEY

Faculty of Arts and Sciences, Department of Mathematics, Adiyaman University, Adiyaman, TURKEY